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High Energy Laser Beam Propagation in the Atmosphere: The Integral Invariants of the Nonlinear Parabolic Equation and the Method of Moments

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Abstract

The method of moments is used to define and derive expressions for laser beam deflection and beam radius broadening for high-energy propagation through the Earth's atmosphere. These expressions are augmented with the integral invariants of the corresponding nonlinear parabolic equation that describes the electric field of high-energy laser beam to propagation to yield universal equations for the aforementioned quantities; the beam deflection is a linear function of the propagation distance whereas the beam broadening is a quadratic function of distance. The coefficients of these expressions are then derived from a thin screen approximation solution of the nonlinear parabolic equation to give corresponding analytical expressions for a target located outside the Earth's atmospheric layer. These equations, which are graphically presented for a host of propagation scenarios, as well as the thin screen model, are easily amenable to the phase expansions of the wave front for the specification and design of adaptive optics algorithms to correct for the inherent phase aberrations. This work finds application in, for example, the analysis of beamed energy propulsion for space-based vehicles.

1.0 Introduction

When delivering large amounts of power through the Earth's atmosphere via millimeter or infrared beams (i.e., laser beams or beams formed at the output of a millimeter wave antenna system), many propagation mechanisms must be addressed that may be potentially deleterious to such power transmission. The most obvious one is the ever-present random variation of the atmospheric refractive index due to local temperature variations known as 'turbulence'. This naturally occurring phenomena is driven by thermal convection of heat from the Earth's surface; once the resulting air motion exceeds a critical value of velocity, laminar flow essentially evolves into turbulent flow and fluctuations in the temperature distribution becomes statistically random (Ref. 1). These temperature fluctuations then act directly on the prevailing refractive index, thus rendering the refractive index a random quantity. These refractive index variations randomly focus and defocus the intervening electromagnetic wave field. Thus, the atmosphere can be considered to be composed of 'lenses' of random focusing and defocusing characteristics that, due to the gross atmospheric motion due to wind, move across the beam. This gives rise to many beam quality variations; the major ones being beam broadening and beam steering. The statistical analysis and modeling of this type of atmospheric propagation as a long and rich history and has resulted in analytical descriptions for the impact of turbulence on the operation of systems relying on such beam propagation. Many models and descriptions exist for the 'engineering analysis' of the operation of transmission systems that rely on the propagation of electromagnetic beam propagation in the atmosphere (Ref. 2).

The scenario discussed above may be considered as 'passive' electromagnetic wave propagation, i.e., the wave field moves through an atmosphere the refractive index of which is determined by other sources, not the field itself. However, as the energy density of the beam increases, absorption of the beam energy by atmospheric gas components results in local heating of the atmospheric which does indeed act directly on the refractive index causing it to decrease in value. This thermal change of the refractive index field then acts on the electromagnetic wave field causing it to also change, and so on. The propagation scenario now becomes an ‘active’ one, whereby the propagating field modifies the very medium it which it exists. This heating process is called ‘thermal blooming’ and substantially differs from that of the passive propagation discussed earlier. Here, a ‘thermal lens’ is created within the atmosphere by the heating due to the energy density of the beam. This ‘self-action’ of the beam will not only bend the beam into regions of higher refractive index (beam steering), but convection within the atmospheric fluid will also arise which is the source of self induced turbulent flow of the medium. The situation is further complicated when one includes the effects of atmospheric wind and aerosols and the abovementioned passive propagation effects. Defocusing and other such associated nonlinear thermal blooming distortions of the beam cross-section will then result. In extreme cases of very large energy densities, the Kerr effect will arise and the propagating beam can subsequently break up into smaller beams, or filaments, which severely constrains the amount of energy density that the beam will be able to possess as it travels through the atmosphere. Unlike the situation of passive propagation, the thermal blooming mechanism introduces nonlinearities into the analysis of the phenomena that substantially complicates a complete mathematical description. Complete analyses of these types of propagation scenarios can only be done numerically, which was a major activity within the United States and Russia in the late 1980s. Other than the usual ‘order-of-magnitude’ estimates using the equations of fluid mechanics and wave propagation, only numerical modeling of the effects of atmospheric thermal nonlinearities abound in the literature. Analytical treatments appropriate for an engineering analysis and assessment of atmospheric propagation systems encountering thermal blooming have been lacking and those that do exist hold only for the geometrical optics case (Refs. 3 to 5).

Two of the more important performance parameters of a high-energy laser-beaming scenario are the accuracy to aim the beam in the desired direction and the size of the beam at the intended target. In most cases, the beam will be focused at the target so the relevant quantity is the size of the focal spot on the target. It is the purpose of this memorandum to derive expressions for the beam deflection from the intended axis and the broadening of the beam focal spot at the target using only the information that can be gleaned for the structure of the nonlinear parabolic equation that describes the propagation process in the presence of thermal nonlinearities without actually solving the equation in its entirety. This will be accomplished by invoking the integral invariants that exist for the nonlinear parabolic equation as well as employing the method of moments to express the statistical moments of the electric field that are used to define the deflection as well as the radius of the laser beam. Once this has been done, a thin screen approximation solution will be given for the propagating wave field of the laser beam through the Earths atmosphere and these results will be applied to derive analytical and graphical results for the beam parameters focused to a target above the atmospheric layer. This thin screen model will allow for the design and specification for the adaptive correction of the phase aberrations that prevail in such propagation situations, and the expressions for the deflection and radius will allow for the evaluation of their corrected values.

2.0 The Moments of the Laser Beam Energy Density and the Nonlinear Parabolic Equation

Given the energy density $\left|E(x,y,z)\right|^2$ of the electromagnetic field distribution of a laser beam in a plane $\vec{r}_\perp = x\hat{x} + y\hat{y}$ transverse to the direction of propagation at a distance z from the origin, one can define the mn^{th} moment of the deposition of this quantity in the transverse plane, viz,

$$\gamma_{mn}(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^n \left|E(x,y,z)\right|^2 dxdy \tag{1}$$

Thus, $\gamma_{00}(z)$ is the total beam energy at the distance z, $\gamma_{10}(z)$ is the first moment or centroid of the beam energy from the z-axis in the x-direction, $\gamma_{01}(z)$ as that in the y-direction, $\gamma_{20}(z)$ as the second moment of the beam energy in the x-direction, etc. Hence, using these spatial moments of the energy density in a plane, one can now form the vector quantity

$$\vec{\Delta}_\perp(z) \equiv \frac{1}{\gamma_{00}(z)}\left(\gamma_{10}(z)\hat{x} + \gamma_{01}(z)\hat{y}\right) \tag{2}$$

that can be defined as the normalized overall transverse displacement of the beam energy from the z-axis (intended direction of propagation) at the distance z, and

$$a_{eff}^2(z) \equiv \frac{1}{\gamma_{00}(z)}\left(\gamma_{20}(z) + \gamma_{02}(z)\right) \tag{3}$$

that can be defined as the square radius of the spread of the beam energy about the z-axis. This particular moment can be centralized by forming the quantity

$$r_{eff}^2(z) = a_{eff}^2(z) - \vec{\Delta}_\perp(z) \tag{4}$$

which now gives the radius of the beam energy (i.e., the beam radius) about the axis shifted by $\vec{\Delta}_\perp(z)$ from the intended direction.

The analysis of these quantities can be a rather difficult problem in the full nonlinear case of high energy laser beam propagation in an absorbing medium which gives rise to a host of thermal nonlinearities collectively known as thermal blooming. It thus becomes of interest to ascertain just how much of the behavior of these performance parameters can be gleaned from the fact that the equation governing the electric field of the laser beam in an absorptive nonlinear medium is given by

$$2ik\frac{\partial E}{\partial z} + \nabla_\perp^2 E + k^2 \varepsilon_T T\left(|E|^2\right)E = 0 \tag{5}$$

without fully solving specific problem scenarios. Here, $\nabla_\perp^2$ is the Laplacian taken along the direction $\vec{r}$ transverse to the direction of propagation z. Equation (5) is the well-known parabolic equation applied to a propagation medium whose variation of permittivity $\Delta\varepsilon$ is governed by the change in temperature T of the medium from its nominal value which is determined by the energy density $|E|^2$ of the field, i.e.,

$$\Delta\varepsilon\left(T, |E|^2\right) = \varepsilon_T T\left(|E|^2\right) \tag{6}$$

where $\varepsilon_T \equiv \partial\varepsilon/\partial T$ is the variation of the permittivity with respect to T; in the case of the Earth atmosphere, $\varepsilon_T < 0$. The functional dependency of the temperature variation on the field energy density is given by the application of the conservation of energy to the atmospheric heat budget, given in its entirety by

$$\rho C_p\left(\frac{\partial T}{\partial t} + \vec{V}\cdot\vec{\nabla}T\right) = \kappa\nabla^2 T + \alpha\frac{c}{8\pi}|E|^2 \tag{7}$$

where ρ is the density of the atmosphere, C_p is its specific heat at constant pressure, κ is its thermal conductivity, $\vec{V}$ is the atmospheric wind velocity, and α is the absorption coefficient. (Strictly speaking, one should also include the set of Navier-Stokes equations in the Boussinesq approximation to account for the viscous and Archimedean forces that occur during the thermally induced motion of the atmosphere. However, for the purposes of this discussion and subsequent analysis, a description at this level is not required.) The implications of the parabolic equation of Equation (5) applied to this nonlinear problem will now be investigated for the evolution of the moments of the beam parameters discussed above.

3.0 The Invariants of the Nonlinear Parabolic Equation

Two integral invariants (sometimes called the 'integrals of motion') of the differential equation of Equation (5) can easily be derived (Refs. 6 to 8). First, multiplying Equation (5) by E^* and subtracting from it the complex conjugate of the same equation gives

$$2ik\left(E^*\frac{\partial E}{\partial z}+E\frac{\partial E^*}{\partial z}\right)+E^*\nabla_\perp^2 E-E\nabla_\perp^2 E^*=0 \tag{8}$$

Using the fact that $|E|^2 \equiv EE^*$ and simplifying where possible yields

$$2ik\frac{\partial|E|^2}{\partial z}+\vec{\nabla}_\perp\cdot\left(E^*\vec{\nabla}_\perp E-E\nabla_\perp E^*\right)=0 \tag{9}$$

Finally, integrating Equation (9) in the transverse plane yields and applying Gauss's Law to the second member of the relationship and letting the surface approach infinity, at which the surface terms approach zero, one finally obtains

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|E|^2 dxdy=\text{const}=\gamma_{00} \tag{10}$$

which is just a statement of the conservation of energy for any transverse plane. (Absorption of the wave field is present which, after all, accounts for the heating and subsequent nonlinear action of the medium. However, the action of the absorption on the overall energy density of the laser beam itself has been neglected in Equation (5). Incorporating it into the subsequent development is straightforward but will not be done here.)

Second, multiplying Equation (5) by $\partial E^*/\partial z$ and adding to it the complex conjugate of the same equation gives

$$\frac{\partial E^*}{\partial z}\nabla_\perp^2 E+\frac{\partial E}{\partial z}\nabla_\perp^2 E^*+k^2\varepsilon_T T\left(|E|^2\right)\frac{\partial|E|^2}{\partial z}=0 \tag{11}$$

As done above, one now integrates this relationship over the transverse plane. Each of the first two terms on the left side of Equation (11) can then be simplified by first using Green's first identity and letting the surface terms vanish for the same reason as employed above:

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\vec{\nabla}_\perp\frac{\partial E^*}{\partial z}\cdot\vec{\nabla}_\perp E+\vec{\nabla}_\perp\frac{\partial E}{\partial z}\cdot\vec{\nabla}_\perp E^*\right)dx\,dy+k^2\varepsilon_T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\left(|E|^2\right)\frac{\partial|E|^2}{\partial z}dxdy=0 \tag{12}$$

Defining the quantity $G\left(|E|^2\right)$ such that

$$T\left(|E|^2\right)=\frac{\partial G\left(|E|^2\right)}{\partial |E|^2} \tag{13}$$

and substituting it into Equation (12) and simplifying yields

$$-\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\vec{\nabla}_\perp E\cdot\vec{\nabla}_\perp E^*\right)dx\,dy+k^2\varepsilon_T\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G\left(|E|^*\right)dxdy=0 \tag{14}$$

which becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\vec{\nabla}_\perp E\cdot\vec{\nabla}_\perp E^*-k^2\varepsilon_T G\left(|E|^2\right)\right]dx\,dy=\text{const}\equiv \mathrm{H} \tag{15}$$

which is essentially the Hamiltonian for the propagating wave field.

4.0 Conservation Laws for the Nonlinear Parabolic Equation

The two integral invariants given above should now be augmented with the conservation laws that prevail quantities that issue from Equation (5). A streamlined method to accomplish this is to employ field theory and find the Lagrangian corresponding to Equation (5) and from it obtain the energy momentum tensor from which the conservation quantities and laws would follow (Ref. 9). However, a direct approach will be used using Equation (5) directly. To this end, the first such conservation law was already derived and is given by Equation (9). Defining

$$\vec{S}_\perp\equiv-\frac{i}{2k}\left(E^*\vec{\nabla}_\perp E-E\vec{\nabla}_\perp E^*\right) \tag{16}$$

which is the energy flux vector of the field, one has

$$\frac{\partial |E|^2}{\partial z}+\vec{\nabla}_{\vec{\rho}}\cdot\vec{S}_\perp=0 \tag{17}$$

This is one of two prevailing conservation laws.

Consider now the other equation which governs the energy flux vector $\vec{S}_\perp$; the idea here is to obtain an equation for $\vec{S}_\perp$ analogous to that of Equation (17) for $|E|^2$. Using Equation (16),

$$\frac{\partial\vec{S}_\perp}{\partial z}=-\left(\frac{i}{2k}\right)\left[\frac{\partial E^*}{\partial z}\vec{\nabla}_\perp E+E^*\vec{\nabla}_\perp\left(\frac{\partial E}{\partial z}\right)-\frac{\partial E}{\partial z}\vec{\nabla}_\perp E^*-E\vec{\nabla}_\perp\left(\frac{\partial E^*}{\partial z}\right)\right] \tag{18}$$

Employing Equation (5) and simplifying gives

$$\frac{\partial\vec{S}_\perp}{\partial z}=\left(\frac{1}{2k}\right)^2\left[-K^*\vec{F}+E^*\vec{\nabla}_\perp K-K\vec{F}^*+E\vec{\nabla}_\perp K^*\right] \tag{19}$$

where

$$\vec{F} \equiv \vec{\nabla}_\perp E, \qquad K \equiv \vec{\nabla}_\perp \cdot \vec{F} + k^2 \varepsilon_T T E \tag{20}$$

Making use of the identity

$$\vec{\nabla}_\perp \left(E^* K \right) = \vec{F}^* K + E^* \vec{\nabla}_\perp K \tag{21}$$

as well as its complex conjugate allows Equation (19) to become

$$\frac{\partial \vec{S}_\perp}{\partial z} = \left(\frac{1}{2k} \right)^2 \left[-2\vec{F} K^* - 2\vec{F}^* K + \vec{\nabla}_\perp \left(E^* K \right) + \vec{\nabla}_\perp \left(E K^* \right) \right] \tag{22}$$

Two more identities are now needed:

$$\nabla_\perp^2 |E|^2 = 2\vec{F} \cdot \vec{F} + E^* \vec{\nabla}_\perp \cdot \vec{F} + E \vec{\nabla}_\perp \cdot \vec{F}^* \tag{23}$$

and

$$\left(\vec{F} \vec{\nabla}_\perp F^* + \vec{F}^* \vec{\nabla}_\perp F \right)_i = \frac{\partial}{\partial x_j} \left(F_i F_j^* \right) + \frac{\partial}{\partial x_j} \left(F_i^* F_j \right) - \frac{\partial}{\partial x_i} \left(\vec{F} \cdot \vec{F}^* \right) \tag{24}$$

(The latter comes from the fact that $\vec{F} \times \vec{\nabla}_\perp \times \vec{F}^* = 0$ since $\vec{F} = \vec{\nabla}_\perp E$, etc.) After a bit of algebraic manipulation, one finds that

$$\left(\frac{\partial \vec{S}_\perp}{\partial z} \right)_i = \left(\frac{1}{2k} \right)^2 \left[-2 \frac{\partial}{\partial x_j} \left(F_i F_j^* + F_i^* F_j \right) + \frac{\partial}{\partial x_j} \left(\nabla_\perp^2 |E|^2 \right) \delta_{ij} \right. \\ \left. + 2k^2 \varepsilon_T \left(\frac{\partial}{\partial x_j} \left(T |E|^2 \right) - T \frac{\partial}{\partial x_j} |E|^2 \right) \delta_{ij} \right] \tag{25}$$

Finally, using Equation (13) and simplifying, Equation (25) gives

$$\left(\frac{\partial \vec{S}_\perp}{\partial z} \right)_i = \left(\frac{1}{k} \right)^2 \frac{\partial}{\partial x_j} \left[\frac{1}{4} \left(\nabla_\perp^2 |E|^2 \right) \delta_{ij} - \frac{1}{2} \left(\vec{\nabla}_\perp E \right)_i \left(\vec{\nabla}_\perp E^* \right)_j \right. \\ \left. - \frac{1}{2} \left(\vec{\nabla}_\perp E^* \right)_i \left(\vec{\nabla}_\perp E \right)_j + \frac{1}{2} k^2 \varepsilon_T \left(T |E|^2 - G\left(|E|^2 \right) \right) \delta_{ij} \right] \tag{26}$$

Equation (26) can be written as the tensor relation

$$\frac{\partial \vec{S}_\perp}{\partial z} - \vec{\nabla}_\perp \cdot \vec{\vec{Q}} = 0 \tag{27}$$

where $\vec{\vec{Q}}$ is the stress tensor

$$\left(\vec{Q}\right)_{ij} \equiv \left(\frac{1}{k}\right)^2 \left[\frac{1}{4}\left(\nabla_\perp^2 |E|^2\right)\delta_{ij} - \frac{1}{2}\left(\vec{\nabla}_\perp E\right)_i \left(\vec{\nabla}_\perp E^*\right)_j \right.$$
$$\left. -\frac{1}{2}\left(\vec{\nabla}_\perp E^*\right)_i \left(\vec{\nabla}_\perp E\right)_j + \frac{1}{2} k^2 \varepsilon_T \left(T|E|^2 - G\left(|E|^2\right)\right)\delta_{ij}\right] \tag{28}$$

The conservation laws of Equation (17) and (27), as well as Equations (10) and (15), which prevail for Equation (5) are now sufficient to obtain expressions for the beam parameters specified by Equations (2) to (4).

5.0 The Displacement and Broadening of a Laser Beam Experiencing Thermal Blooming

The displacement from bore sight of a laser beam which induces thermal nonlinearities during propagation can now be easily calculated using Equation (2) involving the first order spatial moment of the energy density. To this end, differentiating Equation (2) (noting the constancy of γ_{00} by Equation (10)), and using Equation (17) yields

$$\frac{d\vec{\Delta}_\perp(z)}{dz} = -\frac{1}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(x\left(\vec{\nabla}_\perp \cdot \vec{S}_\perp\right)\hat{x} + y\left(\vec{\nabla}_\perp \cdot \vec{S}_\perp\right)\hat{y}\right) dx\, dy \tag{29}$$

Integrating by parts and noting that the surface terms approach zero at infinity gives

$$\frac{d\vec{\Delta}_\perp(z)}{dz} = \frac{1}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{S}_\perp \, dx\, dy \equiv \vec{A}(z) \tag{30}$$

This is sufficient to describe the displacement in that

$$\frac{d^2\vec{\Delta}_\perp(z)}{dz^2} = \frac{1}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial \vec{S}_\perp}{\partial z} dxdy = \frac{1}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{\nabla}_\perp \cdot \vec{Q} dxdy = 0 \tag{31}$$

upon using Equation (27); the last term vanishes upon using Gauss's Theorem and, once again, letting the fields and their derivatives vanish at infinity. Thus, $\vec{A}(z) = \vec{A} = \text{const}$ and the overall beam displacement is given simply by integrating Equation (30), i.e.,

$$\vec{\Delta}_\perp(z) = \vec{A}z \tag{32}$$

where it is assumed from hear on that $\vec{\Delta}_\perp(0) = 0$, i.e., the beam is initially pointed down its intended axis. Thus, the beam deflection from its intended axis is a linear function of distance travelled.

Similarly, the associated overall beam broadening r_{eff}^2 can be found by first differentiating Equation (3) and using Equation (17),

$$\frac{da_{eff}^2(z)}{dz} = -\frac{1}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(x^2 + y^2\right)\left(\vec{\nabla}_\perp \cdot \vec{S}_\perp\right) dx\, dy \tag{33}$$

Integrating by parts gives

$$\frac{da_{eff}^2(z)}{dz} = \frac{2}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{r}_\perp \cdot \vec{S}_\perp \, dx\, dy \equiv B(z) \tag{34}$$

Differentiating once again and using Equation (27) yields

$$\frac{d^2 a_{eff}^2(z)}{dz^2} = \frac{2}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{r}_\perp \cdot \left(\vec{\nabla}_\perp \cdot \vec{\vec{Q}}\right) dx\, dy \tag{35}$$

Integrating by parts and letting the surface term vanish,

$$\frac{d^2 a_{eff}^2(z)}{dz^2} = -\frac{2}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\hat{n} \cdot \vec{\vec{Q}}\right) dx\, dy = -\frac{2}{\gamma_{00}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{i=1}^{2} Q_{ii}\, dx\, dy \tag{36}$$

where $\hat{n}$ is the surface normal. In order to assure that this result is a constant and that no further derivatives are needed, one must evaluate the required trace of the tensor $\vec{\vec{Q}}$. Using Equation (28), one has

$$\sum_{i=1}^{2} Q_{ii} = \frac{1}{2}\left(\frac{1}{k}\right)^2 \left[\vec{\nabla}_\perp \cdot \left(E^* \vec{\nabla}_\perp E + E \vec{\nabla}_\perp E^*\right) - 2\vec{\nabla}_\perp E \cdot \vec{\nabla}_\perp E^* + 2k^2 \varepsilon_T \left(T\,|E|^2 - G\left(|E|^2\right)\right)\right] \tag{37}$$

Introducing this into Equation (36) and using the Gauss Theorem to rid of the divergence term finally gives

$$\frac{d^2 a_{eff}^2(z)}{dz^2} = -\left(\frac{1}{\gamma_{00}}\right)\left(\frac{1}{k}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[-2\vec{\nabla}_\perp E \cdot \vec{\nabla}_\perp E^* + 2k^2 \varepsilon_T T\,|E|^2 - 2k^2 \varepsilon_T G\left(|E|^2\right)\right] dx\, dy \tag{38}$$

This expression can be stated in terms of the invariant of Equation (15), viz,

$$\frac{d^2 a_{eff}^2(z)}{dz^2} = \left(\frac{1}{\gamma_{00}}\right)\left(\frac{1}{k}\right)^2 \left[2H - 2k^2 \varepsilon_T \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[T\,|E|^2 - 2G\left(|E|^2\right)\right] dx\, dy\right] \tag{39}$$

Hence, this expression is constant so long as the integrals vanish within the brackets. Thus, in the case of the Kerr effect, from Equation (6), one has that the temperature is replaced by $T \Rightarrow A|E|^2$ where A is some parameter independent of the electric field intensity, then by Equation (13), $G\left(|E|^2\right) = (1/2)A|E|^4$ and the integrand is identically equal to zero. However, in the case of thermally induced nonlinearities, the temperature field is related to the electric field intensity through Equation (7). In the case of forced convection due to wind and where large Peclet numbers prevail (see Appendix A), $\partial T/\partial x = \beta\left|E(x,y,z)\right|^2$ (for a wind velocity in the x-direction), and this will give rise to much more complicated functional dependencies. A general analysis of this problem is beyond the scope of this particular work but as shown in Appendix A, in the case of forced convection,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[T\,|E|^2 - 2G\left(|E|^2\right)\right] dx\, dy \approx 0 \tag{40}$$

Hence,

$$\frac{d^2 a_{eff}^2(z)}{dz^2} = \left(\frac{1}{\gamma_{00}}\right)\left(\frac{1}{k}\right)^2 2H = \text{const} \equiv C \tag{41}$$

Integrating this expression and remembering Equation (34) gives for the overall effective beam radius

$$a_{eff}^2(z) = \frac{1}{2}Cz^2 + B(0)z + a_{eff}^2(0) \tag{42}$$

and from Equations (4) and (32), for the effective radius about the deflected axis,

$$r_{eff}^2(z) = \frac{1}{2}Cz^2 + B(0)z + a_{eff}^2(0) - \left(\bar{A}z\right)^2 \tag{43}$$

Hence, Equations (32) and (43) give the universal behavior of the deflection and broadening of a laser beam governed by Equation (5) without solving the entire propagation problem. The constants $\bar{A}$, B(0), and C are related to the beam parameters via Equations (30), (34), and (41), respectively.

As a quick check on the validity of Equation (43), consider a Gaussian beam wave propagating in free space. Thus, $T\left(|E|^2\right) = 0$ in Equation (5) and a solution of the resulting equation can be given in the form

$$E(\bar{r}, z) = \frac{A_0}{1 + i\alpha_d z}\exp\left[-\left(\frac{k\alpha_d}{2}\right)\frac{r^2}{1 + i\alpha_d z}\right] \tag{44}$$

where A_0 is the initial maximum amplitude of the wave in the output aperture and

$$\alpha_d \equiv \alpha_1 + i\alpha_2, \quad \alpha_1 \equiv \frac{2}{kW_0^2}, \quad \alpha_2 \equiv \frac{1}{F} \tag{45}$$

with W_0 being the waist size of the beam as it leaves the aperture (the effective radius is $a_{eff}(0) = W_0/\sqrt{2}$) and F being the focal length of the beam. (The beam parameters α_d, α_1, and α_2 are not to be confused with the absorption coefficient α of Equation (7).) Hence, one has

$$|E|^2 = \frac{A_0^2}{\Lambda(z)}\exp\left[-\frac{k\alpha_1}{\Lambda(z)}r^2\right] \tag{46}$$

where

$$\Lambda(z) \equiv (1 - \alpha_2 z)^2 + \alpha_1^2 z^2 \tag{47}$$

and from Equation (10),

$$\gamma_{00} = \pi\frac{W_0^2}{2}A_0^2 \tag{48}$$

which is the total energy contained within the beam. Employing Equation (16) gives

$$\vec{S}_\perp(z) = -|E|^2 \left(\frac{\alpha_2 - \left(\alpha_1^2 + \alpha_2^2\right) z}{\Lambda(z)} \right) (x\hat{x} + y\hat{y}) \tag{49}$$

Equation (30) then yields

$$\vec{A} = 0 \tag{50}$$

and from Equations (34) and (41)

$$B(z) = -\left(\frac{2}{k\alpha_1}\right)\left(\alpha_2 - \left(\alpha_1^2 + \alpha_2^2\right) z\right) \tag{51}$$

$$C = \frac{2\left(\alpha_1^2 + \alpha_2^2\right)}{k\alpha_1} \tag{52}$$

Using these values in Equation (43) finally gives for the effective radius of the beam in a transverse plane at a distance z from the source aperture

$$r_{eff}^2(z) = \frac{W_0^2}{2}\left[(1 - \alpha_2 z)^2 + \alpha_1^2 z^2\right] = \frac{W_0^2}{2}\Lambda(z) \tag{53}$$

which is the value found using from Equation (46) by equating the exponential argument to -1.

6.0 Application of the Foregoing to High Energy Laser Beam Propagation Through the Atmosphere

The displacement and widening of a very high power laser beam due to atmospheric thermal nonlinearities are two very important performance parameters in the case of transmission of, e.g., laser energy from the Earth's surface to power a spacecraft in a low Earth orbit (LEO) for subsequent propulsion into, e.g., a geosynchronous orbit (GEO). In this instance, a thin screen approximation can be made whereby only a very small length L of the entire longitudinal propagation path distance D is intersected by the atmosphere, i.e., $L << D$. To simplify matters, consider instead of the field E, its amplitude and phase representation

$$E(\bar{r}, z) = e(\bar{r}, z)\exp\left[-i\phi(\bar{r}, z)\right] \tag{54}$$

Substituting this into Equation (5) and separating the real and imaginary portions of the resulting equation gives

$$2k\frac{\partial \phi}{\partial z} + \frac{\nabla_\perp^2 e}{e} + \left(\vec{\nabla}_\perp \phi\right)^2 + k^2 \varepsilon_T T\left(|E|^2\right) = 0 \tag{55}$$

$$2k\frac{\partial e}{\partial z} - 2\vec{\nabla}_\perp \phi \cdot \vec{\nabla}_\perp e - \left(\nabla_\perp^2 \phi\right) e = 0 \tag{56}$$

Similarly, using Equation (54) in Equation (16) gives

$$\vec{S}_\perp = -\frac{e^2\vec{\nabla}\phi}{k} \tag{57}$$

and using this intermediate result in Equations (30), (34), and (41), one has

$$\vec{A} = -\frac{1}{\gamma_{00}}\left(\frac{1}{k}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|e|^2\left(\vec{\nabla}_\perp\phi\right)dxdy \tag{58}$$

$$B = -\frac{2}{\gamma_{00}}\left(\frac{1}{k}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|e|^2\left(\vec{r}_\perp\cdot\vec{\nabla}_\perp\phi\right)dxdy \tag{59}$$

$$C = \left(\frac{2}{\gamma_{00}}\right)\left(\frac{1}{k}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left|\left(\vec{\nabla}_\perp e\right)\right|^2 + |e|^2\left(\vec{\nabla}_\perp\phi\right)^2 - k^2\varepsilon_T G\left(|E|^2\right)\right]dxdy \tag{60}$$

where from Equation (10),

$$\gamma_{00} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|e|^2 dxdy \tag{61}$$

The system of Equations (55) and (56) can be considerably simplified by using the fact that the layer through which the laser beam will encounter thermal nonlinearities is very thin with respect to the entire propagation path. Hence, letting only the phase of the field be perturbed within this thin layer and, thus, neglecting amplitude effects, one must solve the phase equation across this thin atmospheric layer, i.e.,

$$2k\frac{\partial\phi}{\partial z} + \left(\vec{\nabla}_\perp\phi\right)^2 + k^2\varepsilon_T T\left(|E|^2\right) = 0 \tag{62}$$

where the perturbing temperature field $T\left(\left|E(\vec{r},z)\right|^2\right)$ is given by Equation (A.3) of Appendix A. As shown in Appendix B, one has in this "thin screen" approximation

$$\phi = \frac{k\left(x^2+y^2\right)}{2\Lambda(z)}\left(\alpha_2 - \left(\alpha_1^2+\alpha_2^2\right)z\right) + \sqrt{\pi}\Phi\left(P_0\right)\exp\left(-k\alpha_1 y^2\right)\left(1+\mathrm{erf}\left(\sqrt{k\alpha_1}x\right)\right) \tag{63}$$

where

$$\Phi\left(P_0\right) \equiv -\frac{P_0}{P_T}\left(\frac{L}{2kW_0^2}\right) > 0 \tag{64}$$

which convolves the initial power P_0 within the laser beam and the power P_T that is characteristic of atmospheric thermal effects, viz,

$$P_T \equiv \frac{\sqrt{2}\pi\rho V C_p}{k^2 W_0\left|\varepsilon_T\right|\alpha} \tag{65}$$

The negative sign occurs in the definition of Equation (64) since, as noted earlier, $\varepsilon_T < 0$.

The evaluation of Equations (58)-(60) commences with Equation (61); by using the first relation of Equation (B.3) of Appendix B evaluated at the output plane of the laser in Equation (61) and performing the indicated integrals in plane polar coordinates gives the same result as Equation (48), i.e.,

$$\gamma_{00} = \frac{\pi A_0^2}{k\alpha_1} \tag{66}$$

The calculation of the vector $\vec{A}$ requires the evaluation of the transverse gradient $\vec{\nabla}_\perp\phi$. Using Equation (63) one has

$$\begin{aligned}\vec{\nabla}_\perp\phi = {} & \frac{k\left(\alpha_2 - \left(\alpha_1^2 + \alpha_2^2\right)z\right)}{\Lambda(z)}\left(\hat{x}x + \hat{y}y\right) \\ & + 2\sqrt{\pi}\Phi\left(P_0\right)\left[\hat{x}\sqrt{\frac{k\alpha_1}{\pi}}\exp\left(-k\alpha_1 x^2\right) - \hat{y}\,k\alpha_1 y\left(1 + \operatorname{erf}\left(\sqrt{k\alpha_1}\,x\right)\right)\right]\exp\left(-k\alpha_1 y^2\right)\end{aligned} \tag{67}$$

Substituting this expression into Equation (58) as well as that of Equation (B.3) evaluated at $z = 0$ (since $\vec{A} = \text{const}$ by Equations (30) and (31) and so can be evaluated at any longitudinal location), one finds that all terms involving first powers of x or y vanish upon integration in the transverse plane leaving

$$\vec{A} = -\sqrt{\frac{\alpha_1}{k}}\Phi\left(P_0\right)\hat{x} \tag{68}$$

Hence, by Equation (32), the beam is deflected by

$$\vec{\Delta}_\perp = -z\sqrt{\frac{\alpha_1}{k}}\Phi\left(P_0\right)\hat{x} \tag{69}$$

indicating that the beam moves against the atmospheric wind direction. This phenomenon is easily explained by the fact that the beam wants to move into regions of higher refractive index, i.e., into cooler regions about the beam axis.

Since the beam deflection adds an asymmetry of the beam along the x-axis, it proves to be beneficial to separate out the x and y contributions of the remaining B and C parameters of Equations (59) and (60). To this end, Equation (59) can be written

$$B = B_x + B_y \tag{70}$$

where

$$B_x = -\frac{2}{\gamma_{00}}\left(\frac{1}{k}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|e|^2\left(x\hat{x}\cdot\vec{\nabla}_\perp\phi\right)dxdy \tag{71}$$

$$B_y = -\frac{2}{\gamma_{00}}\left(\frac{1}{k}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|e|^2\left(y\hat{y}\cdot\vec{\nabla}_\perp\phi\right)dxdy \tag{72}$$

Using Equation (67) and once again noting that terms containing first powers of x and y vanish upon integration, one obtains

$$B_x = -\frac{\alpha_2}{k\alpha_1}, \qquad B_y = -\frac{\alpha_2}{k\alpha_1} + \frac{1}{k}\sqrt{\frac{\pi}{2}}\Phi(P_0) \tag{73}$$

One must now separate Equation (60) into *x* and *y* component contributions. Thus, without loss of generality, one can write

$$C = C_x + C_y \tag{74}$$

where

$$C_x = \left(\frac{2}{\gamma_{00}}\right)\left(\frac{1}{k}\right)^2 \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left|\left(\hat{x}\cdot\vec{\nabla}_\perp e\right)\right|^2 + |e|^2\left(\hat{x}\cdot\vec{\nabla}_\perp \phi\right)^2 \right] dxdy - Q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[k^2 \varepsilon_T G\left(|E|^2\right)\right] dxdy \right\} \tag{75}$$

and

$$C_y = \left(\frac{2}{\gamma_{00}}\right)\left(\frac{1}{k}\right)^2 \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left|\left(\hat{y}\cdot\vec{\nabla}_\perp e\right)\right|^2 + |e|^2\left(\hat{y}\cdot\vec{\nabla}_\perp \phi\right)^2 \right] dxdy - (1-Q)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[k^2 \varepsilon_T G\left(|E|^2\right)\right] dxdy \right\} \tag{76}$$

where the $G\left(|E|^2\right)$ terms have been separated out for convenience. It is very important to note that $G\left(|E|^2\right)$ cannot be arbitrarily apportioned between the arbitrary *x* and *y* components of the coefficient *C*. The $G\left(|E|^2\right)$ term has its origins with the conservation of energy given by Equation (15) and is thus associated with the deposition of energy within the propagating laser beam. The beam deflection adds asymmetry to the beam and energy will be deposited along the *x* and *y* directions differently. In what is to follow, a fraction *Q* of the energy density will be taken to correspond to that along the *x* axis and the remaining 1 – *Q* fraction along the *y* axis. An approximation procedure will be given below to determine the division of energy among these components and to find a value for *Q*.

Consider, at the outset, the evaluation of the first integrand terms of Equations (75) and (76). Once again, working in the $z = 0$ plane, one has from the first of Equation (B.3)

$$\left(\hat{x}\cdot\vec{\nabla}_\perp e\right)^2 = A_0^2 k^2 \alpha_1^2 x^2 \exp\left(-k\alpha_1\left(x^2+y^2\right)\right) \tag{77}$$

and a similar relation for the *y* component. One thus has

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\left(\hat{x}\cdot\vec{\nabla}_\perp e\right)\right|^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\left(\hat{y}\cdot\vec{\nabla}_\perp e\right)\right|^2 dxdy = \frac{A_0^2\pi}{2} \tag{78}$$

Using Equation (67), one also has

$$\begin{aligned} |e|^2\left(\hat{x}\cdot\vec{\nabla}\phi\right)^2 = A_0^2 \exp\left(-k\alpha_1\left(x^2+y^2\right)\right)\Big[& \left(k\alpha_2 x\right)^2 + 4k\alpha_2 x\Phi(P_0)\sqrt{k\alpha_1} \\ & \cdot\exp\left(-k\alpha_1 x^2\right)\exp\left(-k\alpha_1 y^2\right) + 4\Phi^2(P_0)k\alpha_1 \cdot\cdot\exp\left(-2k\alpha_1 x^2\right)\exp\left(-2k\alpha_1 y^2\right)\Big] \end{aligned} \tag{79}$$

and similarly,

$$|e|^2\left(\hat{y}\cdot\bar{\nabla}\phi\right)^2 = A_0^2\exp\left(-k\alpha_1\left(x^2+y^2\right)\right)y^2\Big[\left(k\alpha_2\right)^2 - 4k^2\alpha_1\alpha_2\sqrt{\pi}\Phi\left(P_0\right)$$
$$\cdot\exp\left(-k\alpha_1 y^2\right)\left(1+\operatorname{erf}\left(\sqrt{k\alpha_1}x\right)\right) + +4k^2\alpha_1^2\pi\Phi^2\left(P_0\right)\exp\left(-2k\alpha_1 y^2\right)\left(1+\operatorname{erf}\left(\sqrt{k\alpha_1}x\right)\right)^2\Big] \tag{80}$$

These yield

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|e|^2\left(\hat{x}\cdot\bar{\nabla}\phi\right)^2 dxdy = A_0^2\frac{\pi}{2}\frac{\alpha_2^2}{\alpha_1^2} + A_0^2\frac{4}{3}\pi\Phi^2\left(P_0\right) \tag{81}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|e|^2\left(\hat{y}\cdot\bar{\nabla}\phi\right)^2 dxdy = A_0^2\frac{\pi}{2}\frac{\alpha_2^2}{\alpha_1^2} - A_0^2\pi\sqrt{\frac{\pi}{2}}\frac{\alpha_2}{\alpha_1}\Phi\left(P_0\right) + A_0^2\pi^{3/2}\frac{8}{9}\sqrt{\frac{\pi}{3}}\Phi^2\left(P_0\right) \tag{82}$$

The evaluation of the second integrand term in Equation (76) commences with the use of Equation (40), i.e.,

$$k^2\varepsilon_T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G\left(|E|^2\right)dx\,dy = \frac{1}{2}k^2\varepsilon_T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\,|E|^2\,dx\,dy \tag{83}$$

One now employs Equations (46) and (A.3) in the right side of this relation and obtains in the $z = 0$ plane

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\,|E|^2\,dxdy\Big|_{z=0} = \frac{A_0^4}{2}\beta\sqrt{\frac{\pi}{k\alpha_1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-k\alpha_1 x^2\right)\exp\left(-2k\alpha_1 y^2\right)\left(1+\operatorname{erf}\left(\sqrt{k\alpha_1}x\right)\right)dxdy$$
$$= \frac{A_0^4}{2}\beta\left(\frac{1}{k\alpha_1}\right)^{3/2}\frac{\pi^{3/2}}{\sqrt{2}} \tag{84}$$

Using this in Equation (83) and rearranging terms gives

$$k^2\varepsilon_T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G\left(|E|^2\right)dx\,dy = \frac{P_0}{P_T}\left(\frac{\pi A_0^2}{4}\right)\sqrt{\frac{\pi}{2}} \tag{85}$$

It may be noted that Equation (85) is independent of the thickness L of the absorbing medium. This is due to the level of estimation that the ‘thin screen’ approximation yields; in that approximation, only the phase and not the amplitude of the field intensity of the beam is perturbed. It would only be with the next approximation beyond that used in Appendix B that would add a perturbation to the amplitude of the field. This limitation must be kept in mind in the applications.

As discussed above, the division of this quantity among the x and y components cannot be arbitrarily done. The value of Q, introduced above, will be calculated shortly. Thus, substituting Equations (66), (78), (81) and Equation (85) into Equation (75) results in

$$C_x = \frac{\alpha_1}{k}\left(1+\frac{\alpha_2^2}{\alpha_1^2}+\frac{8}{3}\Phi^2\left(P_0\right)\right) + Q\sqrt{\frac{\pi}{2}}\left(\frac{2}{kL}\right)\Phi\left(P_0\right) \tag{86}$$

and, similarly, putting Equations (66), (78), (82), and Equation (85) into Equation (76) gives

$$C_y = \frac{\alpha_1}{k}\left(1+\frac{\alpha_2^2}{\alpha_1^2}-2\frac{\alpha_2}{\alpha_1}\sqrt{\frac{\pi}{2}}\Phi(P_0)+\frac{2}{3}\frac{\pi}{\sqrt{3}}\Phi^2(P_0)\right)+(1-Q)\sqrt{\frac{\pi}{2}}\left(\frac{2}{kL}\right)\Phi(P_0) \tag{87}$$

In applying Equation (43) to calculate the beam radius about the local axis, care must be taken in the specifications for the *B* and *C* parameters using Equations (73), (86) and (87). In fact, the derivation of Equations (73), (86) and (87) require the corresponding parameters in Equation (43) to be multiplied by a factor of 2, i.e.,

$$r_{eff,x}^2(z) = \frac{1}{2}(2C_x)z^2 + (2B_x(0))z + a_{eff}^2(0) - (\bar{A}\cdot\hat{x}z)^2 \tag{88}$$

and

$$r_{eff,y}^2(z) = \frac{1}{2}(2C_y)z^2 + (2B_y(0))z + a_{eff}^2(0) - (\bar{A}\cdot\hat{y}z)^2 \tag{89}$$

Using Equations (88) and (89), one has from the foregoing, after rearrangement of factors and simplification,

$$\alpha_1 k r_{eff,x}^2(z) = \frac{r_{eff,x}^2}{W_0^2} = (1-\alpha_2 z)^2 + \alpha_1^2\left(1+\frac{5}{3}\Phi^2(P_0)\right)z^2 + Q\alpha_1\sqrt{\frac{\pi}{2}}\left(\frac{2}{L}\right)\Phi(P_0)z^2 \tag{90}$$

and

$$\begin{aligned}\alpha_1 k r_{eff,y}^2(z) = \frac{r_{eff,y}^2}{W_0^2} &= (1-\alpha_2 z)^2 + \alpha_1^2\left(1+\frac{16}{9}\frac{\pi}{\sqrt{3}}\Phi^2(P_0)\right)z^2 \\ &\quad + (1-Q)\alpha_1\sqrt{\frac{\pi}{2}}\left(\frac{2}{L}\right)\Phi(P_0)z^2 + 2\alpha_1\sqrt{\frac{\pi}{2}}\Phi(P_0)z(1-\alpha_2 z)\end{aligned} \tag{91}$$

The case of a focused beam wave, i.e., $F = D$ will be considered, without addressing the methodology with which to achieve it in practice. In this instance, the above relations reduce to

$$\left.\frac{r_{eff,x}^2}{W_0^2}\right|_{foc} = \alpha_1^2\left(1+\frac{5}{3}\Phi^2(P_0)\right)D^2 + Q\alpha_1\sqrt{\frac{\pi}{2}}\left(\frac{2}{L}\right)\Phi(P_0)D^2 \tag{92}$$

and

$$\left.\frac{r_{eff,y}^2}{W_0^2}\right|_{foc} = \alpha_1^2\left(1+\frac{16}{9}\frac{\pi}{\sqrt{3}}\Phi^2(P_0)\right)D^2 + (1-Q)\alpha_1\sqrt{\frac{\pi}{2}}\left(\frac{2}{L}\right)\Phi(P_0)D^2 \tag{93}$$

An expression for the value of *Q* can now be obtained. The first member of the right side of Equation (92) is the semi-axis of the beam in the *x* direction and that of Equation (93) is the semi-axis of the beam in the *y* direction. As a first approximation, it can be assumed that the ratio of the lengths of these semi-axes is equal to the corresponding ratio of the energy deposited along these axes, i.e.,

$$\frac{1+\frac{5}{3}\Phi^2\left(P_0\right)}{1+\frac{16}{9}\frac{\pi}{\sqrt{3}}\Phi^2\left(P_0\right)}=\frac{Q}{1-Q} \tag{94}$$

Solving this expression for Q yields

$$Q=Q\left(P_0\right)=\frac{P_T^2+K_1P_0^2\frac{L\alpha_1}{4}}{2P_T^2+\left(K_1+K_2\right)P_0^2\frac{L\alpha_1}{4}} \tag{95}$$

where $K_1=5/3$ and $K_2=16\pi/9\sqrt{3}$. This expression is well behaved at the limits $\lim\limits_{P_0\to 0}Q=1/2$ (indicating that the energy is deposited equally along the axes when no thermal action is prevalent) and $\lim\limits_{P_0\to\infty}Q=0.341$. Once again, it must be remembered that this approximation procedure is in lieu of a second-order solution to the equations of propagation in which the field amplitude of the beam wave is also given as a function of propagation conditions along the beam.

Numerical application of the foregoing to focused beam propagation in a nonlinear thermal medium commences with the rewriting of Equations (92) and (93) in a dimensionless form. Additionally, the expression for the corresponding beam deflection $\vec{\Delta}_\perp$ as given by Equation (69) will be treated. First, dealing with the latter expression, Equation (69) and be rewritten in the dimensionless form (deflection from intended axis normalized by the initial beam radius)

$$\Delta_n\equiv\left|\frac{\vec{\Delta}_\perp}{W_0}\right|=\sqrt{2}\,\frac{D}{kW_0^2}\Phi\left(P_0\right)=\frac{1}{\sqrt{2}}l_d\Phi\left(P_0\right) \tag{96}$$

where $l_{dD}\equiv 2D/kW_0^2$ is the diffraction length associated with the aperture size and the total propagation distance D to the target. Figures 1 and 2 depict the variation of the normalized deflection of the beam versus the beam power in kilowatts, for $l_{dD}=0.02$ and $l_{dD}=0.2$, respectively; these correspond to realistic beam and propagation conditions. The direction of the deflection is along the –*x* axis, i.e., it is directed into the wind where regions of higher refractive index prevail. As heat is removed from the propagation channel along the *x* axis, it continues to accumulate along the *y* axis of the channel. This gives rise to an asymmetry in the beam cross-section as is shown by considering the beam radii.

Similarly, for the beam radii at the intended focus, one can write Equation (92) as

$$\left.\frac{r_{eff,x}^2/W_0^2}{\left(\alpha_1D\right)^2}\right|_{foc}\equiv r_{F,x}^2=\left(1+\frac{5}{3}\Phi^2\left(P_0\right)\right)+Q\sqrt{\frac{\pi}{2}}\left(\frac{2}{\alpha_1L}\right)\Phi\left(P_0\right) \tag{97}$$

The left side is essentially the square of the ratio of the beam radius in the *x* direction normalized to the intended radius of the focal spot on the target, $2D/kW_0$. The same for Equation (93), i.e.,

$$r_{F,y}^2=\left(1+\frac{16}{9}\frac{\pi}{\sqrt{3}}\Phi^2\left(P_0\right)\right)+\left(1-Q\right)\sqrt{\frac{\pi}{2}}\left(\frac{2}{\alpha_1L}\right)\Phi\left(P_0\right) \tag{98}$$

In both of these relations, the quantity $l_{dL} \equiv \alpha_1 L$ is the diffraction length with respect to the aperture size and the thickness of the atmospheric layer L. Figures 3 and 4 depict the beam radii variation versus beam power for two representative values of l_{dL}.

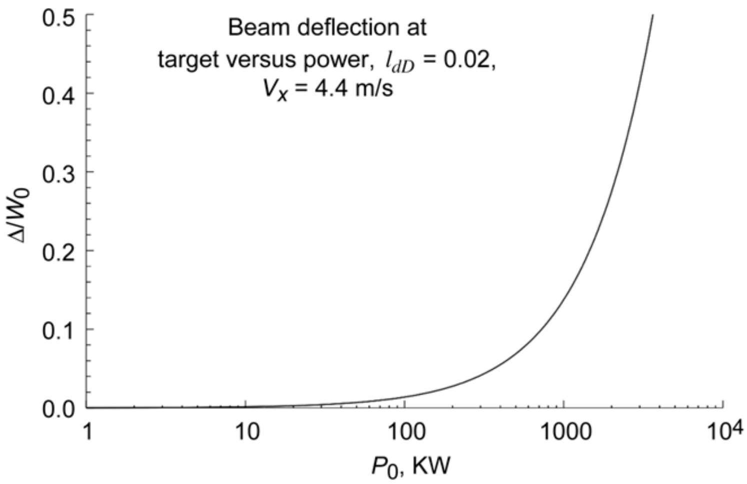

Figure 1.—Beam deflection from intended axis normalized to initial beam radius versus beam power in kilowatts for l_{dD} = 0.02.

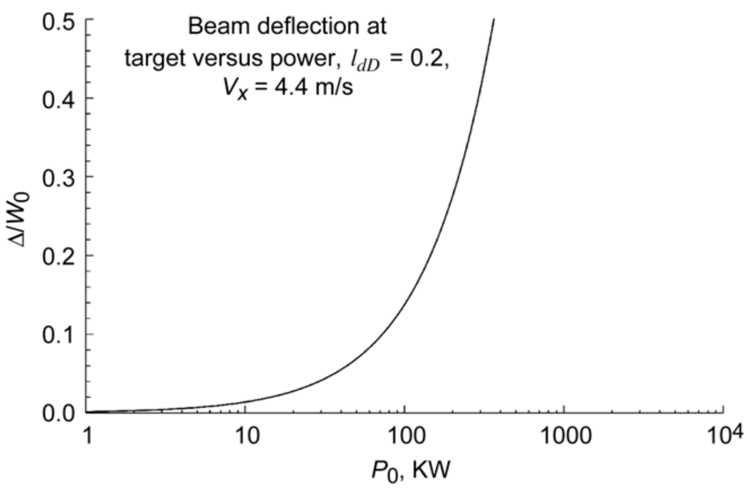

Figure 2.—Beam deflection from intended axis normalized to initial beam radius versus beam power in kilowatts for l_{dD} = 0.2.

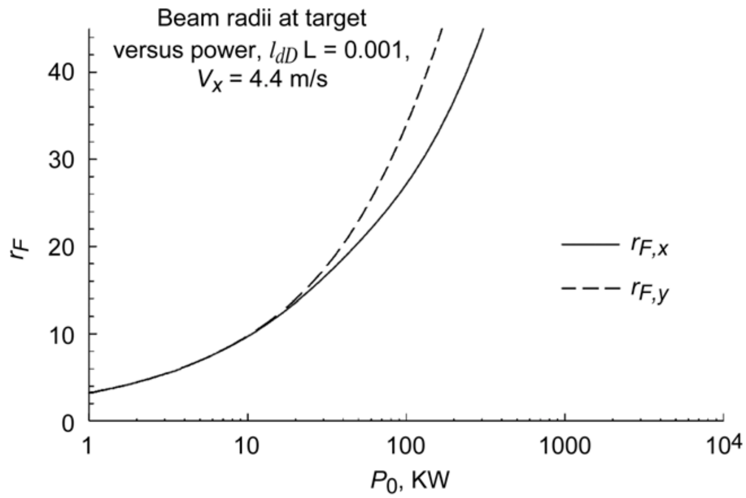

Figure 3.—Beam radii along *x* and *y* axes normalized to focal spot radius at target versus beam power in kilowatts for l_{dD} = 0.001.

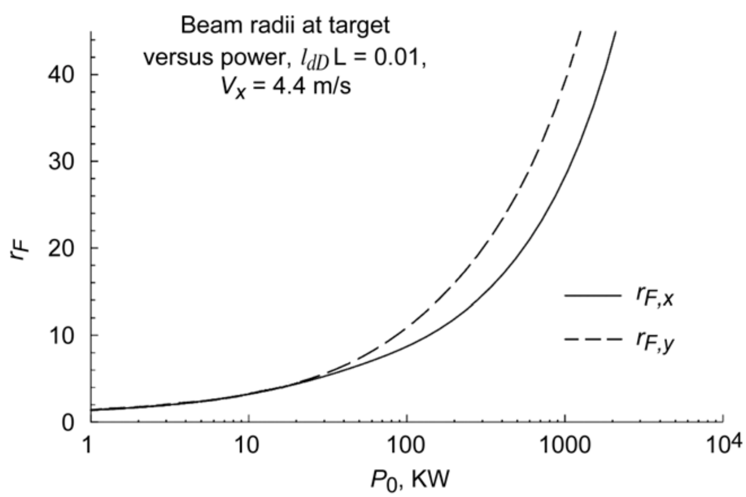

Figure 4.—Beam radii along *x* and *y* axes normalized to focal spot radius at target versus beam power in kilowatts for l_{dD} = 0.01.

In addition to the evaluation of the deleterious effects of the atmosphere on the propagation of high-energy laser beams, the methodology introduced above is amenable to the analysis of the design and specification of adaptive optics approaches whereby the phase front of the transmitted laser beam is appropriately modified to mitigate the atmospheric effects (Ref. 10). For example, due to the elliptical profile of the radiation in the transverse plane, the use of the well-known Zernike polynomials (Ref. 11), which are defined on a unit circle, are not appropriate for this application. Instead, the transmitted phase front $\phi_T(x,y)$ at the output aperture of the transmitter can be expanded in a set of orthogonal polynomials associated with a Gaussian weight function, i.e., the Hermite polynomials $H_n(x)$ and $H_m(y)$. Thus,

$$\phi_T(x,y) = \sum_{n}^{M} \sum_{m}^{N} a_{nm} H_n(x) H_m(y) \tag{99}$$

The expansion coefficients a_{nm} are determined by applying one of several ‘performance metrics’. For example, for power beaming applications, it is desired to shape the phase front so as to minimize the radius of the beam at the target. Such a representation is straightforward to use in Equation (63) and subsequently, in Equations (71), (72), (75) and (76) to obtain the corrected radius values for the beam.

7.0 Summary and Conclusion

The structure of the nonlinear parabolic equation that describes the propagation of a high energy laser beam through a medium with thermal nonlinearities is exploited to describe the moments of the associated electric field in order to capture the universal behavior of the displacement and radius of the beam; these are two of the more significant performance parameters for the operation of the high energy laser transmission system. Once this was done, a simple thin phase screen propagation model of the Earth atmosphere with a wind was advanced and applied to the equations for the beam displacement and radius. The presence of the wind of course broke the symmetry of the beam into one with an elliptical cross section. Expressions were then developed for the behavior of the semi-axes of the beam for the propagation path out of the atmosphere. Graphical results were then presented for the deleterious action of the atmosphere on the propagation of the beam. Since only phase perturbations were admitted into the model, all the effects placed on the propagating beam are phase based and can thus be mitigated at the transmitter. The elliptical cross-section of the beam dictates that the circular Zernike polynomials are not sufficient to describe the phase front and that a Hermite polynomial based phase decomposition be used. Such a model was exploited for the analysis of laser power beaming for space propulsion (Ref. 12).

Appendix A

Equation (7) can be simplified straight away by first specializing the wind velocity to be along the x axis of the coordinate system, $\vec{V} = V\hat{x}$, where $\hat{x}$ is the unit vector. Further, the diffusivity $\nabla^2 T$ will be taken to have contributions only in the direction transverse to the laser beam propagation, i.e., $\nabla^2 T \approx \nabla_\perp^2 T$. Finally, only the stationary heating case will be considered for this particular discussion whereby the time derivative can be dropped. (That is, the CW radiation is taken to be acting long enough for the steady state case to be achieved.) Applying these considerations to Equation (7) allows it to be written

$$V\frac{\partial T}{\partial x} - \chi\nabla_\perp^2 T = \frac{\alpha c}{8\pi\rho C_p}|E|^2 \tag{A.1}$$

where $\chi \equiv \kappa/\rho C_p$ is the associated thermal diffusivity. There are two heat transfer mechanisms that are described by this relation, i.e., forced convection due to the wind given by the first term on the left side of the equation and diffusion by the second term on the left side. The order of magnitude of the corresponding derivatives are $\partial T/\partial x \sim T/r_{eff}$ and $\nabla_\perp^2 T \sim T/r_{eff}^2$ where r_{eff} is the effective laser beam radius. Consider now the ratio of the coefficients of the two terms within the brackets of Equation (A.1), i.e., $\eta \equiv Vr_{eff}/\chi$. This dimensionless ratio is known in fluid mechanics as the Peclet number. Taking the nominal wind velocity $V = 4.4$ m/s (10 mi/hr), $r_{eff} = 10$ m and using the documented value for the thermal diffusivity of the atmosphere $\chi = 2.12\times10^{-5}$ m^2/s, one has that $\eta = 2.1\times10^6 >> 1$ allowing one to neglect the second term in the brackets. In this instance, Equation (A.1) becomes

$$\frac{\partial T}{\partial x} = \beta|E|^2 \tag{A.2}$$

where $\beta \equiv \alpha c/\left(8\pi V\rho C_p\right)$. This equation can now be easily integrated using Equation (46). The result is

$$\begin{aligned} T\left(\left|E\left(\bar{r},z\right)\right|^2\right) &= \beta\int_{-\infty}^{x}\left|E\left(x',y,z\right)\right|^2 dx' \\ &= \beta\frac{A_0^2}{2}\left(\frac{\pi}{k\alpha_1\Lambda(z)}\right)^{1/2}\exp\left[-\frac{k\alpha_1}{\Lambda(z)}y^2\right]\left(1+\operatorname{erf}\left[\left(\frac{k\alpha_1}{\Lambda(z)}\right)^{1/2}x\right]\right) \end{aligned} \tag{A.3}$$

Now in order to substantiate Equation (40) for this case, one proceeds to define

$$I\left(T\left(\left|E(x)\right|^2\right),G\left(\left|E(x)\right|^2\right)\right) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[T|E|^2 - 2G\left(|E|^2\right)\right]dx\,dy \tag{A.4}$$

and attempts to show that the expression is equal to zero. To this end, one forms the derivative

$$\frac{dI}{dx} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\partial T}{\partial x}|E|^2 + \frac{\partial|E|^2}{\partial x}T - 2\frac{\partial G}{\partial x}\right]dx\,dy \tag{A.5}$$

which, if also zero, can be used to argue that Equation (A.4) vanishes do to conditions at $x \to \infty$. From Equation (13),

$$G(\Xi(x)) = \int_{0}^{\Xi(x)} T(\Xi')d\Xi', \qquad \Xi(x) \equiv |E(x)|^2 \tag{A.6}$$

Differentiating Equation (A.6),

$$\frac{\partial G}{\partial x} = \frac{\partial \Xi(x)}{\partial x} T(\Xi) + \int_{0}^{\Xi} \frac{\partial T(\Xi')}{\partial x} d\Xi' \tag{A.7}$$

substituting into Equation (A.5), using Equation (A.2), performing the indicated functional integration, and simplifying finally gives

$$\frac{dI}{dx} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\partial |E|^2}{\partial x} T \right] dx\,dy \tag{A.8}$$

which does not appear to vanish. To be sure, proceeding to integrate Equation (A.8) by parts yields

$$\frac{dI}{dx} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[|E|^2 \frac{\partial T}{\partial x} \right] dx\,dy = -\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |E|^4 \, dx\,dy \tag{A.9}$$

upon remembering Equation (A.2). The $|E|^4$ dependency is reminiscent of the Kerr effect that appears at field intensities much higher than for thermal blooming effect. To be sure, in order to secure numerical estimates, one uses Equation (46) in Equation (A.9) to obtain

$$\frac{dI}{dx} = \beta A_0^4 \frac{\pi}{4} W_0^2 \tag{A.10}$$

To rid of the amplitude factors and put them into a more familiar quantity, i.e., the total power P_0 within the laser beam, one uses the relation

$$P_0 \equiv \left(\frac{cA_0^2}{8\pi} \right) \left(\frac{\pi W_0^2}{2} \right) \tag{A.11}$$

as well as remember the definition of β to write Equation (A.10) as

$$\frac{dI}{dx} = \frac{16\alpha P_0^2}{cW_0^2 V \rho C_p} \tag{A.12}$$

Using the values of the parameters given above and additionally taking $W_0 \approx 1$, this gives $dI/dx \approx 4 \times 10^{-5}$ for $P_0 = 1 \times 10^6$ watts. Hence, dI/dx effectively vanishes for nominal laser power and thus Equation (A.4) can also be taken to vanish, i.e.,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[T\left|E\right|^{2}-2G\left(\left|E\right|^{2}\right)\right]dx\,dy\approx 0 \tag{A.13}$$

A more complete treatment would involve a more general solution to the temperature diffusion equation, Equation (A.1), but will not be dealt with here.

Appendix B

Writing the total phase as the sum of the unperturbed phase ϕ_0 (i.e., defined by Equation (62) with $T\left(|E|^2\right)=0$) and the perturbed component $\tilde{\phi}$, one has, upon introducing $\phi=\phi_0+\tilde{\phi}$ into Equation (62), the system of equations

$$2k\frac{\partial\phi_0}{\partial z}+\left(\vec{\nabla}_\perp\phi_0\right)^2=0 \tag{B.1}$$

$$2k\frac{\partial\tilde{\phi}}{\partial z}+2\vec{\nabla}_\perp\phi_0\cdot\vec{\nabla}_\perp\tilde{\phi}+\left(\vec{\nabla}_\perp\tilde{\phi}\right)^2+k^2\varepsilon_T T\left(|E|^2\right)=0 \tag{B.2}$$

Now, equating Equations (44) and (54), one finds that

$$\begin{aligned} e(\vec{r},z)&=\frac{A_0}{1+i\alpha_d z}\exp\left(-\frac{k\alpha_1}{2\Lambda(z)}r^2\right)\\ \phi_0(\vec{r},z)&=\frac{kr^2}{2\Lambda(z)}\left(\alpha_2-\left(\alpha_1^2+\alpha_2^2\right)z\right)\end{aligned} \tag{B.3}$$

the second relation of which, indeed, can be shown to satisfy Equation (B.1).

A solution for Equation (B.2) commences with multiplying it by $1/k$ and considering $k\to\infty$ (relative to all the other inverse lengths in the problem) which gives

$$2\frac{\partial\tilde{\phi}}{\partial z}+k\varepsilon_T T\left(|E|^2\right)=0 \tag{B.4}$$

One needs to use Equation (A.3) to integrate this expression. To this end, one applies the following ‘thin screen’ approximation; For the case considered here, the perturbing atmospheric layer of thickness L, in which the thermal nonlinearities take place, is such that $L \ll D$ where D is the total distance to the target between LEO and GEO. Thus, within this region, relative to the total propagation distances involved, one can let $\Lambda(z)\approx 1$ for the longitudinal distances z occurring within the atmospheric layer. Thus, one can write

$$T\left(|E|^2\right)\approx\beta A_0^2\sqrt{\pi}\frac{W_0}{\sqrt{2}}\exp\left[-k\alpha_1 y^2\right]\left(\frac{1}{2}\right)\left(1+\operatorname{erf}\left[\sqrt{k\alpha_1}x\right]\right) \tag{B.5}$$

Substituting this result into Equation (B.4) and integrating across the thickness L of the atmosphere yields, upon rearranging factors,

$$\tilde{\phi}=-\left(\frac{P_0}{P_T}\right)\left(\frac{\sqrt{\pi}L}{kW_0^2}\right)\left(\frac{1}{2}\right)\exp\left(-k\alpha_1 y^2\right)\left(1+\operatorname{erf}\left(\sqrt{k\alpha_1}x\right)\right) \tag{B.6}$$

where

$$P_0\equiv\left(\frac{cA_0^2}{8\pi}\right)\left(\frac{\pi W_0^2}{2}\right) \tag{B.7}$$

is the initial power of the beam wave of waist radius W_0 and

$$P_T \equiv \frac{\sqrt{2}\pi\rho V C_p}{k^2 W_0 |\varepsilon_T| \alpha} \tag{B.8}$$

is the characteristic power for the onset of thermal blooming. Defining

$$\Phi(P_0) \equiv -\frac{P_0}{P_T}\left(\frac{L}{2kW_0^2}\right) > 0 \tag{B.9}$$

(since $\varepsilon_T < 0$), Equations (B.3) and (B.6) yield for the total phase of the laser beam leaving the thin atmospheric layer with a wind of velocity V

$$\phi = \frac{k\left(x^2+y^2\right)}{2\Lambda(z)}\left(\alpha_2 - \left(\alpha_1^2+\alpha_2^2\right)z\right) + \sqrt{\pi}\Phi(P_0)\exp\left(-k\alpha_1 y^2\right)\left(1+\operatorname{erf}\left(\sqrt{k\alpha_1}x\right)\right) \tag{B.10}$$

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14. ABSTRACT

The method of moments is used to define and derive expressions for laser beam deflection and beam radius broadening for high-energy propagation through the Earth's atmosphere. These expressions are augmented with the integral invariants of the corresponding nonlinear parabolic equation that describes the electric field of high-energy laser beam to propagation to yield universal equations for the aforementioned quantities; the beam deflection is a linear function of the propagation distance whereas the beam broadening is a quadratic function of distance. The coefficients of these expressions are then derived from a thin screen approximation solution of the nonlinear parabolic equation to give corresponding analytical expressions for a target located outside the Earth's atmospheric layer. These equations, which are graphically presented for a host of propagation scenarios, as well as the thin screen model, are easily amenable to the phase expansions of the wave front for the specification and design of adaptive optics algorithms to correct for the inherent phase aberrations. This work finds application in, for example, the analysis of beamed energy propulsion for space-based vehicles.

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