RELIABILITY AND THROUGHPUT ANALYSIS OF A CONCATENATED CODING SCHEME*

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ABSTRACT

In this paper, the performance of a concatenated coding scheme for error control in ARQ systems is analyzed for both random-noise and burst-noise channels. In particular, the probability of undetected error and the system throughput are calculated. In this scheme, the inner code is used for both error correction and error detection, and the outer code is used for error detection only. Interleaving/deinterleaving is assumed within the outer code. A retransmission is requested if either the inner code or the outer code detects the presence of errors. Various coding examples are considered. The results show that concatenated coding can provide extremely high system reliability (i.e., low probability of undetected error) and high system throughput.

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1. INTRODUCTION

In a companion paper [1], the probability of undetected error of a specific concatenated coding scheme on a memoryless binary symmetric channel (MBSC) was calculated. Two linear block codes, denoted by C_f and C_b, are used in the concatenated code. The inner code C_f, called the frame code, is an (n,k) binary block code with minimum distance d_f. The frame code is designed to correct t or fewer errors and to simultaneously detect $\lambda(\lambda > t)$ or fewer errors, where $t + \lambda + 1 < d_f$ [2]. The outer code is an (n_b, k_b) binary block code with

$$n_b = mk \ , \tag{1}$$

where m, a positive integer, is the number of frames. The outer code is designed for error detection only.

No interleaving/deinterleaving within the concatenated coding scheme was assumed in [1]. In this paper, we modify the coding scheme of [1] by assuming interleaving/deinterleaving within an outer code word. In addition, we extend the analysis to include burst-noise channels.

The encoding of the concatenated code is achieved in two stages (see Fig. 1). A message of k_b bits is first encoded into a codeword of n_b bits in the outer code C_b. Then this codeword is interleaved to depth m. After interleaving, the n_b-bit block is divided into m k-bit words for encoding by the frame code C_f. Each n-bit code word is called a frame. The two dimensional block fromat is depicted in Fig. 2.

Decoding consists of error correction and error detection on each frame and error detection on the m decoded k-bit segments. When a frame is received, it is first decoded based on the frame code C_f. The n-k parity bits are then removed from the decoded frame. If there are t or fewer transmission errors in a received frame, the errors will be corrected, and the decoded segment is error-free. If there are more than t errors in the received frame, the errors will be either detected or undetected. If the errors are detected, the decoder stops decoding immediately and requests a retransmission of the entire block. On the other hand, if the errors in a frame are undetected, the decoded segment will be stored in a buffer and the decoder begins to decode the next frame. After the m frames of a block have been decoded, the m k-bit decoded segments are then deinterleaved. Error detection is then performed on these deinterleaved segments based on the outer code C_b. If no errors are detected, the m decoded segments are assumed to be error-free, and are accepted by the receiver. If the presence of errors is detected, the m decoded segments are discarded and the receiver requests a retransmission of the entire block.

The error control scheme described above is actually a combination of forward-error-correction (FEC) and automatic-repeat-request (ARQ). In this paper, we analyze the performance of this error control scheme. Specifically, the system reliability and the system throughput are calculated. The system reliability is measured in terms of the probability of undetected error after decoding.

First, by assuming the inner channel to be an MBSC with a bit error rate (BER) ε, we look at the outer channel created by the combination of the interleaver, the frame code, and the inner channel. Then we develop precise expressions for both the probability of undetected error and the system throughput. Various coding examples are considered, and one case studied in [1] is included for comparison. Our results indicate that concatenated coding can provide high throughputs and extremely low undetected error probabilities at moderate values of ε, and for the example considered in [1] the probability of undetected error is slightly lower with interleaving than without interleaving. In addition, interleaving simplifies the performance analysis compared to the analysis without interleaving, which requires a detailed knowledge of the algebraic structure of both the inner code and the outer code. This allows us to easily compare the performance of several different coding examples.

Finally, the analysis is extended to a Gilbert type burst-noise channel [3-5]. The burst-noise channel contains two states. Each state represents an MBSC with BER ε_j, $j = 1,2$, and $\varepsilon_2 >> \varepsilon_1$. The probabilities of undetected error on burst-noise channels, as expected, degrade compared with those on MBSCs with the same average BER, while the system throughputs remain almost the same. For moderate values of average BER, low probabilities of undetected error are still achievable.

2. SYSTEM PERFORMANCE ON AN MBSC

The Outer Channel Model. Let $P_c^{(f)}(\varepsilon)$ denote the probability of correct decoding for the frame code. Suppose that a bounded-distance decoding algorithm is employed. Bounded-distance decoding corrects all received n-bit frames with t or fewer errors. When an n-bit frame with more than t errors is detected, no attempt is made to correct the errors. Since there are $\binom{n}{i}$ distinct ways in which i errors may occur among n bits,

$$P_c^{(f)}(\varepsilon) = \sum_{j=0}^{t} \binom{n}{j} \varepsilon^j (1-\varepsilon)^{n-j} \tag{2}$$

for bounded-distance decoding.

For a code word $\underline{v}$ in the frame code C_f, let $w(\underline{v})$ denote the Hamming weight of $\underline{v}$. If a decoded frame contains an undetectable error pattern, this error pattern must be a nonzero codeword in C_f. Let $\underline{e_o}$ be an undetectable error pattern after decoding. The probability $P_f(w,\varepsilon)$ that a decoded frame contains a nonzero error pattern $\underline{e_o}$ after decoding is given by [1,6-8]

$$P_f(w,\varepsilon) = \sum_{i=0}^{t} \sum_{j=0}^{\min(t-i,n-w)} \binom{w}{i}\binom{n-w}{j} \varepsilon^{w-i+j}(1-\varepsilon)^{n-w+i-j} , \tag{3}$$

where $w = w(\underline{e_o})$, and ε is the BER of the inner channel. If $\varepsilon \ll \frac{1}{n}$, then

$$P_f(w,\varepsilon) \approx \binom{w}{t}\varepsilon^{w-t}(1-\varepsilon)^{n-w+t} . \tag{4}$$

Let $P_{ud}^{(f)}(\varepsilon)$ denote the probability of undetected error for the frame code. Let $\{A_w^{(f)}, d_f \leq w \leq n\}$ be the weight distribution of C_f. It follows from (3) and (4) that

$$P_{ud}^{(f)}(\varepsilon) = \sum_{w=d_f}^{n} A_w^{(f)} P_f(w,\varepsilon) , \tag{5}$$

and

$$\begin{aligned} P_{ud}^{(f)}(\varepsilon) &\approx A_{d_f}^{(f)} P_f(d_f,\varepsilon) \\ &\approx A_{d_f}^{(f)} \binom{d_f}{t} \varepsilon^{d_f-t}(1-\varepsilon)^{n-d_f+t} , \text{ for } \varepsilon \ll \frac{1}{n} . \end{aligned} \tag{6}$$

Now consider any one of the m frames. If the decoded frame contains undetected errors, the BER ε_a after decoding is given by

$$\varepsilon_a = \frac{1}{n} \sum_{w=d_f}^{n} w A_w^{(f)} P_f(w,\varepsilon) . \tag{7}$$

For $\varepsilon \ll \frac{1}{n}$,

$$\varepsilon_a \approx \frac{1}{n} d_f A_{d_f}^{(f)} P_f(d_f,\varepsilon) \tag{8}$$

is a good approximation to ε_a. Let E be defined as the event that a frame contains undetected errors. Let $\varepsilon_{a/E}$ denote the BER embedded in a decoded frame conditioned on the occurrence of event E. It follows from (7) that

$$\varepsilon_{a/E} = \varepsilon_a/P_r\{E\} = \varepsilon_a/P_{ud}^{(f)}(\varepsilon). \tag{9}$$

For $\varepsilon \ll \frac{1}{n}$, substituting (6) and (8) into (9) yields

$$\varepsilon_{a/E} \approx \frac{1/n\ d_f A_{d_f}^{(f)} P_f(d_f,\varepsilon)}{A_{d_f}^{(f)} P_f(d_f,\varepsilon)} = \frac{d_f}{n} . \tag{10}$$

Now define S to be a random variable such that when h of the m frames contain undetected errors, and the remaining m-h frames are decoded correctly, $S = h$, $h = 0,1,2,\ldots,m$. It follows from (2) and (5) that

$$P_r\{S = h\} = \binom{m}{h}[P_{ud}^{(f)}(\varepsilon)]^h[P_c^{(f)}(\varepsilon)]^{m-h} . \tag{11}$$

Note that (11) is not a binomial distribution because $P_{ud}^{(f)}(\varepsilon) + P_c^{(f)}(\varepsilon) < 1$, i.e., some received frames with more than t errors are detected by the frame code.

After deinterleaving of the m decoded segments (with the n-k parity bits removed from each frame), the BER embedded in the n_b-bit block, conditioned on $S = h$, is given by

$$\varepsilon_o(h) = \varepsilon_{a/E} \cdot \frac{h}{m} , \quad h = 0,1,2,\ldots,m. \tag{12}$$

We call the channel specified by (11) and (12) the outer channel, and it is depicted in Fig. 3. Note that $\varepsilon_o(0) = 0$. This channel can be viewed as a block interference (BI) channel, as described in [9]. Δ_h, $h = 0,1,2,\ldots,m$, is called the <u>h^{th} component channel</u> of the BI channel. Each block of n_b bits (n_b is the length of the outer code) is transmitted over one of the m+1 component channels. The random variable S determined which component channel is used to transmit a given n_b-bit block.

<u>The Probability of Undetected Error and the System Throughput.</u> Let $\{A_i^{(b)}, d_b < i < n_b\}$ be the weight distribution of the outer code, where d_b is the minimum distance of C_b. Let $P_{ud}^{(b)}(\varepsilon)$ be the probability of undetected error for the outer code C_b. If the n_b-bit block is transmitted over the h^{th} component channel Δ_h of the outer channel, it follows from (12) that

$$P_{ud}^{(b)}(\varepsilon_o(h)) = \sum_{i=d_b}^{n_b} A_i^{(b)}(\varepsilon_o(h))^i(1 - \varepsilon_o(h))^{n_b-i} \quad . \tag{13}$$

Let $P_{ud}(\varepsilon)$ be the average probability of undetected error of the concatenated code. From (11) and (13) we obtain

$$\begin{aligned} P_{ud}(\varepsilon) &= \sum_{h=0}^{m} \Pr\{S = h\} P_{ud}^{(b)}(\varepsilon_o(h)) \\ &= \sum_{h=1}^{m} \{\binom{m}{h}[P_{ud}^{(f)}(\varepsilon)]^h[P_c^{(f)}(\varepsilon)]^{m-h} \\ &\cdot \sum_{i=d_b}^{n_b} A_i^{(b)}(\varepsilon_o(h))^i(1-\varepsilon_o(h))^{n_b-i}\} \ , \end{aligned} \tag{14}$$

where $P_c^{(f)}(\varepsilon)$ and $P_{ud}^{(f)}(\varepsilon)$ are geven by (2) and (5), respectively.

The system throughput is defined as the ratio of the average number of information bits successfully accepted by the receiver per unit time to the total number of bits that can be transmitted per unit time [2]. It is determined by the retransmission strategy, which may be one of the three basic types: stop-and-wait, go-back-N, or selective-repeat. All three basic ARQ schemes achieve the same reliability; however, they have different throughputs. Suppose that selective-repeat ARQ is used as the retransmission strategy. The specific manner in which the receiver signals to the transmitter for a retransmission will not be considered. It will be assumed, however, that this backword signal is error free, and that repeat retransmissions of a block are possible. Then the throughput of the concatenated coding system is [2]

$$\eta = \frac{k}{n} \cdot \frac{k_b}{n_b} \cdot (P_{ud}(\varepsilon) + P_c(\varepsilon)), \tag{15}$$

where $P_c(\varepsilon)$ is the probability of accepting a correct block. Note that a transmitted block will be received correctly if and only if all m frames are decoded correctly. Therefore,

$$P_c(\varepsilon) = [P_c^{(f)}(\varepsilon)]^m = [\sum_{j=0}^{t} \binom{n}{j}\varepsilon^j(1-\varepsilon)^{n-j}]^m . \tag{16}$$

For the usual situation where $P_{ud}(\varepsilon) \ll P_c(\varepsilon)$, it follows from (15) and (16) that

$$\eta \approx \frac{k}{n} \cdot \frac{k_b}{n_b} \cdot [\sum_{j=0}^{t} \binom{n}{j} \varepsilon^j(1-\varepsilon)^{n-j}]^m . \tag{17}$$

It can easily be seen that η increases monotonically as t increases; but for small ε, η is only a weakly increasing function of t.

In order to find the relationship between t and $P_{ud}(\varepsilon)$, we see from (14) that

$$P_{ud}(\varepsilon) \approx m \cdot P_{ud}^{(f)}(\varepsilon) \cdot [P_c^{(f)}(\varepsilon)]^{m-1}$$

$$\cdot \{\sum_{i=d_b}^{n_b} A_i^{(b)}(\varepsilon_o(1))^i(1-\varepsilon_o(1))^{n_b-i}\} , \text{ for } \varepsilon \ll \frac{1}{n} . \tag{18}$$

Using (6), (10), and (12), $P_{ud}(\varepsilon)$ can be further approximated as

$$P_{ud}(\varepsilon) = K \cdot \binom{d_f}{t} \varepsilon^{d_f-t}(1-\varepsilon)^{n-d_f+t} \cdot [P_c^{(f)}(\varepsilon)]^{m-1} , \quad \text{for } \varepsilon \ll \frac{1}{n} , \tag{19}$$

where

$$K \stackrel{\Delta}{=} m \cdot A_{d_f}^{(f)} \cdot \{\sum_{i=d_b}^{n_b} A_i^{(b)} (\frac{d_f}{m \cdot n})^i (1 - \frac{d_f}{m \cdot n})^{n_b-i}\}$$

is a constant which is independent of t. Let Q(t) denote the right hand side of (19). Then

$$\frac{Q(t+1)}{Q(t)} \approx \frac{(d_f-t)}{(t+1)} \cdot \frac{1}{\varepsilon} \gg n , \text{ for } \varepsilon \ll \frac{1}{n} . \tag{20}$$

That is, for $\varepsilon \ll \frac{1}{n}$, when t increases by 1, $P_{ud}(\varepsilon)$, the probability of undetected error, will increase by approximately ε^{-1}. Thus $P_{ud}(\varepsilon)$ is a strongly increasing function of t. For this reason, a large value of t is not desirable in such a system.

Coding Examples. In this subsection we present some concatenated code examples whose purpose is to give a feeling for actual system performance. Recall that the concatenated coding scheme described above is used in ARQ systems, and that the major advantage of ARQ is that it requires simple decoding while achieving high system reliability and throughput. Therefore, only codes which require simple decoding are chosen as examples.

Example 1. This concatenated code example has been proposed for a NASA telecommand system, and was also considered in [1]. The frame code C_f is a distance-4 Hamming code with generator polynomial

$$g(x) = (x+1)(x^6+x+1) = x^7 + x^6 + x^2 + 1 , \tag{21}$$

where $x^6 + x + 1$ is a primitive polynomial of degree 6. The natural length of this code is 63. This code is used for single error correction (t=1), and is also used to detect all error patterns of weight two and some higher odd weight error patterns. The outer code is a distance-4 shortened Hamming code with generator polynomial

$$\begin{aligned} g(x) &= (x+1)(x^{15} + x^{14} + x^{13} + x^{12} + x^4 + x^3 + x^2 + x + 1) \\ &= x^{16} + x^{12} + x^5 + 1 , \end{aligned} \tag{22}$$

where $x^{15} + x^{14} + x^{13} + x^{12} + x^4 + x^3 + x^2 + x + 1$ is a primitive polynomial of degree 15. This code is the X.25 standard for packet-switched data networks [10]. The natural length of this code is $2^{15} -1 = 32,767$. In this example, a shortened code of maximum length 3,584 bits is considered. This code is used for error detection only. We assume that the number of information bytes (IB) in a frame is between 3 and 7, that is, the inner code can also be shortened.

To obtain a precise result for $P_{ud}(\varepsilon)$, a computer program was written to help determine the reliability of the proposed concatenated coding scheme. We found that if only one frame contains a weight 4 undetected error pattern, then this error pattern can always be detected by the outer code. Thus (14) can be modified as follows:

$$P_{ud}(\varepsilon) = \binom{m}{1}\overline{P}_{ud}^{(f)}(\varepsilon)[P_c^{(f)}(\varepsilon)]^{m-1} \cdot \sum_{i=d_b}^{n_b} A_i^{(b)} (\overline{\varepsilon}_o(1))^i(1-\overline{\varepsilon}_o(1))^{n_b-i}$$

$$+ \sum_{h=2}^{m} \{\binom{m}{h}[P_{ud}^{(f)}(\varepsilon)]^h[P_c^{(f)}(\varepsilon)]^{m-h}$$

$$\cdot \sum_{i=d_b}^{n_b} A_i^{(b)} (\varepsilon_o(h))^i(1 - \varepsilon_o(h))^{n_b-i}\} , \tag{23}$$

where

$$\overline{P}_{ud}^{(f)}(\varepsilon) = \sum_{w=d_f+1}^{n} A_w^{(f)} P_f(w,\varepsilon) , \tag{24.1}$$

and

$$\overline{\varepsilon}_o(1) = \frac{(1/n) \sum_{w=d_f+1}^{n} wA_w^{(f)} P_f(w,\varepsilon)}{\overline{P}_{ud}^{(f)}(\varepsilon)} \cdot \frac{1}{m} . \tag{24.2}$$

Results for the probability of undetected error $P_{ud}(\varepsilon)$, based on (23), and the system throughput η, are plotted in Fig. 4 for $m = 64$, $IB = 7$, and for $m = 24$, $IB = 4$, respectively, where we have used the method in [11] to obtain

$$P_{ud}^{(b)}(\varepsilon_o(h)) = \sum_{i=d_b}^{n_b} A_i^{(b)} (\varepsilon_o(h))^i(1 - \varepsilon_o(h))^{n_b-i} .$$

Comparing the results here with those obtained in [1], we see that interleaving slightly improves the system reliability. For example, for $m = 64$, $IB = 7$, and $\varepsilon = 10^{-5}$, $P_{ud}(\varepsilon) = 6.7 \times 10^{-22}$ with interleaving, while $8.05 \times 10^{-22} < P_{ud}(\varepsilon) < 8.78 \times 10^{-19}$ without interleaving [1].

The example described above can be altered by allowing the frame code to do error detection only (i.e., $t = 0$). In this case, $P_{ud}(\varepsilon)$ and η are shown in Fig. 5.

Example 2. The same frame code and outer code are employed as in Example 1. However, the inner channel is assumed to be an AWGN channel with BPSK modulation and the frame code is decoded by using the Viterbi decoding algorithm with repeat request and infinite demodulator output quantization [12]. Let u, a positive real number, be the retransmission metric threshold of the algorithm [12]. Let $P_{ud}^{(f)}$, $P_d^{(f)}$, and ε_a denote the probability of undetected error, the probability of detected error, and the BER after decoding, respectively, for the frame code. Then [12]

$$P_{ud}^{(f)} \leq k\, Q\left(\sqrt{\frac{2E_N}{N_o} d_f} + u\sqrt{\frac{2E_N}{N_o}}\right) \exp\left(\frac{E_N}{N_o} d_f\right) T(X)\Big|_{X = \exp\left(-\frac{E_N}{N_o}\right)}, \quad (25)$$

$$P_d^{(f)} \leq k\, Q\left(\sqrt{\frac{2E_N}{N_o} d_f} - u\sqrt{\frac{2E_N}{N_o}}\right) \exp\left(\frac{E_N}{N_o} d_f\right) T(X)\Big|_{X = \exp\left(-\frac{E_N}{N_o}\right)}, \quad (26)$$

$$\varepsilon_a \leq Q\left(\sqrt{\frac{2E_N}{N_o} d_f} + u\sqrt{\frac{2E_N}{N_o}}\right) \exp\left(\frac{E_N}{N_o} d_f\right) \frac{\partial T(X,Y)}{\partial Y}\Big|_{\substack{Y = 1 \\ X = \exp\left(-\frac{E_N}{N_o}\right)}}, \quad (27)$$

where

$$Q(X) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} dz, \quad (28)$$

$$T(X) = \sum_{i=d_f}^{n} A_i^{(f)} X^i, \quad (29)$$

$$\frac{\partial T(X,Y)}{\partial Y}\Big|_{Y=1} = \sum_{i=d_f}^{n} B_i^{(f)} X^i, \quad (30)$$

E_N/N_o is the channel symbol signal energy-to-noise power density ratio, and $B_i^{(f)}$ is the total number of nonzero information bits in all codewords of weight i. From (25) and (26) we see that the probability of correct decoding for the frame code is

$$\begin{aligned} P_c^{(f)} &= 1 - P_{ud}^{(f)} - P_d^{(f)} \\ &\approx 1 - P_d^{(f)}, \quad \text{for } P_{ud}^{(f)} \ll P_d^{(f)}. \end{aligned} \quad (31)$$

The probability of undetected error of the concatenated code, P_{ud}, and the system throughput, η, can be computed by using (25) - (31) in (23) and (15). In (23), $\varepsilon_o(h)$, h = 2,3,...,m, is given by (9) and (12), and

$$\overline{P}_{ud}^{(f)} < k\, Q\left(\sqrt{\frac{2E_N}{N_o} d_f} + u\sqrt{\frac{2E_N}{N_o}}\right) \exp\left(\frac{E_N}{N_o} d_f\right)\left(\sum_{i=d_f+1}^{n} A_i^{(f)} X^i\right)\Big|_{X = \exp(-\frac{E_N}{N_o})} , \tag{32}$$

$$\overline{\varepsilon}_o(1) = \frac{Q\left(\sqrt{\frac{2E_N}{N_o} d_f} + u\sqrt{\frac{2E_N}{N_o}}\right) \exp\left(\frac{E_N}{N_o} d_f\right)\left(\sum_{i=d_f+1}^{n} B_i^{(f)} X^i\right)\Big|_{X = \exp(-\frac{E_N}{N_o})}}{\overline{P}_{ud}^{(f)}} . \tag{33}$$

Both P_{ud} and η are shown in Fig. 6 for $u = 4$, where E_N/N_o and ε are related by the equation

$$\varepsilon = Q\left(\sqrt{\frac{2E_N}{N_o}}\right) . \tag{34}$$

The influence of the value of u on the system performance is obvious. For larger values of u, from (25), (26), and (31), the probabilities $P_{ud}^{(f)}$ and $P_c^{(f)}$ become smaller, and consequently the probability of undetected error and the system throughput are lower.

Example 3. The outer code is again a shortened distance-4 Hamming code with generator polynomial given by (22). The frame code is an (n,n-1) single-parity-check code. The frame code has a minimum distance of 2, and is used for error detection only. The frame code can detect all odd weight error patterns. The weight distribution of the frame code can be calculated from

$$A_{2i} = \binom{n-1}{2i} + \binom{n-1}{2i-1} , \quad i = 1,2,3,\ldots, \frac{n}{2} , \tag{35.1}$$

$$A_j = 0 \quad \text{for all odd } j , \tag{35.2}$$

where $\binom{n}{k} = 0$ for $k < 0$ and $k > n$, and x denotes the integer part of x.

Because the outer code can detect three or fewer errors, if only one frame contains a weight 3 or less undetected error pattern, then this error pattern can always be detected by the outer code. Hence, (23) is used to compute the probability of undetected error, where

$$\overline{P}_{ud}^{(f)}(\varepsilon) = \sum_{j=2}^{\frac{n}{2}} A_{2j}\, \varepsilon^{2j}(1 - \varepsilon)^{n-2j} \tag{36}$$

is the probability of undetected error when the undetected error pattern has weight greater than 3, and

$$\overline{\varepsilon}_o(1) = \frac{\sum_{j=2}^{\frac{n}{2}} \frac{2j}{n} A_{2j}\varepsilon^{2j}(1-\varepsilon)^{n-2j}}{\overline{P}_{ud}^{(f)}(\varepsilon)} \cdot \frac{1}{m} \ . \tag{37}$$

Fig. 7 shows the probability of undetected error $P_{ud}(\varepsilon)$ and the system throughput η for this example.

From Figures 4-7, we observe that the performance of a particular scheme depends strongly upon the channel noise conditions. Therefore, we cannot say that a particular one of the above schemes is "best". However, we can draw several conclusions which will be discussed below.

From Figures 4 and 5, we can see the tradeoffs between the probability of undetected error and the system throughput obtained by varying the number of correctable errors t in the frame code. Smaller values of t always result in a lower probability of undetected error, and, therefore, a higher system reliability. But as the channel BER gets higher, the system throughput degrades rapidly for small t. The system throughput is less affected by t if the channel BER is small.

Figure 6 shows the advantages of a Viterbi decoded (soft decision) frame code over an algebraically decoded (hard decision) frame code. The Viterbi decoding algorithm makes the system much more flexible in trading between system reliability and throughput by simply changing the value of u. Varying u can be viewed as a generalized method of "varying t" for algebraic decoding of the frame code. From comparing Figures 4-7, we see that lower inner code rates provide higher system reliabilities but lower system throughputs. We conclude that, at moderately lower BER's, the concatenated coding scheme is capable of achieving high system throughputs and extremely low undetected error probabilities.

3. SYSTEM PERFORMANCE ON A BURST-NOISE-CHANNEL

Channels with memory often occur in practice. Errors on these channels tend to occur in bursts, and hence they are called burst-noise-channels. In this section we extend the performance analysis of the concatenated coding scheme to burst-noise-channels.

The Inner Channel Model. The generalized Gilbert type channel [3-5], as shown in Fig. 8, is used as our inner channel model. There are two states in the model. Each state represents a BSC. State 1 is the "quiet" state, where the BER is ε_1. State 2 is the "noisy" state, where the BER is ε_2, and $\varepsilon_2 \gg \varepsilon_1$. The transition probabilities between states are $P_1 = \Pr\{1 \rightarrow 2\}$ and $P_2 = \Pr\{2 \rightarrow 1\}$ (see Fig. 8). The probabilities of remaining in states 1 and 2 are $q_1 = 1-P_1$ and $q_2 = 1-P_2$, respectively. To simply the model's treatment, we assume that one transition time in the model corresponds to the transmission of one frame of length n bits, i.e., the noisy bursts last for a multiple of the transmission time of a frame. This is a reasonable assumption for channels where burst lengths are usually long compared to the transition time of one frame. The average burst length is then [3]

$$\bar{L} = \frac{1}{P_2} \quad \text{frames,} \tag{38}$$

or

$$\bar{\ell} = \bar{L}n = \frac{1}{P_2} n \quad \text{bits,} \tag{39}$$

the average BER is

$$\bar{\varepsilon} = \frac{1}{P_1+P_2} (P_2\varepsilon_1 + P_1\varepsilon_2) \ , \tag{40}$$

and the steady state probability of being in the noisy state is

$$P_n = \frac{\bar{\varepsilon}-\varepsilon_1}{\varepsilon_2-\varepsilon_1} \ . \tag{41}$$

Four parameters govern the model. They can be chosen to be $\bar{L}$, $\bar{\varepsilon}$, P_n, and the high-to-low BER ratio $\varepsilon_2/\varepsilon_1$.

The Outer Channel Model. Let $P_c^{(f)}(\varepsilon_j)$, $P_{ud}^{(f)}(\varepsilon_j)$, ε_{aj}, and $\varepsilon_{aj/E}$, j=1,2, denote the probability of correct decoding for the frame code, the probability of undetected error for the frame code, the BER in a decoded frame, and the BER embedded in the decoded frame conditioned on the decoded frame containing undetected errors, respectively, when the frame is transmitted in state j. (In the following, we will always use the subscript j, j=1,2, to denote that a frame is transmitted in state j.) Then $P_c^{(f)}(\varepsilon_j)$, $P_{ud}^{(f)}(\varepsilon_j)$, ε_{aj}, and $\varepsilon_{aj/E}$ are given by (2), (5), (7), and (9), respectively, with ε replaced by ε_j, j=1,2.

Now define $E_{\ell,h}$ $0 \le \ell \le h \le m$, to be an event such that h of the m decoded frames contain undetected errors (the other m-h decoded frames are error free) and ℓ of the h frames containing undetected errors are transmitted in state 2 of the inner channel. Let $P_r\{E_{\ell,h}\}$ be the probability that event $E_{\ell,h}$ occurs. Then, after deinterleaving of the m segments (with the n-k parity bits removed from each decoded frame), the BER embedded in the n_b-bit block, conditioned on the occurrence of event $E_{\ell,h}$, is given by

$$\varepsilon_o(E_{\ell,h}) = [\ell \cdot \varepsilon_{a2/E} + (h-\ell)\, \varepsilon_{a1/E}]/m ,$$
$$0 \le \ell \le h \le m . \tag{42}$$

We call the channel specified by (42) and the probability distribution $P_r\{E_{\ell,h}\}$ the outer channel (see Fig. 9).

The Probability of Undetected Error and the System Throughput. If the n_b-bit block is transmitted over the component channel $\Delta_{\ell,h}$ of the outer channel, the probability of undetected error of the outer code is

$$P_{ud}^{(b)}(\varepsilon_o(E_{\ell,h})) = \sum_{i=d_b}^{n_b} A_i^{(b)} (\varepsilon_o(E_{\ell,h}))^i (1 - \varepsilon_o(E_{\ell,h}))^{n_b-i} . \tag{43}$$

Based on the above outer channel model, the average probability of undetected error of the concatenated code can be expressed as

$$P_{ud} = \sum_{h=0}^{m} \sum_{\ell=0}^{h} P\{E_{\ell,h}\}\, P_{ud}^{(b)}(\varepsilon_o(E_{\ell,h})) . \tag{44}$$

For large m, the computation of (44) is very complex and time consuming. To reduce the computational work to a manageable load, we seek an approximation to (44). Define

$$\varepsilon_{max} = \max[\varepsilon_{a1/E}, \varepsilon_{a2/E}] \; . \tag{45}$$

It follows from (42) that

$$\varepsilon_o(E_{\ell,h}) \leq h \cdot \varepsilon_{max}/m \overset{\Delta}{=} \varepsilon_o(h) \; , \tag{46}$$

and equality holds when ε_1 and ε_2 are equal, i.e., when the inner channel is an MBSC. Assuming that $P_{ud}^{(b)}(\varepsilon)$ is an increasing function of ε, $0 \leq \varepsilon \leq \frac{1}{2}$, we obtain from (44) and (46)

$$\begin{aligned} P_{ud} &\leq \sum_{h=0}^{m} \sum_{\ell=0}^{h} \Pr\{E_{\ell,h}\} \, P_{ud}^{(b)}(\varepsilon_o(h)) \\ &= \sum_{h=0}^{m} P_{ud}^{(b)}(\varepsilon_o(h)) \sum_{\ell=0}^{h} \Pr\{E_{\ell,h}\} \\ &= \sum_{h=0}^{m} P_{ud}^{(b)}(\varepsilon_o(h)) \, \beta(h) \; , \end{aligned} \tag{47}$$

where

$$\beta(h) \overset{\Delta}{=} \sum_{\ell=0}^{h} \Pr\{E_{\ell,h}\} \; , \; 0 \leq h \leq m \; , \tag{48}$$

is the probability that h of the m decoded frames contain undetected errors (and the remaining m-h decoded frames are error free).

$\beta(h)$ can be readily computed by a recursive method. To find $\beta(h)$, we model the decoded frame status as a Markov chain. In state j, j=1,2, the decoded frame contains an undetected error with probability $P_{ud}^{(f)}(\varepsilon_j)$ and is error free with probability $P_c^{(f)}(\varepsilon_j)$. Define G(h,m) = Pr{h of the m decoded frames contain undetected errors/the inner channel starts in state 1} and B(h,m) = Pr{h of the m decoded frames contain undetected errors/the inner channel starts in state 2}. By applying a similar argument as in [5], we obtain

$$\beta(h) = \frac{p_2}{p_1+p_2} G(h,m) + \frac{p_1}{p_1+p_2} B(h,m) \ , \ 0 \leq h \leq m \ . \tag{49.1}$$

$G(h,m)$ and $\beta(h,m)$ can be found recursively from

$$G(h,m) = G(h,m-1)\ q_1 P_c^{(f)}(\varepsilon_1) + B(h,m-1) p_1 P_c^{(f)}(\varepsilon_1)$$

$$+ G(h-1,m-1) q_1 P_{ud}^{(f)}(\varepsilon_1) + B(h-1,m-1) p_1 P_{ud}^{(f)}(\varepsilon_1) \ , \tag{49.2}$$

$$B(h,m) = B(h,m-1) q_2 P_c^{(f)}(\varepsilon_2) + G(h,m-1) p_2 P_c^{(f)}(\varepsilon_2)$$

$$+ B(h-1,m-1) q_2 P_{ud}^{(f)}(\varepsilon_2) + G(h-1,m-1) p_2 P_{ud}^{(f)}(\varepsilon_2) \ , \tag{49.3}$$

where

$$G(0,1) = P_c^{(f)}(\varepsilon_1) \quad , \quad B(0,1) = P_c^{(f)}(\varepsilon_2) \ ,$$

$$G(1,1) = P_{ud}^{(f)}(\varepsilon_1) \quad , \quad B(1,1) = P_{ud}^{(f)}(\varepsilon_2) \ , \tag{49.4}$$

and $G(h,m) = B(h,m) = 0$ when $h < 0$ or $h > m$.

Note that if $\varepsilon_{a1/E} \approx \varepsilon_{a2/E}$, the upper bound of (47) is very close to (44). Fortunately, this is usually the case for $0 < \varepsilon_1 < \varepsilon_2 < \frac{1}{2}$, especially for small ε_1 and ε_2, for then $\varepsilon_{a1/E} \approx \varepsilon_{a2/E} \approx d_f/n$.

To evaluate the system throughput, again assume that selective-repeat ARQ is used. In order to simplify the problem, we assume that retransmissions do not depend on the previous inner channel state. This is a reasonable assumption if the channel round-trip delay is large. Then the system throughput is given by (15), where P_c, the probability of correct decoding, can be found from

$$P_c = \frac{p_2}{p_1+p_2} G(m) + \frac{p_1}{p_1+p_2} B(m) \ , \tag{50.1}$$

and where

$$G(m) = G(m-1)\ q_1 P_c^{(f)}(\varepsilon_1) + B(m-1) p_1 P_c^{(f)}(\varepsilon_1) \ , \tag{50.2}$$

$$B(m) = B(m-1)\ q_2 P_c^{(f)}(\varepsilon_2) + G(m-1) p_2 P_c^{(f)}(\varepsilon_2) \ , \tag{50.3}$$

$$G(1) = P_c^{(f)}(\varepsilon_1) \quad , \quad B(1) = P_c^{(f)}(\varepsilon_2) \; . \tag{50.4}$$

Coding Examples on a Burst-Noise-Channel. For the inner channel, we choose $P_n = 0.1$, $\overline{L} = 5$, and $\varepsilon_2/\varepsilon_1 = 10$ and 1000 for our examples.

Example 4. The same frame and outer codes are used as in Example 1. The probability of undetected error, P_{ud}, and the system throughput, η, are plotted in Figures 10.a and 10.b for t = 1, and in Figures 11.a and 11.b for t = 0, respectively.

Example 5. The same coding scheme is used as in Example 3. P_{ud} and η are shown in Figures 12.a and 12.b.

The performance of the concatenated coding scheme on burst-noise-channels depends greatly on the channel parameters, especially on the high-to-low BER ratio, $\varepsilon_2/\varepsilon_1$. As shown in Figures 10.a-12.b, for a given average BER $\overline{\varepsilon}$, with the other parameters fixed, as the $\varepsilon_2/\varepsilon_1$ ratio becomes large, the system performance becomes poor. Our results indicate that on a burst-noise-channel, for a given average BER, the system reliability degrades significantly, while the system throughput remains about the same, compared with the same coding scheme on an MBSC. For moderate values of average BER, high system reliability and throughput are still achievable.

4. CONCLUSIONS

In this paper, the performance of a concatenated coding scheme for error control in data communications is analyzed. By developing a block interference channel model for the outer channel, both the undetected error probability and the system throughput of the concatenated coding scheme were calculated for burst-noise channels as well as random-noise channels. The performance of several specific coding examples was compared. Results indicate that high throughputs and extremely low undetected error probabilities are achievable using this scheme.

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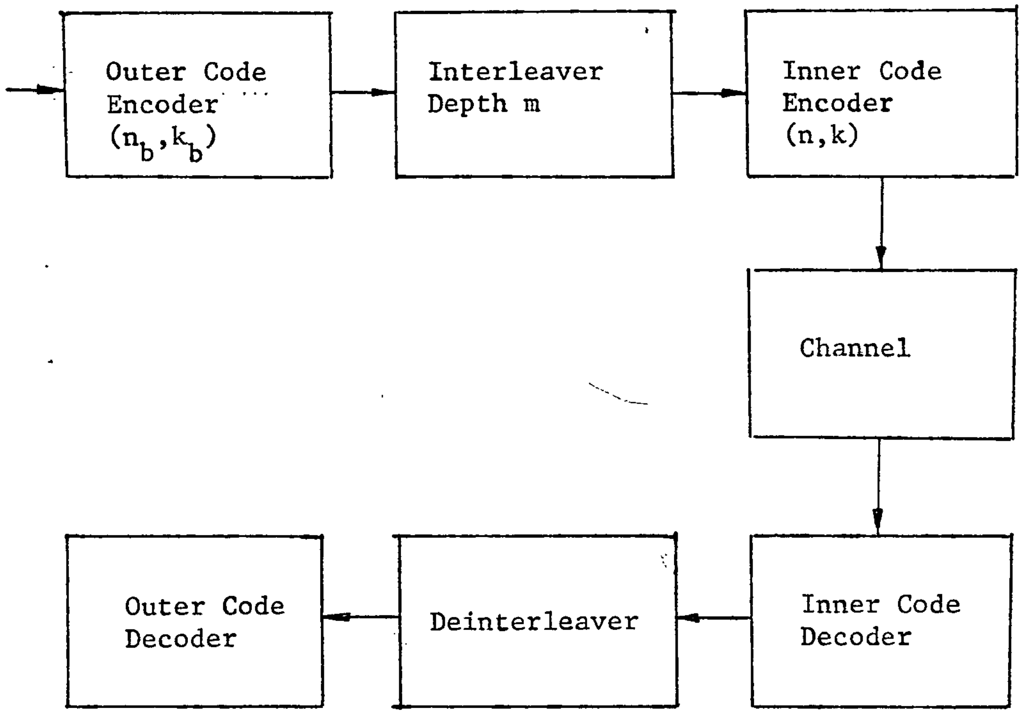

Figure 1 A concatenated coding system

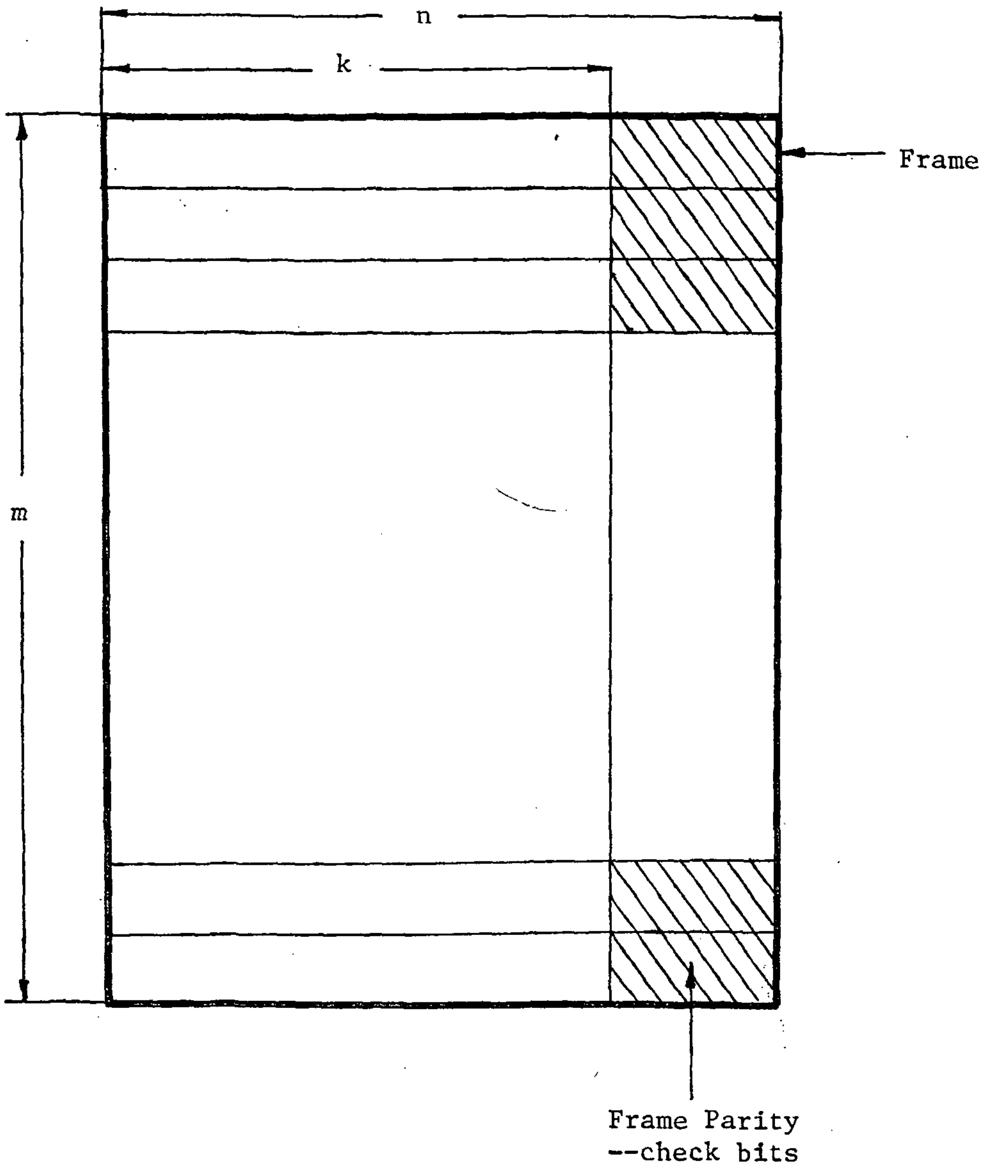

Figure 2 Block format of a concatenated code word

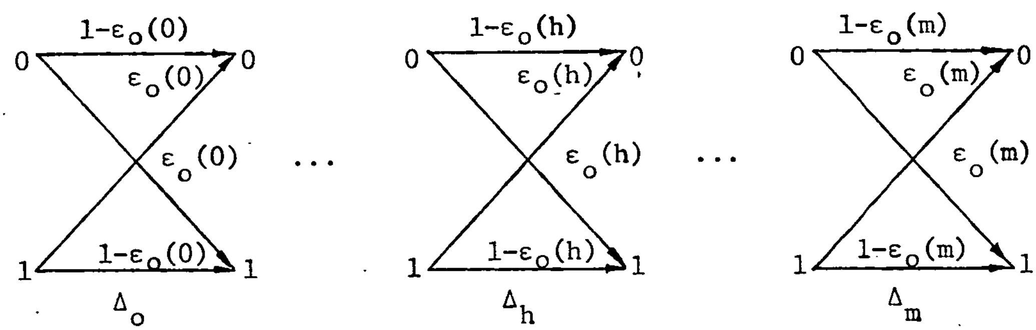

Figure 3 The outer channel resulting from decoding the inner code on an MBSC.

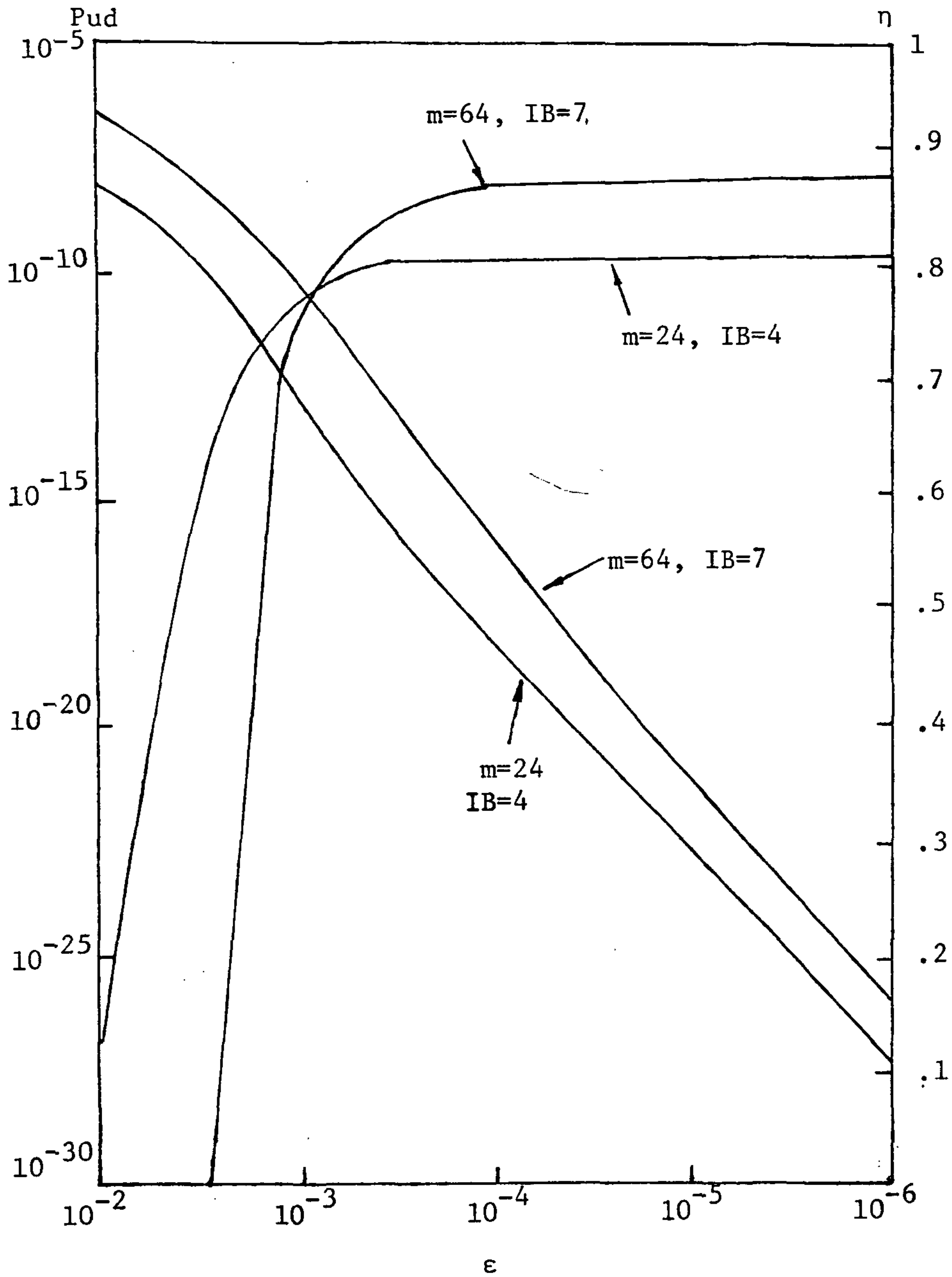

Figure 4 Performance of the concatenated code of Example 1 (t=1).

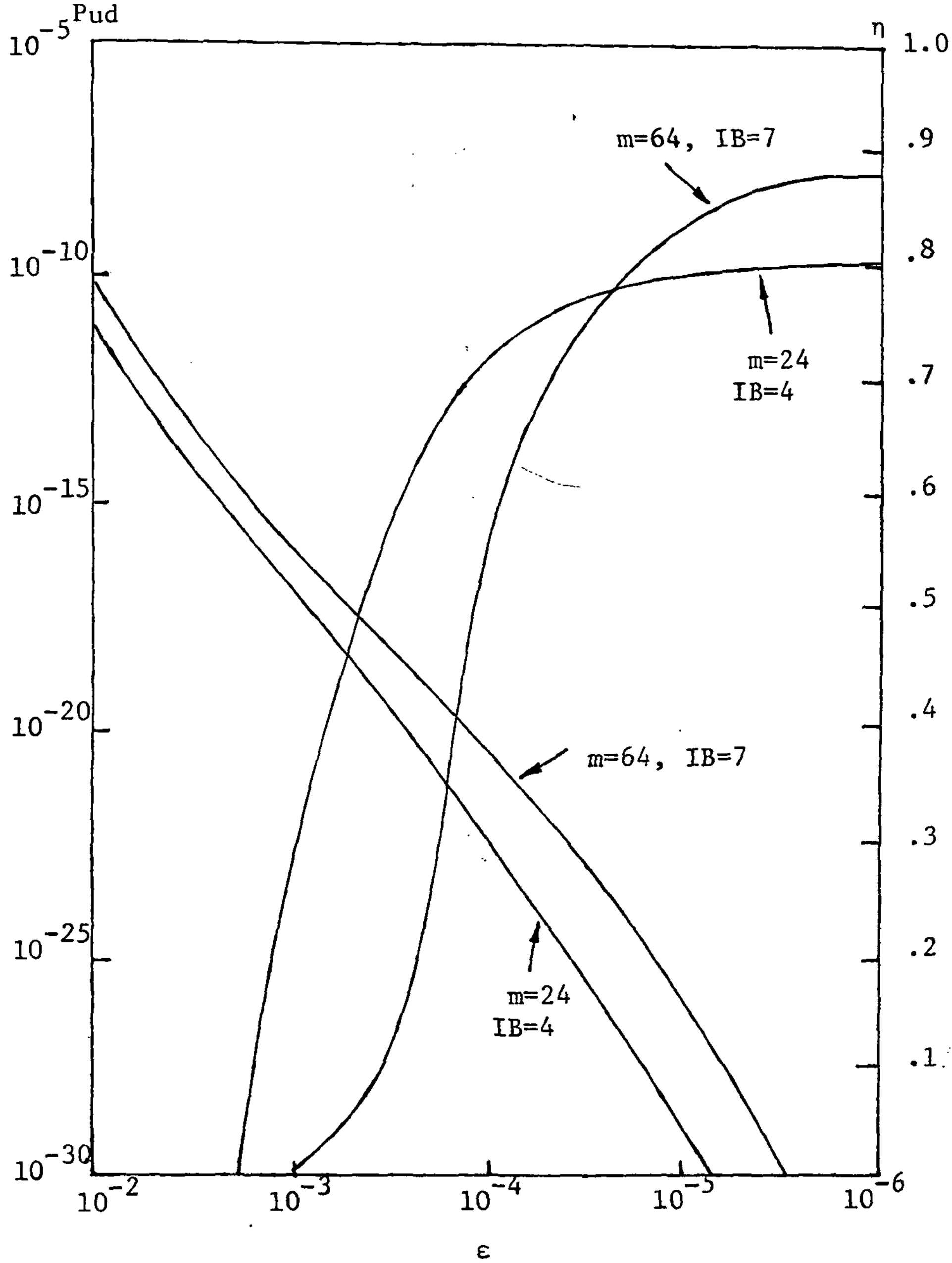

Figure 5 Performance of the concatenated code of Example 1 (t=0).

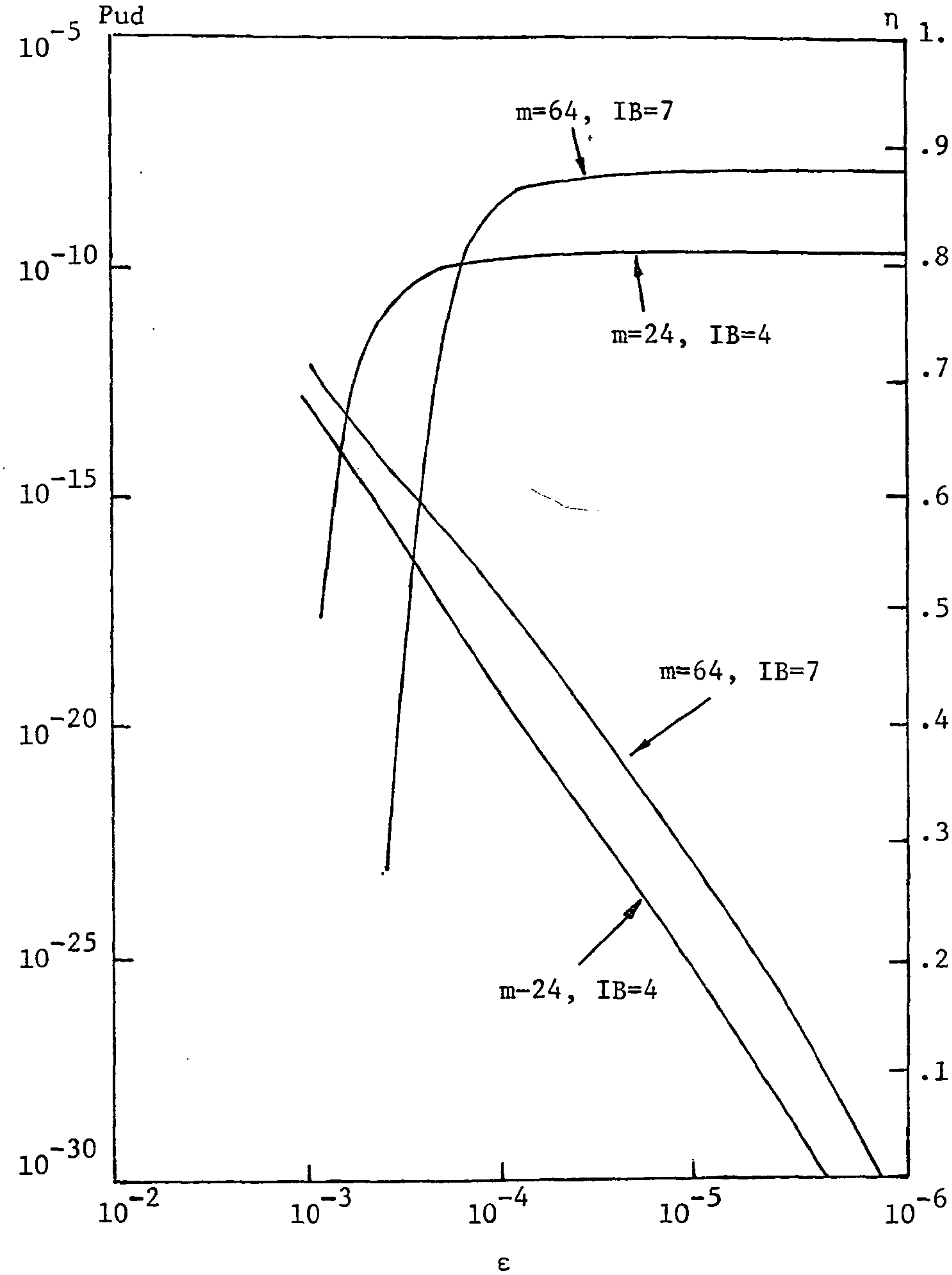

Figure 6 Performance of the concatenated code of Example 2 with u=4.

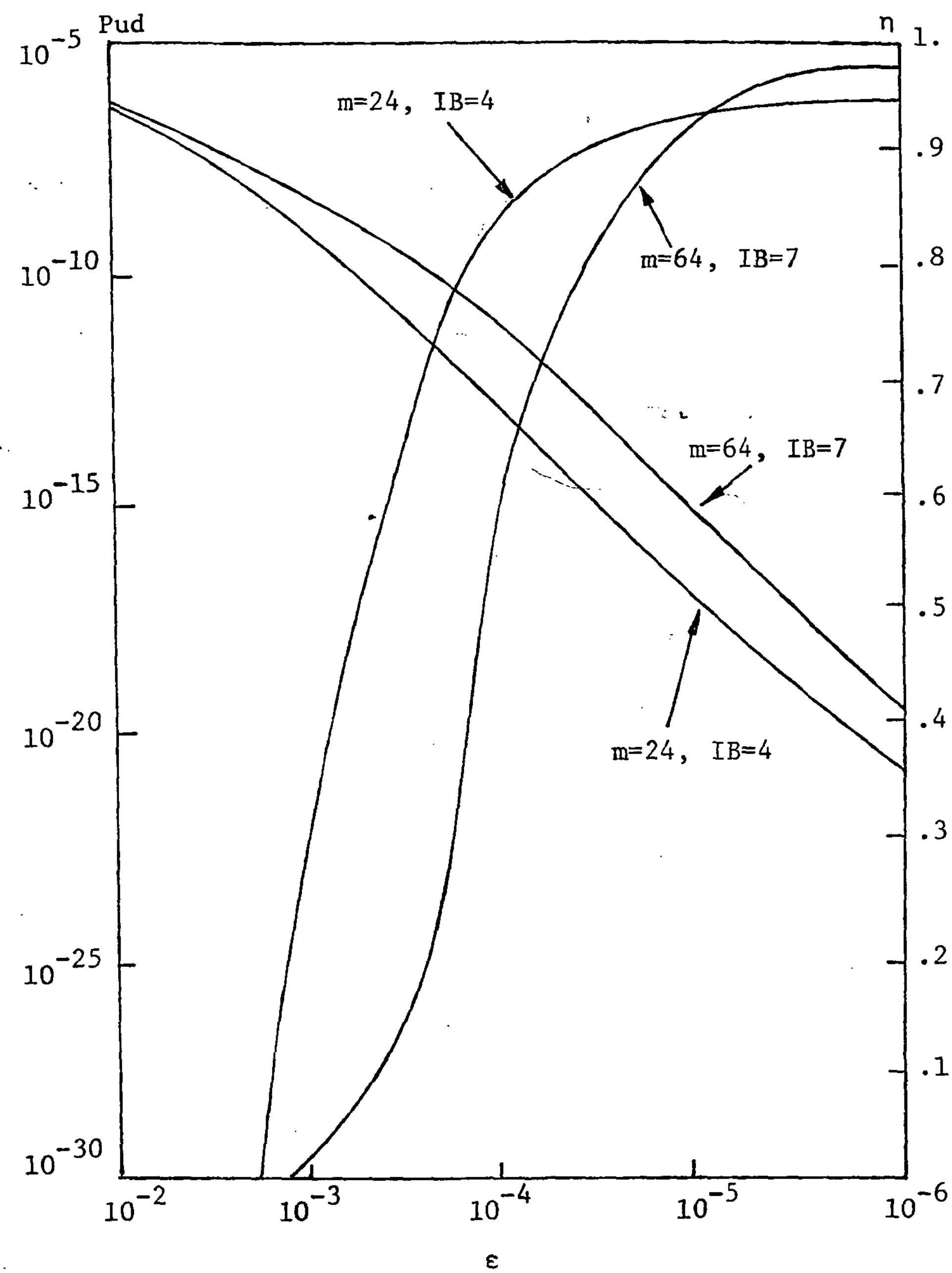

Figure 7 Performance of the concatenated code of Example 3.

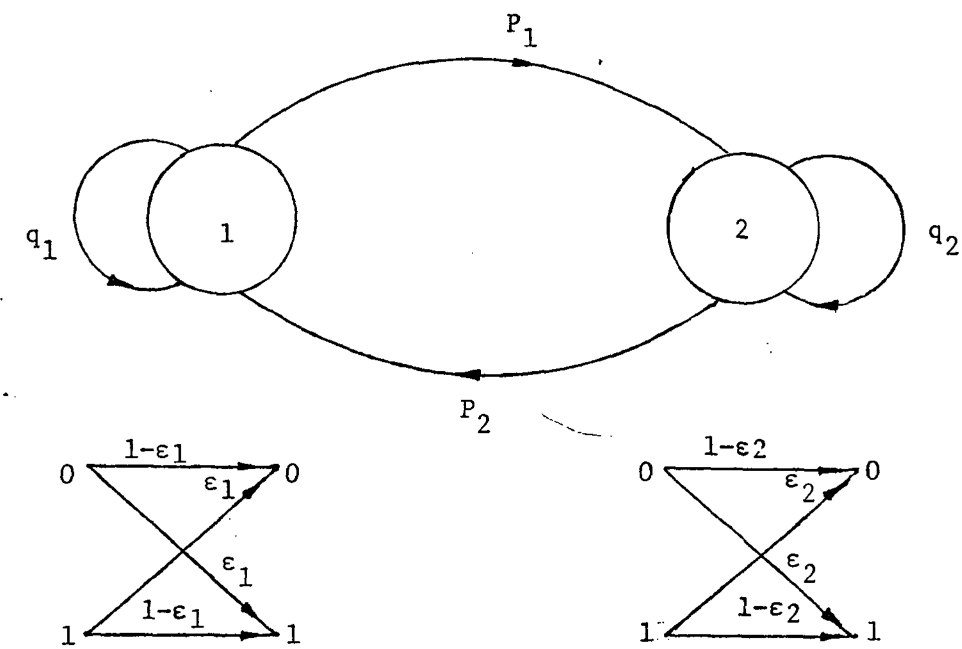

Figure 8 The burst-noise inner channel.

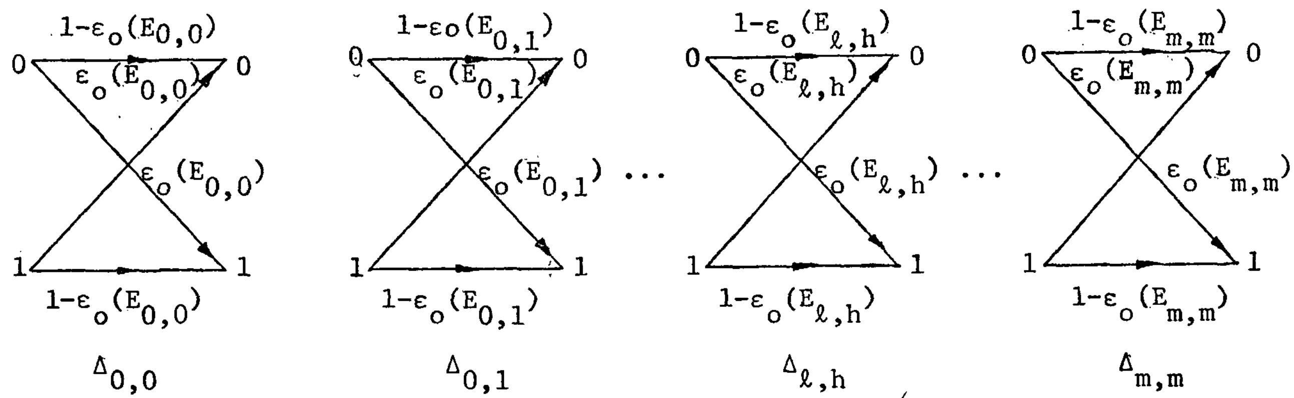

Figure 9 The outer channel resulting from decoding the inner code on a burst-noise channel.

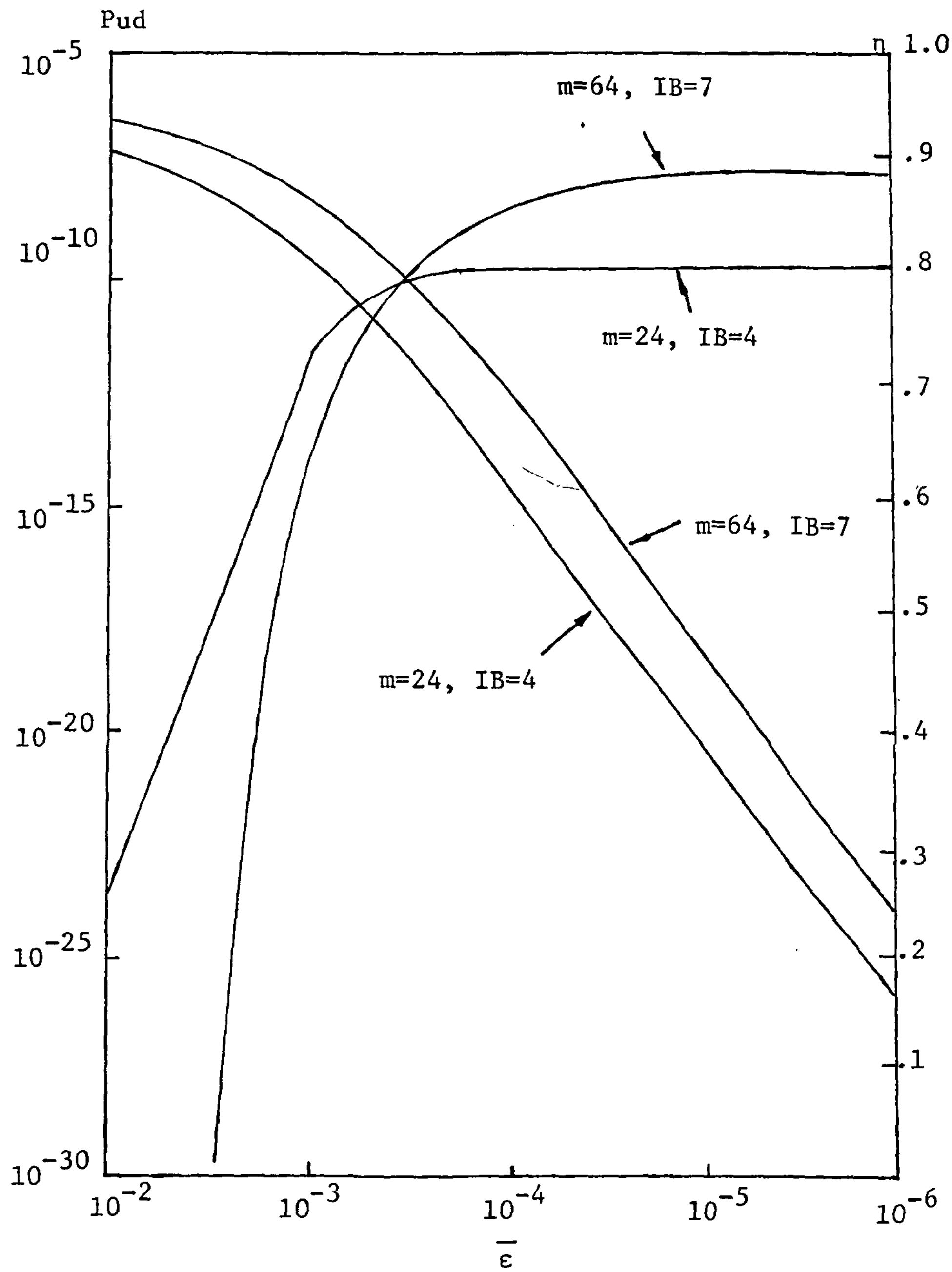

Figure 10.a Performance of the concatenated code of Example 4 with P_n=0.1, $\bar{L}$=5, $\varepsilon_2/\varepsilon_1$=10, t=1.

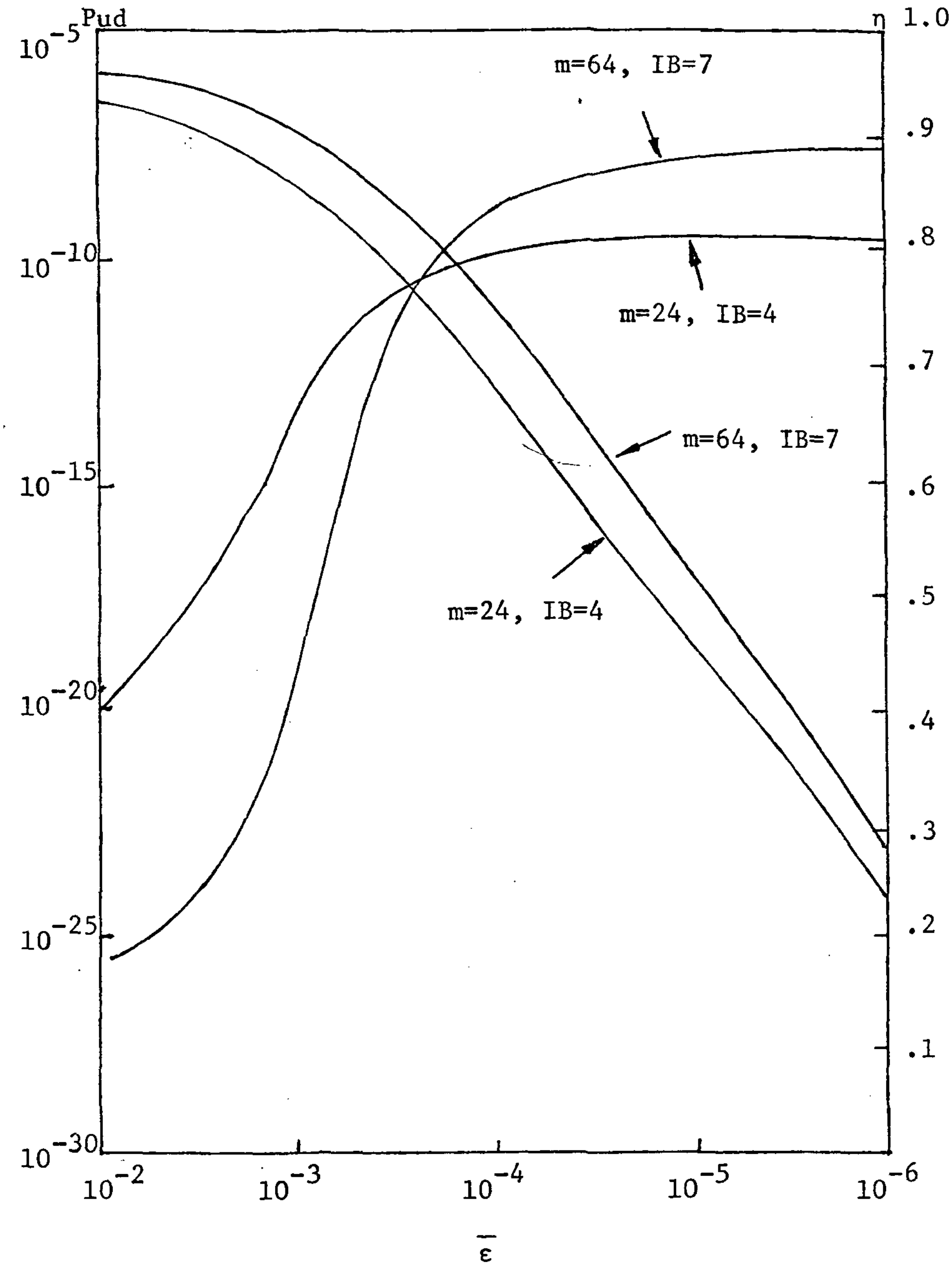

Figure 10.b Performance of the concatenated code of Example 4 with P_n=0.1, $\bar{L}$=5, $\varepsilon_2/\varepsilon_1$=1000, t=1.

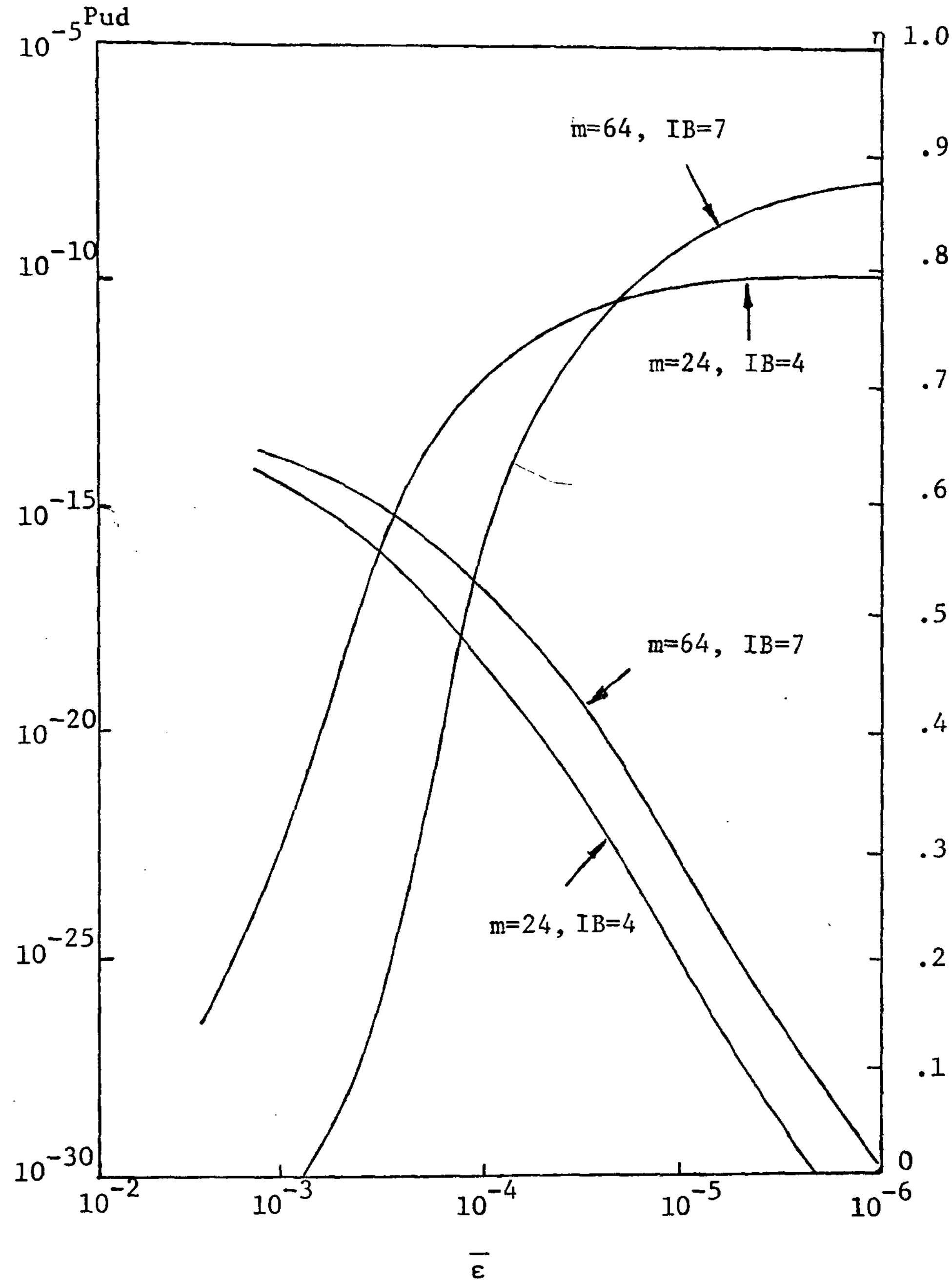

Figure 11.a Performance of the concatenated code of Example 4 with P_n=0.1, $\bar{L}$=5, $\varepsilon_2/\varepsilon_1$=10, t=0.

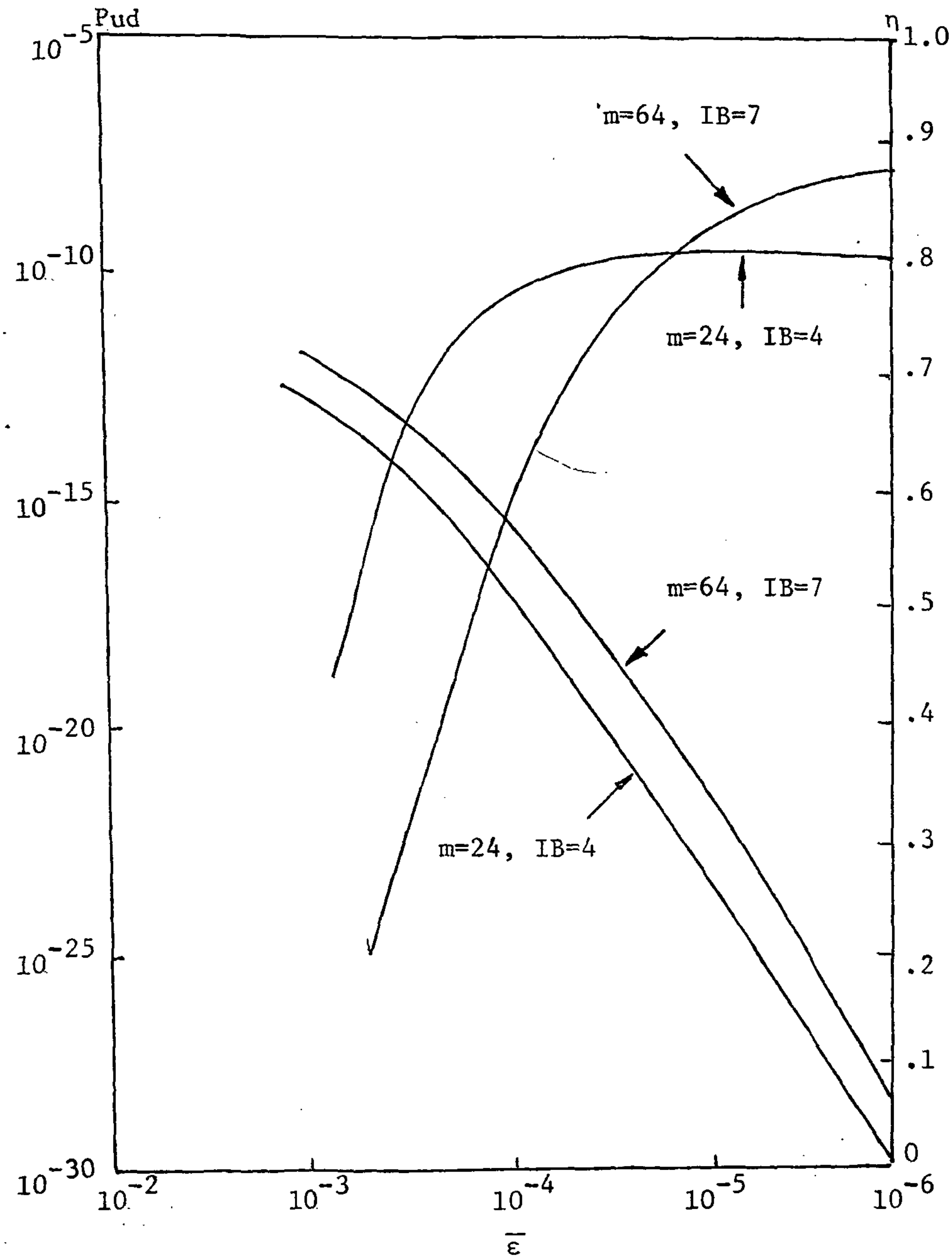

Figure 11.b Performance of the concatenated code of Example 4 with P_n=0.1, $\bar{L}$=5, $\varepsilon_2/\varepsilon_1$=1000, t=0.

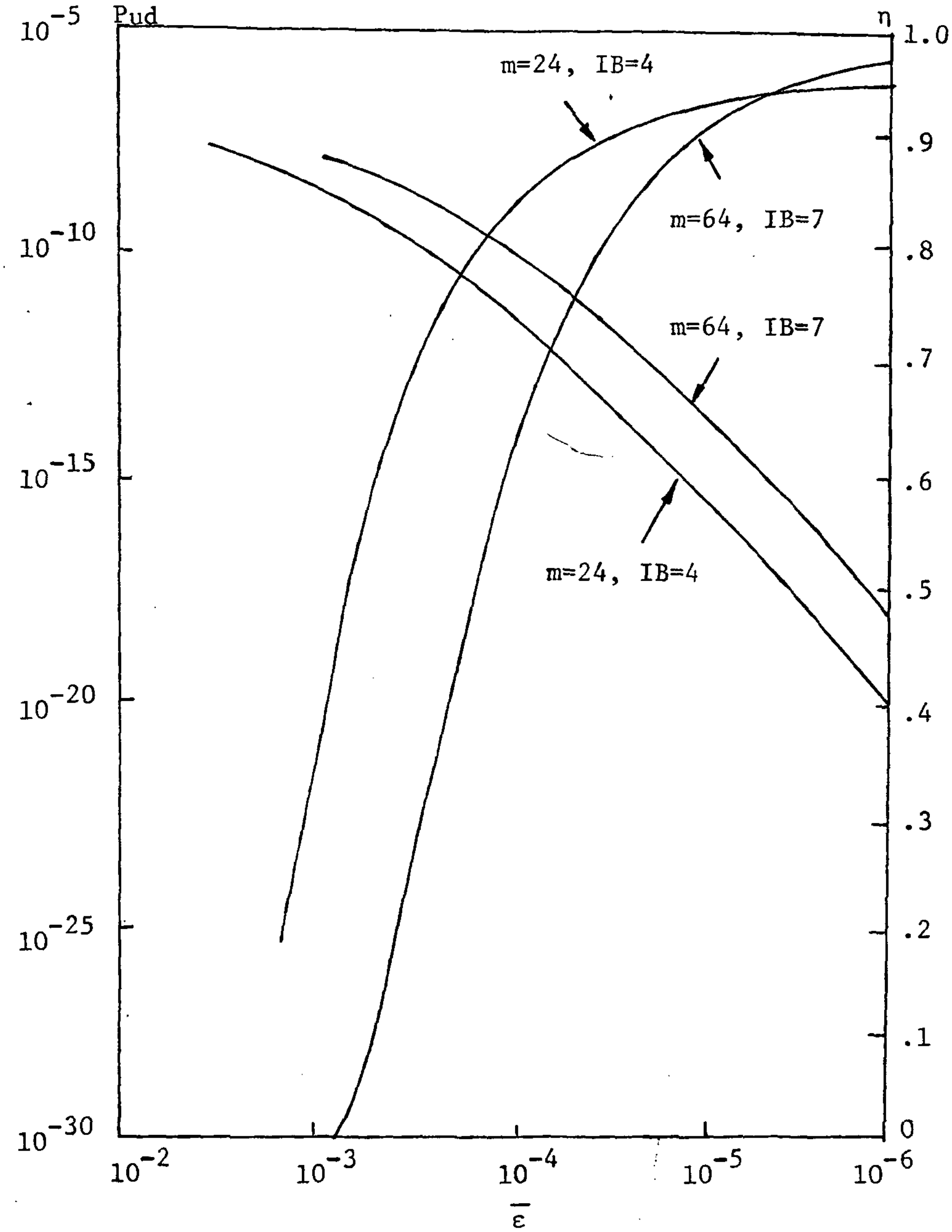

Figure 12.a Performance of the concatenated code of Example 5 with P_n=0.1, $\bar{L}$=5, $\varepsilon_2/\varepsilon_1$=10.

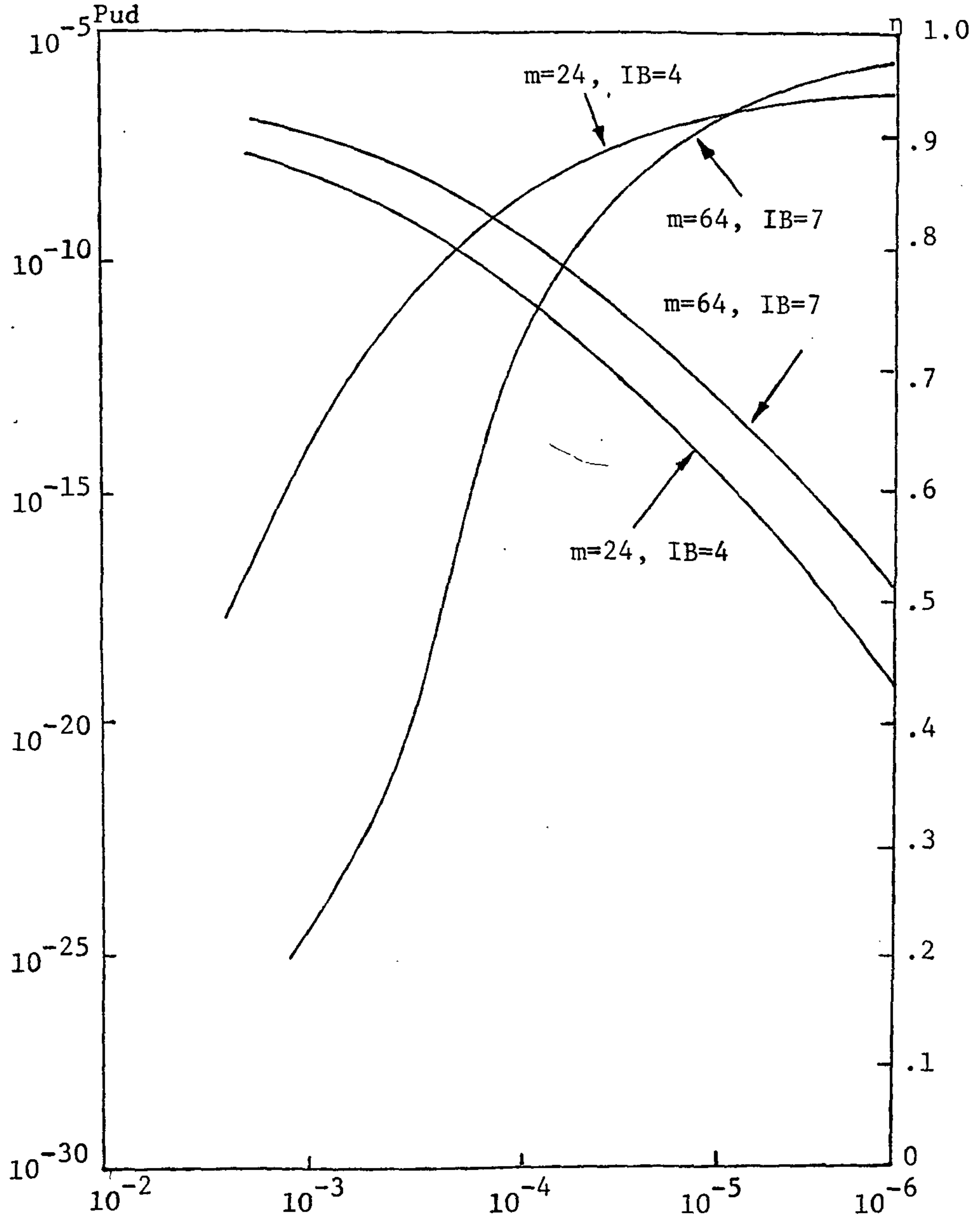

Figure 12.b Performance of the concatenated code of Example 5 with P_n=0.1, $\overline{L}$=5, $\varepsilon_2/\varepsilon_1$=1000.