PARALLEL PROCESSORS AND NONLINEAR STRUCTURAL DYNAMICS ALGORITHMS AND SOFTWARE

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PREFACE

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Abstract

An explicit-explicit subcycling procedure for the finite element analysis of structural dynamics is developed. This procedure has relaxed the usual constraint of requiring integer time step ratios for adjacent nodal groups. This allows for greater advantage to be taken of local stability criteria, and thus improves the efficiency of the explicit time integrator. Example problems are included to demonstrate the accuracy and stability of the method.

I. Introduction

Explicit-explicit subcycling is oftentimes used in dynamic finite element problems in order to improve computational efficiency when the element size varies over the mesh. This is accomplished by separating the elements or nodes into groups and assigning a different time step to each group. The time step of each group depends only on the frequencies of the elements in that group. This results in great computational savings since less stiff elements can be assigned to the same group. In this way a time step is used for each group which is much closer to the critical time steps of the elements in the group. Mixed time integration was first introduced by Belytschko and Mullen[1] where they presented an "implicit-explicit" method for the second order equations which arise from the semidiscretization of the continuum equation. This was a nodal partition where both implicit and explicit methods were used in the time integration process. The stability of this type of procedure was studied in [2,3]. Hughes and Liu[4,5] later introduced an "implicit-explicit" method which divided the mesh into groups of elements rather than nodes, known as an element partition. Both methods were found to substantially improve computational efficiency. An "explicit-explicit" subcycling procedure was then introduced by Belytschko and Mullen[6]. In this procedure each group of nodes was integrated explicitly, but different time steps were used for each group.

Mixed time integration procedures for first order semidiscretizations have also been proposed. Belytschko et al.[7] present sufficient conditions for stability of their element partition by using the element eigenvalue inequality directly on the evolution equations. The stability analysis of nodal partitions is complicated by the fact that their amplification matrices are unsymmetric. Smolinski et al.[8] avoid solving any unsymmetric systems in their nodal partition, and are able to develop stability conditions for semidiscretizations of the diffusion equation.

A major drawback to all these previous methods is that integer time step ratios are required. That is, if one group with a time step of Δt shares nodes or elements with a second group, then the only possibilities for the time step of the second group are $[(n)\Delta t]$ or $[\Delta t / n]$ where n is a positive integer. This introduces two difficulties when dividing the mesh into groups. First, it can severely restrict the division of the mesh into more than just a few groups, and therefore decrease the efficiency of the mixed time integration procedure being used. Second, since the time step is restricted to just a few possibilities, a time step may have to be used for a group that is much smaller than the stable time steps of the elements in the group.

Recently, Mizukami[9] has relaxed this restriction for first order ordinary differential equations which result from semidiscretization of the diffusion equation. He presents an explicit-explicit nodal partition that allows for much greater flexibility when assigning time steps to the different nodal groups. He then gives a stability proof of the method, however, this proof only applies to a particular class of elements. Donea and Laval[10] have recently extended this proof of stability to arbitrary elements.

The purpose of this paper is to present an explicit-explicit subcycling procedure for structural dynamics which does not require integer time step ratios for adjacent groups. It will be shown that this procedure requires very little additional storage and appears to be stable for a wide variety of example problems.

II. Nomenclature and definitions

n	=	time step iteration counter
Δt_M	=	master time step
Δt_I	=	nodal time step
Δt_I^{old}	=	nodal time step for the previous step
$\Delta t_e^{critical}$	=	critical time step of an element
Δt_e	=	time step used to update an element
Δt_{MLS}	=	master level time step
t_M	=	master clock
t_I	=	nodal clock
t_e	=	element clock
t_{MLT}	=	master level time
$\mathbf{d}$	=	global vector of displacements
$\mathbf{v}$	=	global vector of velocities
$\mathbf{a}$	=	global vector of accelerations
$\mathbf{d}^e$	=	element level vector of displacements
$\mathbf{v}^e$	=	element level vector of velocities
$\boldsymbol{\sigma}^e$	=	Cauchy stress tensor for an element
$\dot{\boldsymbol{\sigma}}^e$	=	stress rate tensor for an element
$\boldsymbol{\varepsilon}^e$	=	strain tensor for an element
$\dot{\boldsymbol{\varepsilon}}^e$	=	strain rate tensor for an element
$\mathbf{f}$	=	global vector of nodal forces
$\mathbf{f}_{ext}$	=	global vector of external nodal forces
$\mathbf{f}_{int}$	=	global vector of internal nodal forces
$\mathbf{f}_{int}^e$	=	element level vector of internal nodal forces

W_{ext} = external work done on the system

W_{kin} = kinetic energy of the system

W_{int} = internal energy of the system

ΔW_{int} = element level increment in internal energy

$\mathbf{B}$ = gradient operator

$\mathbf{L}^e$ = element assembly operator

$\mathbf{M}$ = lumped mass matrix

m_I = diagonal component of the mass matrix for node I

Ω^e = domain of the element

NUMEL = total number of elements

NUMEL = total number of nodes

MOD(i,j) = remainder of the division of i by j

III. Governing equations

The equations of motion for structural dynamics are given by

$$\mathbf{Ma} + \mathbf{f} = \mathbf{0} \tag{3.1}$$

where,

$\mathbf{M}$ = mass matrix

$\mathbf{a}$ = global vector of accelerations

$\mathbf{f}$ = global vector of nodal forces.

The nodal forces are comprised of external and internal parts.

$$\mathbf{f} = \mathbf{f}_{int} - \mathbf{f}_{ext} \tag{3.2}$$

where,

$\mathbf{f}_{ext}$ = global vector of external nodal forces

$\mathbf{f}_{int}$ = global vector of internal nodal forces.

The internal nodal forces for a single element are calculated from the element stresses and can be written

$$\mathbf{f}^{e}_{int} = \int_{\Omega^e} \mathbf{B}^T \boldsymbol{\sigma}^e \, d\Omega \tag{3.3}$$

where,

$\mathbf{f}^e_{int}$ = element internal force vector

Ω^e = domain of the element

$\mathbf{B}$ = gradient operator

$\boldsymbol{\sigma}^e$ = Cauchy stress tensor for an element.

These internal forces are then assembled to the global vector:

$$\mathbf{f}_{int} = \mathbf{f}_{int} + (\mathbf{L}^e)^T \mathbf{f}^e_{int} \tag{3.4}$$

where,

$\mathbf{L}^e$ = element assembly operator.

For nonlinear analysis a strain rate formulation will be used to calculate the stress of an element.

$$\dot{\boldsymbol{\sigma}}^e = \mathbf{C}\,\dot{\boldsymbol{\varepsilon}}^e \tag{3.5}$$

$$\boldsymbol{\sigma}^e = \boldsymbol{\sigma}^e + \Delta t_e\,\dot{\boldsymbol{\sigma}}^e \tag{3.6}$$

where,

$\dot{\boldsymbol{\varepsilon}}^e$ = strain rate tensor for an element, such as the velocity strain or rate-of-deformation

$\dot{\boldsymbol{\sigma}}^e$ = frame-invarient stress rate tensor for an element

$\mathbf{C}$ = constitutive matrix

Δt_e = time step used for the stress update of an element.

The strain rate tensor is given by the strain displacement relation of the element as follows:

$$\dot{\boldsymbol{\varepsilon}}^e = \mathbf{B}\, \mathbf{v}^e \tag{3.7}$$

where $\mathbf{v}^e$ is the element velocity vector.

In this paper, central difference time integration with variable time steps will be used to to advance the nodal velocities and displacements in time. The equations are given by the following:

$$v_{I,n+1/2} = v_{I,n-1/2} + \frac{1}{2}\left(\Delta t_I + \Delta t_I^{old}\right) a_{I,n} \tag{3.8}$$

$$d_{I,n+1} = d_{I,n} + \Delta t_I\, v_{I,n+1/2} \tag{3.9}$$

where,

$a_{I,n}$ = acceleration of node I at time $(n)\Delta t$

$v_{I,n+1/2}$ = velocity of node I at time $(n+1/2)\Delta t$

$d_{I,n+1}$ = displacement of node I at time $(n+1)\Delta t$

Δt_I = time step of node I

Δt_I^{old} = time step of node I for the previous step.

Since the time step for a node could change for each step, it is necessary to average the time steps as shown in equation (3.8) for the velocity update. Note that equation (3.6) has been written so that different time steps may be used for different elements. This

modification of the usual equations has also been made on equations (3.8) and (3.9) so that different time steps can be used for each node. The use of different time steps for different nodes and for different elements is common to all subcycling procedures. For the procedure to be presented here, however, the element update time steps of equation (3.6) are not necessarily the same as the nodal update time steps of equations (3.8) and (3.9). This added feature is unique to this new procedure and allows for the use of non integer time step ratios of adjacent nodes as will be explained in the following sections.

IV. Subcycling Procedure

The basic idea underlying this subcycling procedure is the use of a master clock which is incremented by the smallest time step which occurs within the mesh; a master clock was also used by Belytschko and Liu[11]. This smallest time step will be referred to as the master time step. While the master clock is updated by the master time step, the nodal clocks are incremented by the time step of each node. Whenever a nodal clock time is equal to the master clock the velocity and displacement of the node are updated with equations (3.8) and (3.9) using the time step of the node. The time step of each node must be an integer multiple of the master time step and cannot be greater than the smallest critical time step of the elements surrounding the node. Although the nodal time steps must be integer multiples of the master time step, the attractive feature of this procedure is that it is not necessary for the time steps of two adjacent nodes to form an integer ratio.

The times at which each element is updated are also governed by the nodal clocks of the nodes of the element. Each element is updated every time one of the nodes of the element will undergo a velocity and displacement update. For the purpose of this element update, it is necessary to use an element clock for each element. This element clock is set equal to the master clock each time the element is updated. In this way a record is kept of when each element was last updated. The time step for the incremental update of the element stresses, equation (3.7), is then given by the difference between the master clock and the element clock. It is important to realize that this element update time step does not necessarily coincide with the time step of any of the element's nodes.

As can be seen, the structure of this procedure is very similar to the subcycling algorithm described by Belytschko and Liu[11] and by Belytschko[12]. The major difference is that in the previous algorithm the times at which an element was updated was governed by element clocks while in this new procedure they are governed by the nodal clocks of the nodes of the element. Figure 1 gives the nodal influence diagram for the

partition described in references [11] and [12]. In this figure an arrow represents the influence of one node on the velocity and displacement calculation of another node, a solid dot represents a time at which the velocity and displacement of a node are updated, and an open dot represents a time at which the displacement of a node would have to be interpolated between two known values. In order for a partition to work correctly the element stresses must be calculated whenever a node of the element is updated. In figure 1 it can be seen that the stresses of element 1 are updated every $6\Delta t$, while the stresses of elements 2 and 3 are updated every $2\Delta t$ as is shown by the bold horizontal lines. Because the time steps of these two nodes form an integer ratio, element stresses are calculated whenever they are needed for nodal updates. However, if this procedure is used in the case of non integer time step ratios as shown in figure 2, then the appropriate element stresses would not be available to update each node's acceleration, velocity, and displacement. For example, to update the nodal values of node 2 at time $t = (n+3)\,\Delta t$, it is necessary to have the element stresses at this time for all elements that are adjacent to node 2. From figure 2 it is seen that the stress values of element 2 are known at times $(n)\,\Delta t$, $(n+2)\,\Delta t$, and $(n+4)\,\Delta t$, but not at the required time, $(n+3)\,\Delta t$. If nodal clocks are used instead, as shown in figure 3, the values of the element stresses are calculated whenever they are needed for a nodal update so that even in the case of non integer time step ratios it is possible to correctly update the nodal values.

V. Implementation of the Subcycling Procedure

A. Nodal Grouping

The first step in the nodal grouping procedure is to assign a time step to each node. This is accomplished as follows:

1. Assign a very large time step to each node, Δt_I.
2. Loop on the elements, (JE).

   Calculate the critical time step of the element, $\Delta t_e^{critical}(JE)$.

   Loop on the nodes of this element, (I).

$$\Delta t_I = \text{Minimum}(\Delta t_I , \Delta t_e^{critical}(JE)). \tag{5.1}$$

In this way each node is given the time step of the smallest element time step of the elements connected to the node. Next, it is necessary to determine a master time step so that all nodal time steps are an integer multiple of the master time step. This is most easily accomplished by letting the master time step be given by the smallest nodal time step.

$$\Delta t_M = \text{Minimum} (\Delta t_I). \tag{5.2}$$

The remaining nodal time steps are then reduced by the equation

$$\Delta t_I = \Delta t \, [\text{Integer}(\Delta t_I / \Delta t)]. \tag{5.3}$$

By equation (5.3) it is obvious that the master time step must be chosen so that it is smaller than or equal to all nodal time steps. It can also be seen from equation (5.3) that all nodal time steps are integer multiples of the master time step. Although equation (5.2) was used to determine the master time step, it should be noted that the master time step can be set to any value as long as the following two restrictions are satisfied.

1. All nodal time steps are integer multiples of the master time step.

2. A nodal time step is not greater than the smallest critical time step of the elements surrounding the node.

Next, the master level time step must be calculated. This identifies the time increment that occurs between steps when all nodes are updated and is defined, therefore, as the least common multiple of all nodal time steps.

$$\Delta t_{MLS} = \text{Least common multiple } (\Delta t_I). \tag{5.4}$$

Once this value is calculated the master level time can be calculated by adding the master level time step to the current master time.

$$t_{MLT} = t_M + \Delta t_{MLT}. \tag{5.5}$$

This master level time identifies the next time that all nodal and element clocks will be the same value, that is t_{MLT}. This means that all nodes and element stresses will have been updated to this master time level. It is useful to have all nodal values and element stresses updated to the same time level at some point in

the analysis so that the energy balance of the system may be checked.Once the master level time is determined the nodal grouping is complete.

To illustrate the nodal grouping procedure, an example is given in figure 4, along with the element time steps, nodal time steps, master time step, master level time step, and master level time.

B. Nodal Force Calculations

After the master time step, nodal time steps, and the master level time have all been assigned as described in the previous section, the time stepping procedure can begin. The first step in the procedure is to calculate the nodal forces. The external nodal forces at each node are calculated as usual, however the calculation of internal nodal forces for each element must be modified. Before an element is updated the clocks of all the nodes of the element are checked by the condition

$$\text{Is} \quad (t_I = t_M) \quad \text{for node I} \tag{5.6}$$

where,

t_I = nodal clock

t_M = master clock.

If condition (5.6) is true for any node of a given element, then the nodal clock is equal to the master clock and the node will be updated during this step. This nodal update will require the correct element stresses at this time for any element connected to this node. Therefore, this element must be updated during this step. This can be seen more easily with the help of an example. If it is

assumed that condition (5.6) is true for node I of an element, JE, then the velocity and displacement of node I will be updated by equations (3.8) and (3.9) which depend on the nodal acceleration, a_I. The calculation of a_I requires current internal nodal force values at node I, so that any element connected to node I must undergo a stress update at this time, so that the element stress coincides with the current time, t_M. Therefore, if condition (5.6) is true for any node of an element, the stress of the element must be updated. For the purpose of this stress update an element time step must be calculated. This element time step is given by the equation

$$\Delta t_e = t_M - t_e \tag{5.7}$$

where,

Δt_e = element time step

t_e = element clock.

This will give the time increment that has occurred since the last stress update of this element. Note that this element time step is not necessarily the same for an element each time it is updated. This can be seen by examining element 2 of figure 3. Once the element time step is calculated, the stress update of the element can take place. This update is given by equations (3.5) through (3.7) with the element update time step just calculated, Δt_e. In this way the element is updated to the current master clock time, t_M, using the appropriate time increment for each element. After the stresses have been updated the nodal forces are calculated in the usual manner by equations (3.3) and (3.4). The final step in the internal nodal

force calculation is to increment the clocks of all elements that have been updated with the equation

$$t_e = t_M. \tag{5.8}$$

C. Nodal Updates

The next step in the time stepping procedure is to update the nodal variables. Again, central difference time integration is used, however, the procedure given by equations (3.1), (3.8) and (3.9) must be modified slightly. The calculation of the nodal acceleration at time $(n)\Delta t$ is the same as before and is given by

$$a_{I,n} = \frac{(f_{I,n}^{ext} - f_{I,n}^{int})}{m_I} \tag{5.9}$$

for node I. To update the velocity of node I at time $(n)\Delta t$, the validity of the following condition is checked:

$$\text{Is} \quad (t_I = t_M) \quad \text{for node I ?} \tag{5.10}$$

If condition (5.10) is true for node I, then equations (3.8) and (3.9) are used to update the velocity and displacement of node I. If condition (5.10) is false, then the nodal values are not changed. Once the new velocities and displacements have been calculated, the nodal clock is incremented by the nodal time step.

$$t_I = t_I + \Delta t_I. \tag{5.11}$$

VI. Energy Calculations

A major drawback to explicit time integration schemes is the fact that they are conditionally stable. When performing explicit calculations a limit must be placed on the time increment being used in order to prevent a loss of stability. In linear calculations this presents no problem since in this case the stability criteria remain the same for the duration of the analysis. However, in nonlinear calculations, the stability conditions of the mesh are constantly changing due to changes in problem geometry or material properties. Therefore, some means of checking the stability of the problem must be introduced. In linear problems, when an instability occurs it is characterized by exponential oscillatory growth in the nodal displacements. A periodic check on displacements can usually guarantee that no instability has occurred in the problem. However, in nonlinear situations where plastic deformations may dissipate large amounts of energy, the decreased modulus may enable the system to regain stability (known as an arrested instability, Belytschko[13]). In these situations simply checking for large unrealistic results may not always provide assurance that no instability has occurred, since an instability may be confined to a small region of the mesh and the problem may even regain stability at a later time. If this happens, the results obtained for the problem may seem quite reasonable, but the earlier instability in the problem will render the final results totally inaccurate. An alternative method for checking the stability of explicit finite element calculations is the use of an energy balance check. Hughes and Belytschko[14] gives this energy balance check in the form

$$| W_n^{kin} + W_n^{int} - W_n^{ext} | \leq \delta \, \|W\| \tag{6.1}$$

where,

W_n^{ext} = external work performed on the system up to time $(n)\Delta t$

W_n^{int} = internal energy of the system at time $(n)\Delta t$

W_n^{kin} = kinetic energy of the system at time $(n)\Delta t$

δ = tolerance

and $\|W\|$ is some measure of the total energy of the system. The use of any one of the terms on the left hand side of equation (6.1) alone would not provide a useful measure of this energy. For impulsively loaded problems when an initial velocity is prescribed, there would be no external work. For vibration problems the values of the kinetic energy and internal energy would oscillate and become very small at some point in the problem. Therefore, a measure of the total energy of the system should include all three terms.

$$\|W\| = |W_n^{ext}|+|W_n^{int}|+|W_n^{kin}|. \tag{6.2}$$

The use of absolute value sign in the left hand side of equation (6.1) is not really necessary since an instability is usually characterized by a growth in kinetic and internal energies which are nonnegative except for roundoff errors; therefore, this left hand side would always be positive in the case of an instability.

In time integration without subcycling equation (6.1) is checked at each time step to insure that the work done on the system and the energy of the system are nearly equal. Since a numerical instability in a problem always implies a loss of energy balance, it can be assumed that if equation (6.1) is true at every step, there is no such instability in the problem. For the case with subcycling, this energy balance cannot be checked at every step. This is because the internal energy of each element is not updated at every step. This can be seen in the equation for the increment of internal energy for an element.

$$\Delta W_n^{int,e} = \frac{\Delta t_e}{2}\int_{\Omega^e} (\boldsymbol{\sigma}^e_{n-m}+\Delta\boldsymbol{\sigma}^e_n)^T \dot{\boldsymbol{\varepsilon}}^e_n \, d\Omega \tag{6.3}$$

where,

$\Delta W_n^{int,e}$ = increment of internal energy for an element from time (n-m) Δt_M to (n) Δt_M.

In equation (6.3) the subscript m represents the number of master time steps that have passed since this element was last updated. After this increment of internal energy for an element is calculated, it is added to the total internal energy of the system.

$$W_n^{int} = W_n^{int} + \Delta W_n^{int,e}. \tag{6.4}$$

Note that for the subcycling procedure, this element internal energy increment is not calculated for every element at every step. It is only calculated for an element when the stress of the element has been updated. At other steps the internal energy in the element is not known. Therefore, the energy balance can only be checked at steps when the stress of every element is updated. At these steps the contributions of all elements are added to the internal energy of the system and the correct value of the internal energy of the system is known. By definition these steps occur only at the master level times described earlier.

The external work done on the system is calculated at every master time step and is given by the equation

$$W_n^{ext} = W_{n-1}^{ext} + \frac{\Delta t_M}{2}(v^{n-1/2})^T (f_n^{ext} - f_{n-1}^{ext}). \tag{6.5}$$

The kinetic energy can only be computed at half steps since the nodal velocities are only calculated at half steps.

$$W^{kin}_{n+1/2} = \frac{1}{2}\left[(v_{n+1/2})^T M v_{n+1/2}\right]. \tag{6.5}$$

The kinetic energy at each step is then found by averaging the half step values.

$$W^{kin}_{n} = \frac{1}{2}(W^{kin}_{n+1/2}+W^{kin}_{n-1/2}). \tag{6.6}$$

VII. Additional Storage Requirements

The addition of subcycling to an existing finite element code requires only four new arrays of storage. The additional storage requirements are as follows:

t_e = a clock for each element

t_I = a clock for each node

Δt_I = nodal time step for each node

Δt_I^{old} = nodal time step for each node for the previous step

Δt_{MLS} = master level time step.

The necessity of these additional storage requirements and a general overview of the procedure can be understood with the help of the flowchart given in figure 5.

VIII. Numerical Examples

To test the stability and accuracy of the proposed subcycling scheme, two example problems have been examined. Problem 1 consists of a bar subjected to a suddenly applied load at one end while the opposite end is fixed to a rigid wall. The bar is modelled with 32 constant strain rod elements which have been divided into four groups. The problem dimensions and material properties are shown in figure 6, while the nodal grouping data is given in table 1. In case 1 the external force applied to the bar is small so that the problem remains completely elastic, while in case 2 plastic deformation takes place due to the larger external force. The elastic problem is solved with the proposed subcycling procedure. Results of the elastic problem are given in figures 7 and 8. Figure 7 gives the time history of the stress at x=5.5 for both the subcycled and non subcycled cases. The exact solution is also included for reference. Figure 8 gives the energy error for both the subcycled and non subcycled cases. Figures 9 and 10 give similar results for the plastic problem. These results indicate that the subcycling procedure performs well for the proposed problem, since the stresses are similar with and without subcycling, and there is minimal energy balance loss for the subcycled case.

The second example problem is a rotating ring subjected to a suddenly applied internal pressure. The problem dimensions and material properties are given in figure 11, and the finite element meshes are given in figures 12 and 14, Key[15]. Figure 13. gives the time history of the maximum stress of a single element for the mesh shown in figure 12. Figure 15 compares the effective plastic strain of an element as a function of time with and without subcycling. Figure 16 gives the effective plastic strains of two different elements in the subcycled case, while the energy error is shown in figure 17. Figures 15 through 17 all refer to the 100 element mesh of figure 14 with the nodal grouping data in table 2. From the figures it can be seen that the subcycled solution retains the symmetry of the problem and appears to be stable since there is no appreciable loss of energy balance.

VIII. Conclusions

The subcycling, or mixed time integration, procedure described here has great flexibility in exploiting the increased stability of domains with larger of more flexible elements. Thus the difficulties associated with the presence of a few stiff elements are avoided. The procedure is easy to program, requiring on the order of 40 additional FORTRAN statements. Furthermore, it requires little additional storage:

2 (NUMEL) + 2 (NUMNP)

where,

NUMEL = number of elements in the mesh

NUMNP = number of nodes in the mesh.

Stability proofs are not available for these subcycling procedures for second-order systems; stability proofs have been developed only for linear first-order systems, Belytschko, Smolinski, and Liu[7], Mizukami[9]. However, the energy stability checks clearly indicate that the procedure developed here is stable and we have not encountered any instabilities.

Because of the great computational savings which can be achieved by this method, it is an attractive feature for any computer program used for engineering analysis by explicit time integration.

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List of figures

Table 1. Nodal grouping and time steps for example problem 1.

Group	Elements	Nodes	Element length	Subcycling time step	No subcycling time step
A	1-10	1-10	1.0	$10\Delta t$	Δt
B	11-15	11-15	0.4	$4\Delta t$	Δt
C	16-25	16-26	0.1	Δt	Δt
D	26-32	27-33	1.0	$10\Delta t$	Δt

Table 2. Nodal grouping and time steps for example problem 2.

Group	Elements	Nodes	Subcycling time step	No subcycling time step
1	1-28	3-54	$12\Delta t$	Δt
2	29-37	55-72	$8\Delta t$	Δt
3	38-48	73-94	$6\Delta t$	Δt
4	49-55	95-108	$4\Delta t$	Δt
5	56-65	109-128	$3\Delta t$	Δt
6	66-78	129-154	$2\Delta t$	Δt
7	79-100	155-200 and 1-2	Δt	Δt

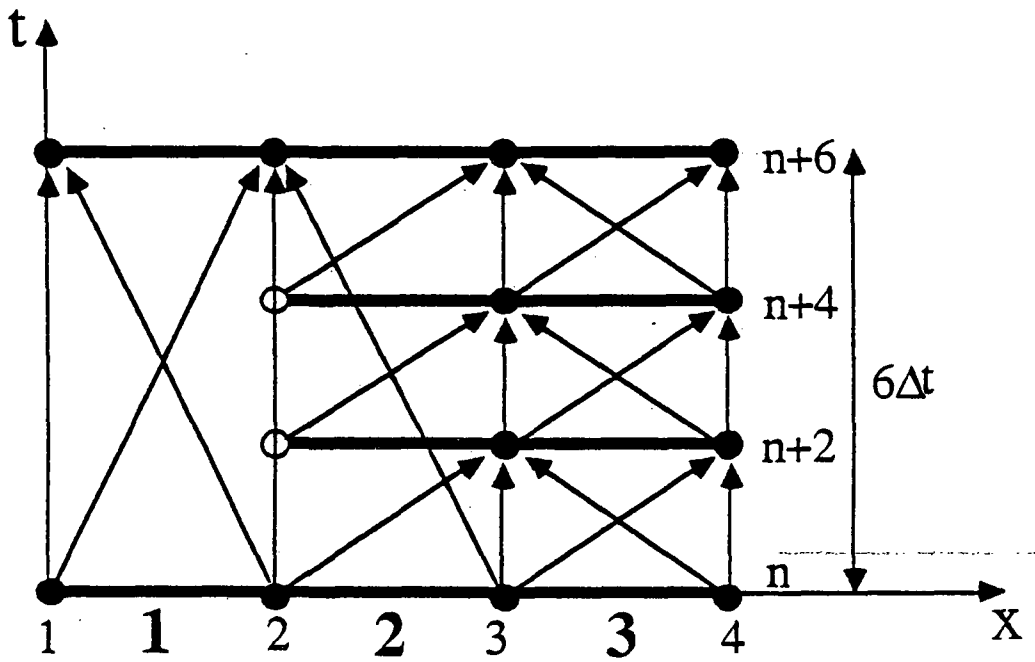


Figure 1

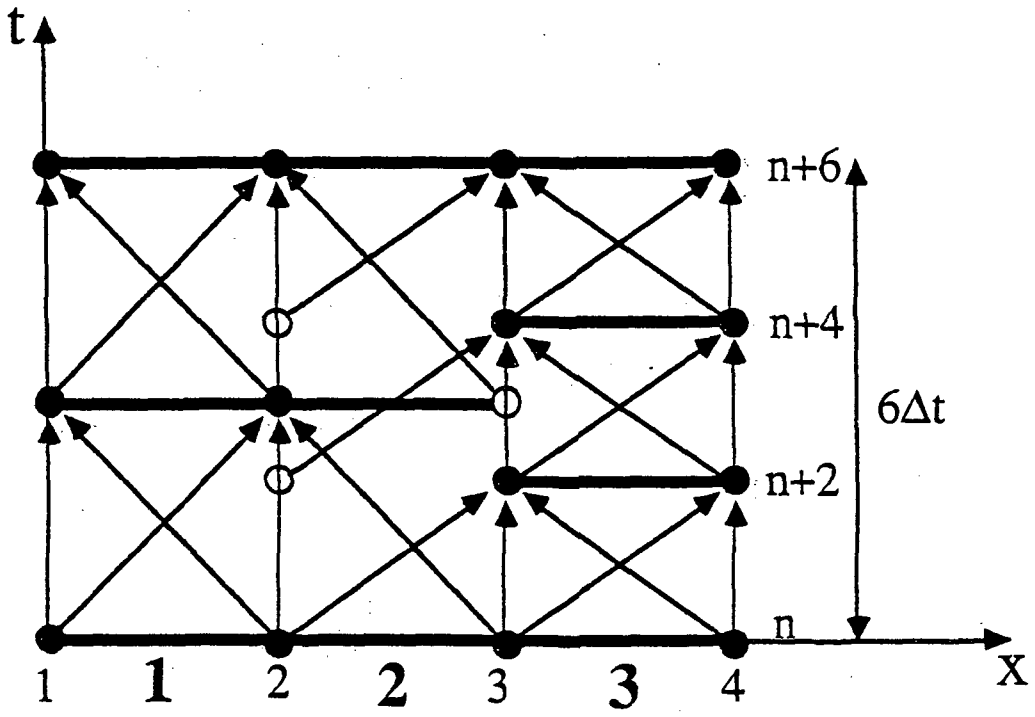


Figure 2

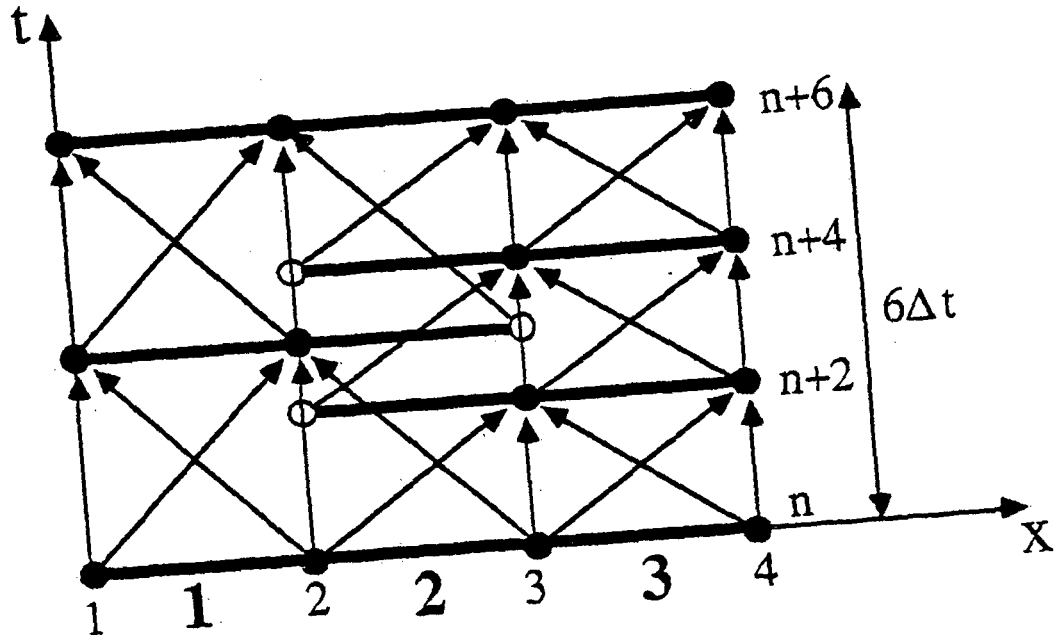
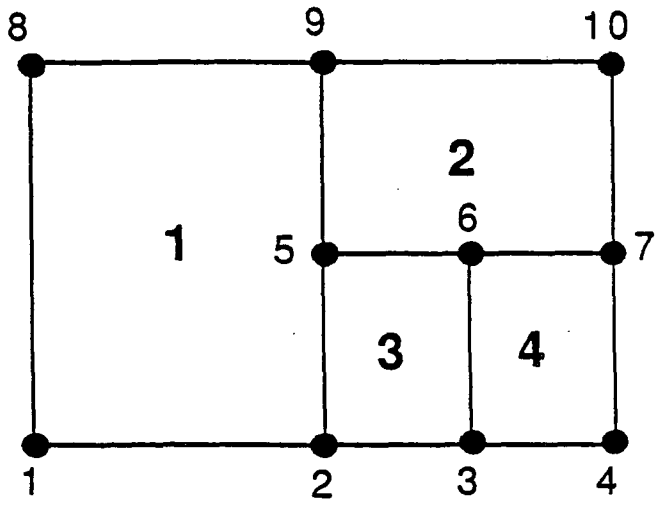


Figure 3



Element time steps	Nodal time steps
$\Delta t_1 = 3.4$	$\Delta t_1, \Delta t_8 = 3.4$
$\Delta t_2 = 1.1$	$\Delta t_2, \Delta t_3, \Delta t_4,$
$\Delta t_3 = 1.1$	$\Delta t_5, \Delta t_6, \Delta t_7 = 1.1$
$\Delta t_4 = 2.3$	$\Delta t_9, \Delta t_{10} = 2.3$

Master time step

$\Delta t = 1.1$

Truncated nodal time steps

$\Delta t_1, \Delta t_8 = 3\Delta t = 3.3$

$\Delta t_2, \Delta t_3, \Delta t_4,$

$\Delta t_5, \Delta t_6, \Delta t_7 = \Delta t = 1.1$

$\Delta t_9, \Delta t_{10} = 2\Delta t = 2.2$

Master level time step

Δt_{MLS} = least common multiple of (1.1,2.2,3.3) = 6.6

Master level time

$t_{MLT} = 6.6$

Figure 4

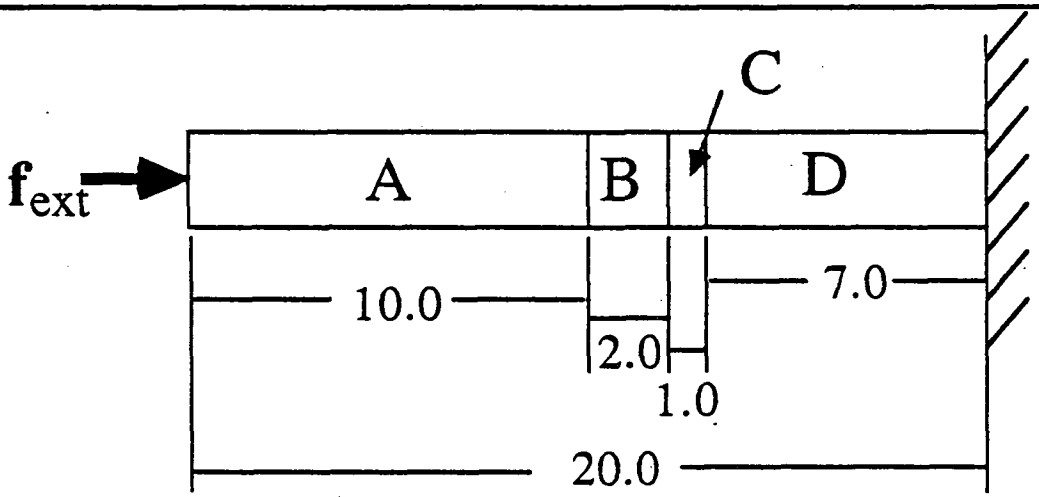
Initialization

1. Read in data and set initial conditions, $t_M = t_e = t_I = 0$
2. Determine the values of Δt_I, Δt_M, and t_{MLS} as described in section VII.A.

Time integration loop

3. Calculate the external nodal forces.
4. Loop on the elements of the mesh (JE)
   If any node of element JE satisfies equation (5.5)
   THEN for element JE
   A. Calculate Δt_M with equation (5.6)
   B. Update the stress by equations (3.4) through (3.6).
   C. Calculate the internal nodal forces and assemble with equation (3.3).
   D. Calculate the increment in internal energy for this element and assemble to the internal energy of the system, equations (6.2) and (6.3).
   E. $\Delta t_e = \Delta t_M$
5. Use equation (1) to calculate the acceleration vector, a_I.
6. Loop on all the nodes of the mesh (I)
   A. If node I satisfies condition (5.9)
      THEN the velocity and displacement of node I are updated by equations (3.7) and (3.8).
   B. $\Delta t_I^{old} = \Delta t_I$
   C. If node I does not satisfy condition (5.9)
      THEN the velocity and displacement of node I are not updated.
7. Calculate the external work done of the system with equation (6.4).
8. Calculate the kinetic energy of the system with equations (6.5) through (6.6)
9. If this is a master level time step, i.e., $t_M = t_{MLT}$
   THEN
   A. Check the energy balance with equation (6.1).
   B. If desired, recalculate the time steps, etc., as described in section VII.A.
10. Output the results.
11. If the maximum time of the problem is not exceeded
    THEN
    A. $t_M = t_M + \Delta t_M$.
    B. GOTO step 3.
12. If the maximum time of the problem is exceeded
    THEN STOP.

Figure 5



Modulus of elasticity = 100.0
Density = 1.0
Plastic modulus = 25.0
Yield point (isotropic hardening) = 1.0

Case	f_{ext}
1	0.2
2	2.0

Figure 6

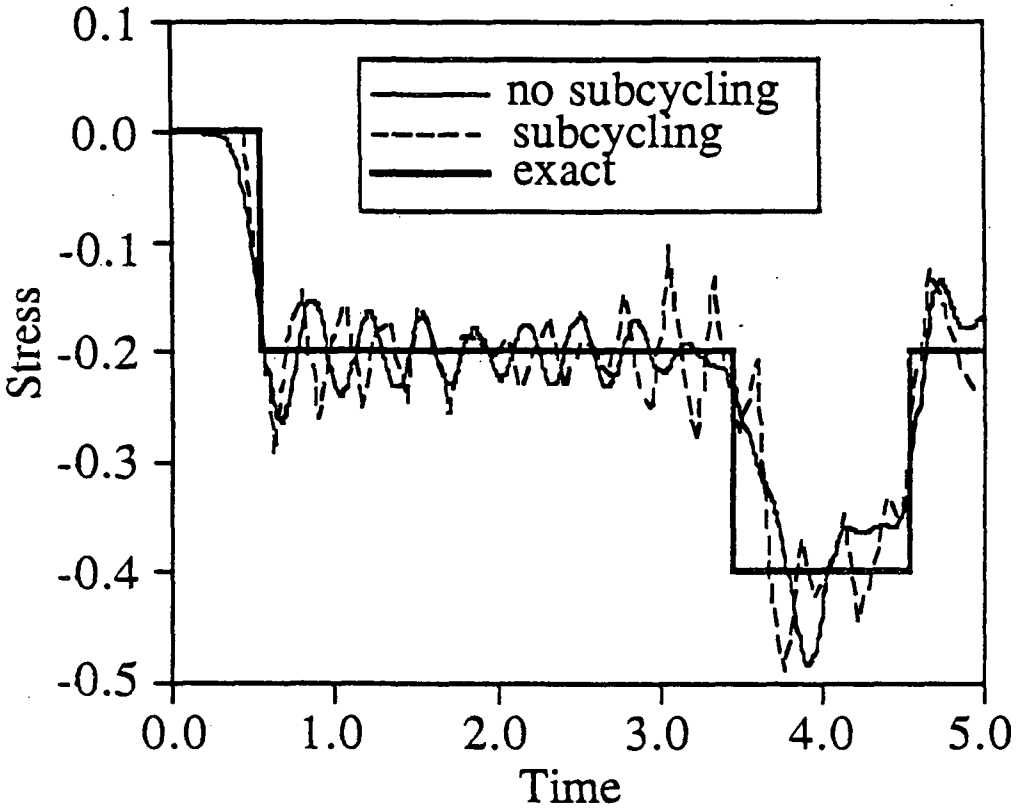

0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
Stress
no subcycling
subcycling
exact
0.0
1.0
2.0
3.0
4.0
5.0
Time

Figure 7

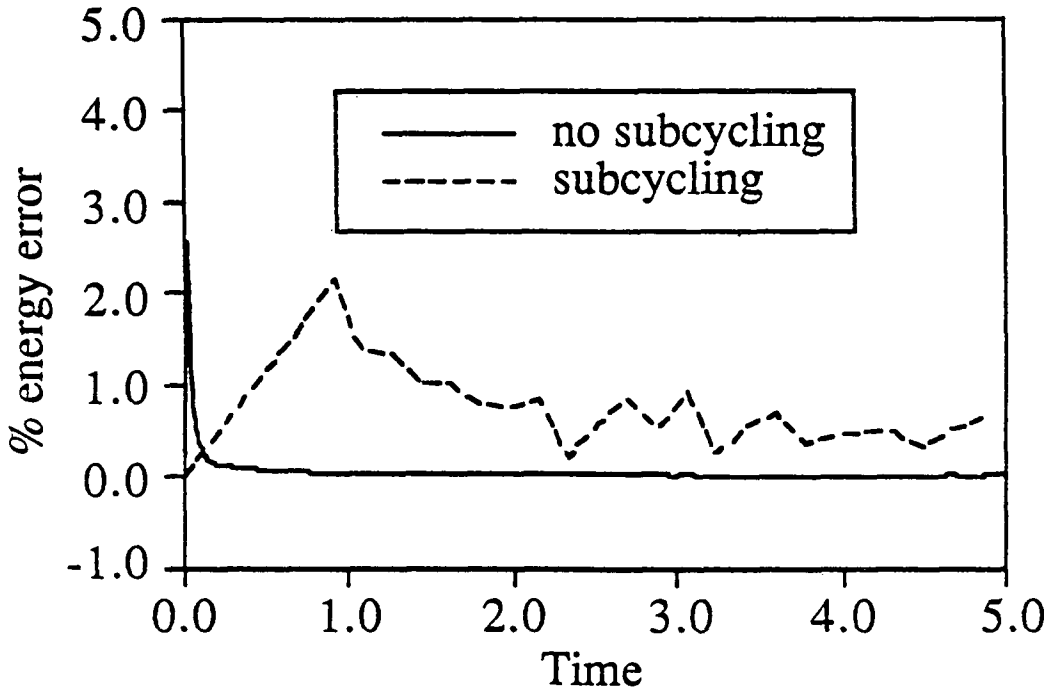

5.0
4.0
3.0
2.0
1.0
0.0
-1.0
% energy error
no subcycling
subcycling
0.0
1.0
2.0
3.0
4.0
5.0
Time

Figure 8

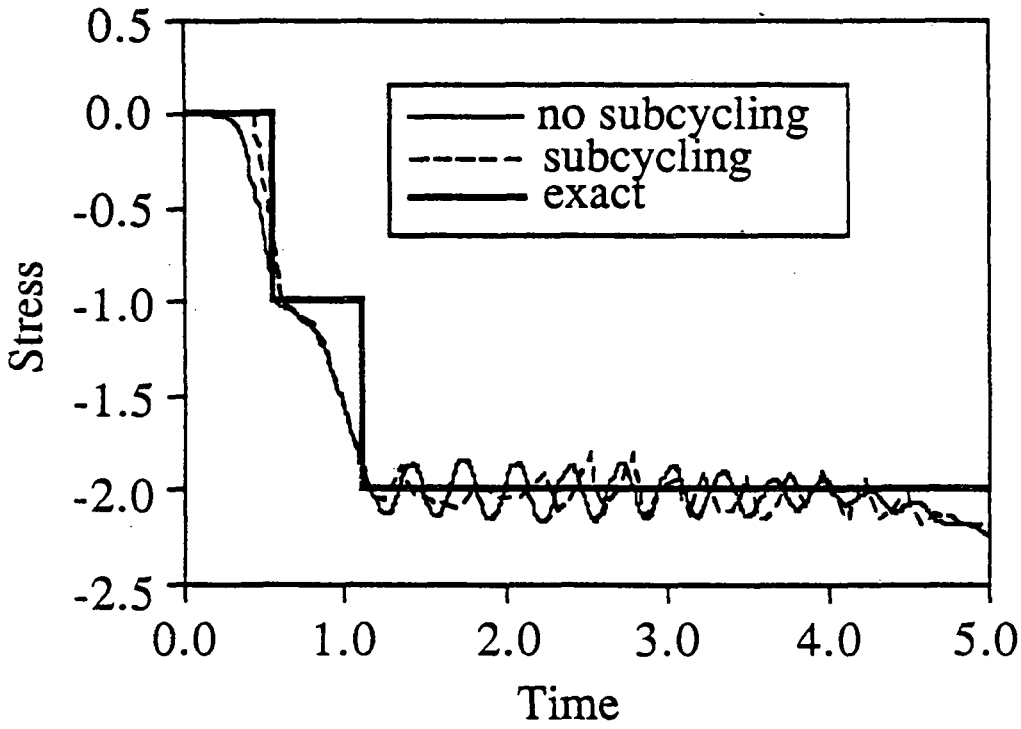

0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
Stress
0.0
1.0
2.0
3.0
4.0
5.0
Time
no subcycling
subcycling
exact

Figure 9

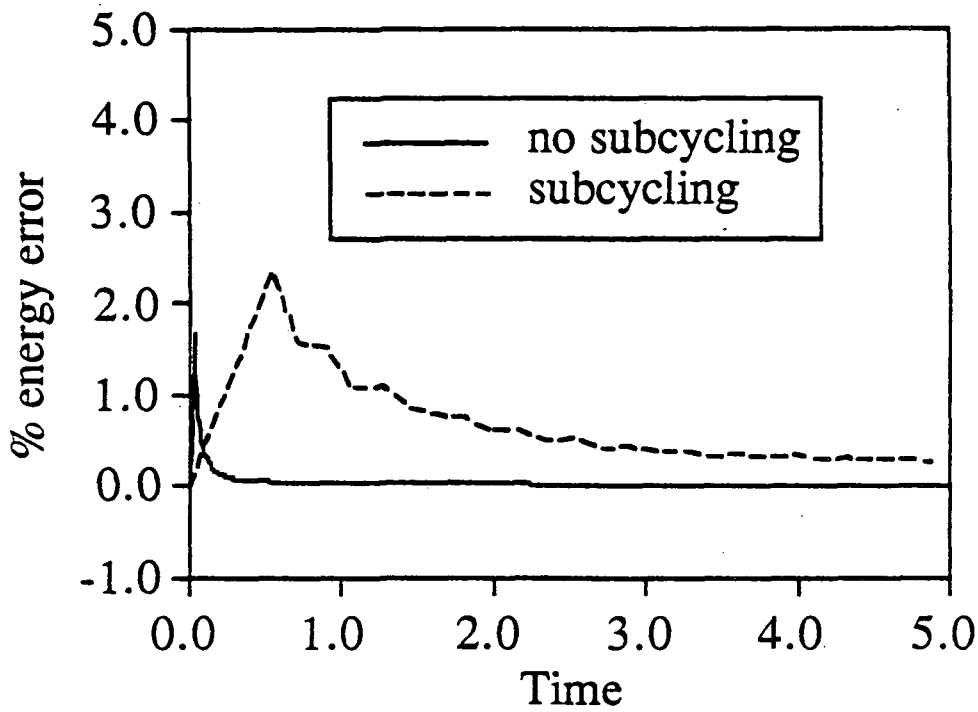
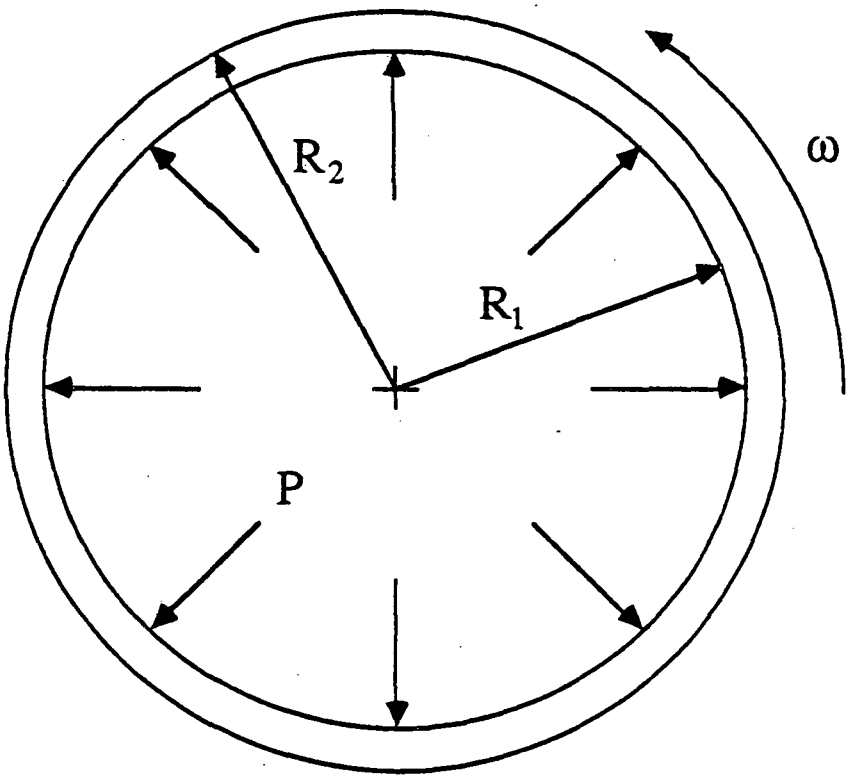

5.0
4.0
3.0
2.0
1.0
0.0
-1.0
% energy error
no subcycling
subcycling
0.0
1.0
2.0
3.0
4.0
5.0
Time

Figure 10



E	= 1.03 e+7 psi	ρ	= 2.508 e-4 lbf sec^2 / in^4
υ	= 0.333	R_1	= 1.05
$E_{tangent}$	= 5.17 e+5 psi	R_2	= 0.95
β	= 1 (isotropic)	ω	= 4000
σ_{yield}	= 4.15 e+4 psi	P	= 9760 psi

Figure 11

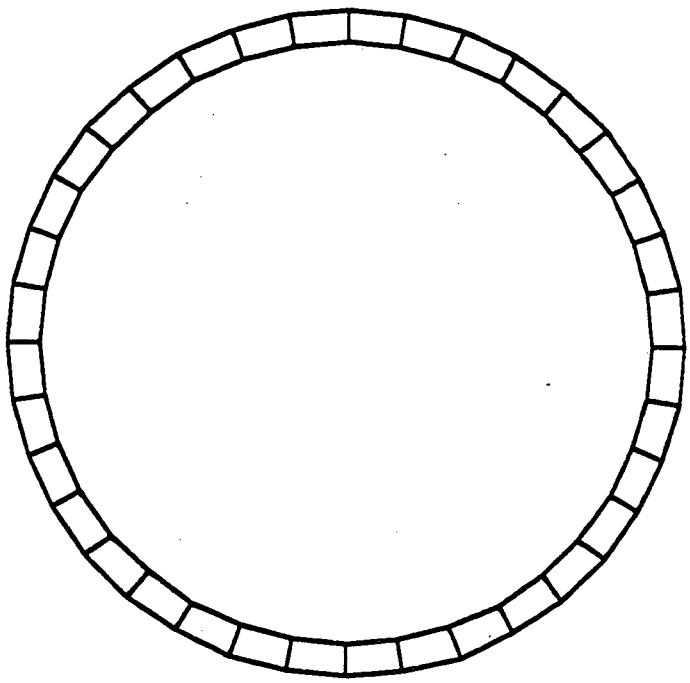


Figure 12

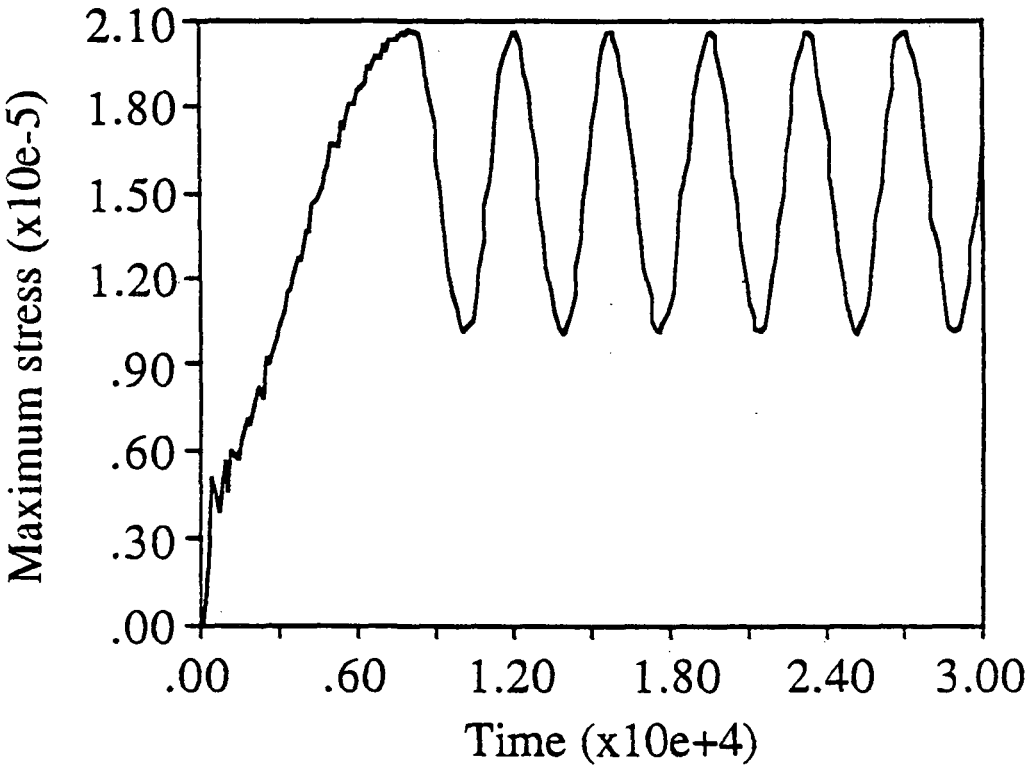

2.10
1.80
1.50
1.20
.90
.60
.30
.00
Maximum stress (x10e-5)
.00
.60
1.20
1.80
2.40
3.00
Time (x10e+4)

Figure 13

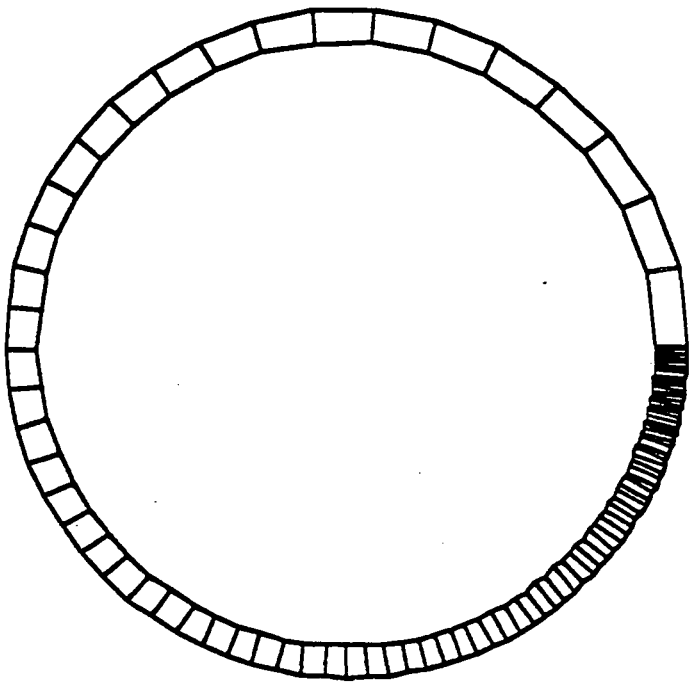


Figure 14

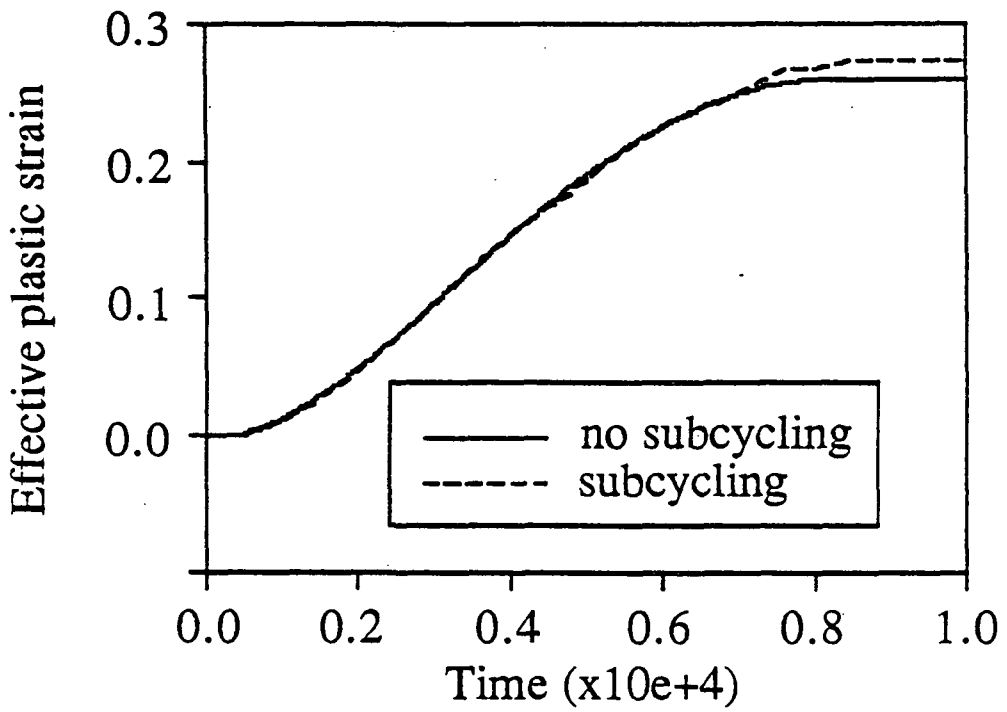

0.3
0.2
0.1
0.0
Effective plastic strain
no subcycling
subcycling
0.0
0.2
0.4
0.6
0.8
1.0
Time (x10e+4)

Figure 15

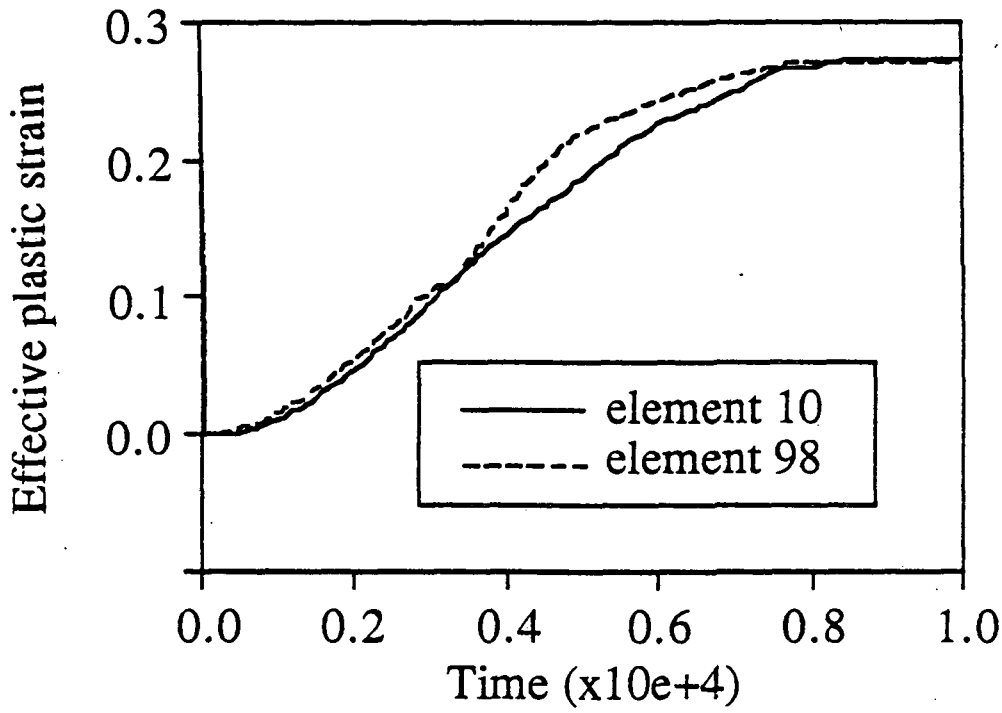

0.3
0.2
0.1
0.0
Effective plastic strain
element 10
element 98
0.0
0.2
0.4
0.6
0.8
1.0
Time (x10e+4)

Figure 16

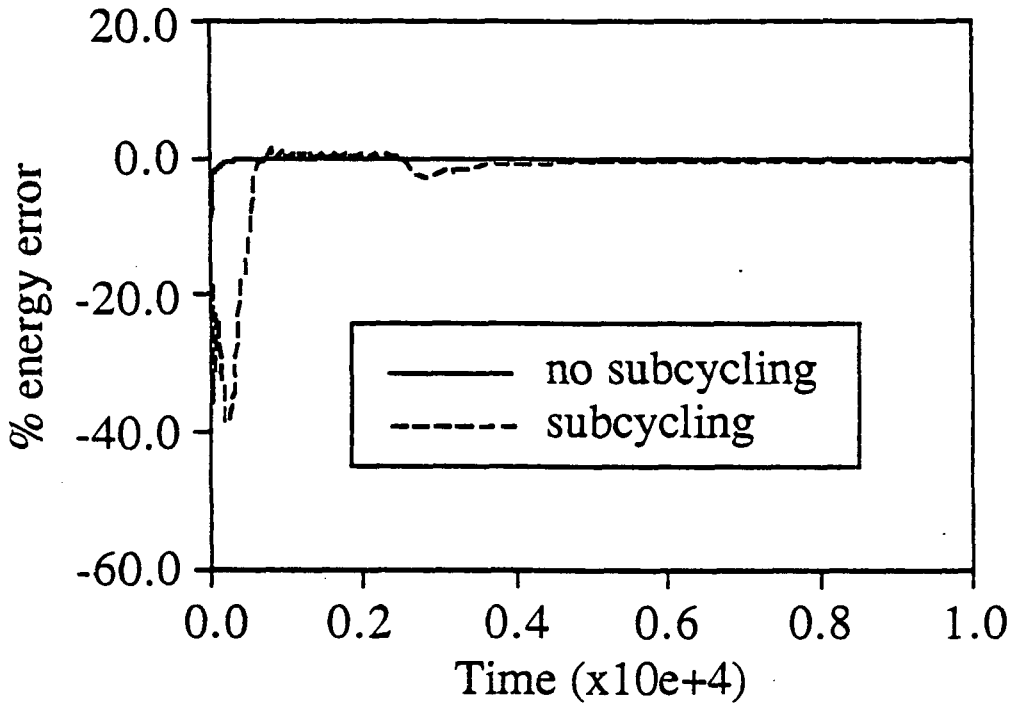

20.0
0.0
-20.0
-40.0
-60.0
% energy error
no subcycling
subcycling
0.0
0.2
0.4
0.6
0.8
1.0
Time (x10e+4)

Figure 17