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Summary

Material characterization parameters obtained from naturally flawed specimens are necessary for reliability evaluation of nondeterministic advanced ceramic structural components. The least squares best fit method is applied to the three parameter uniaxial Weibull model to obtain the material parameters from experimental tests on volume or surface flawed specimens subjected to pure tension, pure bending, four point or three point loading. Several illustrative example problems are provided.

Introduction

The objective of this report is to apply the least squares best fit (LSBF) method to evaluate the parameters used in the uniaxial Weibull three parameter model. These parameters, scale factor σ_o, Weibull modulus m, and threshold (location) parameter σ_u, are material dependent. Weibull two or three parameter models are used to specify a probabilistic distribution for monolithic ceramic materials. The success in the use of the two parameter model rather than the three parameter model depends on the importance of ignoring the threshold (location) parameter. Disregarding the threshold parameter is conservative and simplifies matters. This simplification can be justified only by comparing the predicted behavior of a component with its observed performance.

Equations are developed to obtain the three material parameters from inert volume or surface flawed data. Inert data imply fast fracture (no subcritical crack growth). The inert data are obtained from experimental tests on specimens subjected to either pure tension, pure bending, and four or three point loading (fig. 1). Ideally the data are obtained under conditions representative of the service environment.

Several applications are presented in the section entitled EXPERIMENTAL APPLICATIONS. Experimental data are analyzed for volume flaw failure of silicon nitride (SNW-1000) specimens tested in four point bending (ref. 1). In addition, analysis is made of surface flaw failure data of silicon carbide specimens, annealed in both the longitudinal and transverse direction and tested in three point bending (Private communication from Sung Choi and Jonathan Salem, NASA Lewis Research Center). The four point bend volume flaw data are also used for a four point bend surface flaw analysis to illustrate the application of the developed equations. It is realized that these data are not representative of the physical problem.

Symbols

A	tensile surface area
b	specimen thickness
L	beam length
m	Weibull modulus (shape parameter)
n	number of inert data points
P_f	probability of failure
V	volume in tension
W	specimen depth
x,y,z	Cartesian coordinates
$\delta_{1j},\delta_{2j},\delta_{3j}$	lower limit of integral for j^{th} specimen
$\sigma_{fj}(x,y,z)$	stress distribution in the specimen j at fracture
$\sigma_{fj_{max}}$	maximum principal stress in specimen j at fracture
$\sigma_{fj_{max,comp}}$	computed maximum principal stress in specimen j based on P_{fj} and assumed material parameters
σ_o	scale parameter
σ_u	threshold stress (location parameter)
σ_θ	characteristic strength

Subscripts

assumed	assumed
comp	computed

previous	assumed previous value of assumed
j	specimen designation
T	tensile
v	volume
s	surface

Analysis Based on Three Parameter Uniaxial Weibull Model

The three parameter uniaxial Weibull model is used to describe the material inert strength probabilistic distribution. For both volume and surface flawed specimens, least squares best fit (LSBF) methods are developed to obtain the three material parameters from experimental tests on pure tension, pure bending, and four or three point loaded specimens. The necessary and sufficient condition for a solution is satisfied when the three computed parameters produce the lowest value of the sum of the residuals squared, that is, when $\sum_{j=1}^{n}\left(\sigma_{fj_{max,comp}} - \sigma_{fj_{max}}\right)^2$ = minimum, where n is the number of specimens tested. The $\left(P_{fj}, \sigma_{fj_{max}}\right)$ data points are obtained from the experimental tests where $P_{fj} = (j - 0.3)/(n + 0.4)$. P_{fj} and $\sigma_{fj_{max}}$ are, respectively, the probability of failure and maximum principal tensile stress in the j^{th} specimen at failure. $\sigma_{fj_{max,comp}}$ is the computed maximum failure stress based on the value of P_{fj} and the computed inert strength material parameters.

Pure Tension (Fig. 1(a)), Volume Flaws

$$P_{fj} = 1 - \exp\left[-\int_{V_{Tj}}\left(\frac{\sigma_{fj}(x,y,z) - \sigma_{uv}}{\sigma_{ov}}\right)^{m_v} dV\right] \qquad (1)$$

$$\sigma_{fj}(x,y,z) \geq \sigma_{uv}$$

where V_{Tj} is the volume in tension of the j^{th} specimen with a stress distribution throughout the volume denoted by $\sigma_{fj}(x,y,z)$, σ_{uv} is the threshold stress, σ_{ov} is the scale factor, and m_v is the Weibull modulus. For this case $\sigma_{fj}(x,y,z) = \sigma_{fj_{max}}$ and the tensile gage volume of specimen j is $V_{Tj} = L_2 b_j W_j$. Hence

$$\ln\left[\frac{\ln\left(1 - P_{fj}\right)^{-1}}{V_{Tj}}\right] = m_v \ln\left(\sigma_{fj_{max}} - \sigma_{uv}\right) - m_v \ln \sigma_{ov} \qquad (2)$$

The following system of n linear equations is solved in a LSBF sense:

$$\begin{Bmatrix} \ln\left[\frac{\ln\left(1 - P_{f1}\right)^{-1}}{V_{T1}}\right] \\ \ln\left[\frac{\ln\left(1 - P_{f2}\right)^{-1}}{V_{T2}}\right] \\ \vdots \\ \vdots \\ \vdots \\ \ln\left[\frac{\ln\left(1 - P_{fn}\right)^{-1}}{V_{Tn}}\right] \end{Bmatrix} = \begin{bmatrix} \ln\left(\sigma_{f1_{max}} - \sigma_{uv}\right) & 1 \\ \ln\left(\sigma_{f2_{max}} - \sigma_{uv}\right) & 1 \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ \ln\left(\sigma_{fn_{max}} - \sigma_{uv}\right) & 1 \end{bmatrix} \begin{Bmatrix} m_v \\ -m_v \ln \sigma_{ov} \end{Bmatrix} \qquad (3)$$

In matrix notation {Y} = [A] {X}, where the j^{th} term in the column vector {Y} is $y_j = \ln\left[\frac{\ln\left(1 - P_{fj}\right)^{-1}}{V_{Tj}}\right]$ and vector $\{X\} = \begin{Bmatrix} m_v \\ -m_v \ln \sigma_{ov} \end{Bmatrix}$. The equation that must be satisfied to obtain the LSBF solution is

$$\{X\} = \left[A^T A\right]^{-1}\left[A^T\right]\{Y\} \qquad (4)$$

where superscript T defines the transpose.

The answer is obtained in the following manner: Assume a value for σ_{uv}, and solve for m_v and σ_{ov}. With these values, compute the model failure stresses $\sigma_{fj_{max,comp}}$ at all of the (n) P_{fj} data points, where

$$\sigma_{fj_{max,comp}} = \sigma_{ov}\left[\frac{\ln\left(1 - P_{fj}\right)^{-1}}{V_{Tj}}\right]^{\frac{1}{m_v}} + \sigma_{uv} \qquad (5)$$

Evaluate the sum of the squares of the residuals, where

$$\text{Sum} = \sum_{j=1}^{n}\left(\sigma_{fj_{max,comp}} - \sigma_{fj_{max}}\right)^2 \qquad (6)$$

Repeat the process for another value of σ_{uv}. Compute the new sum of the squares of the residuals (eq. (6)). Continue until the parameters $(m_v, \sigma_{ov}, \sigma_{uv})$ produce the minimum value of the sum of the residuals squared.

Pure Tension (Fig. 1(a)), Surface Flaws

$$P_{fj} = 1 - \exp\left[-\int_{A_{Tj}} \left(\frac{\sigma_{fj}(x,y,z) - \sigma_{us}}{\sigma_{os}}\right)^{m_s} dA\right] \tag{7}$$

$$\sigma_{fj}(x,y,z) \geq \sigma_{us}$$

For pure tension, $\sigma_{fj}(x,y,z) = \sigma_{fj_{max}}$, and the area in tension is $A_{Tj} = 2\,L_2(W_j + b_j)$ where L_2 is the gage length, W is the width, and b is the thickness. Hence,

$$\ln\left[\frac{\ln(1 - P_{fj})^{-1}}{A_{Tj}}\right] = m_s \ln\left(\sigma_{fj_{max}} - \sigma_{os}\right) - m_s \ln \sigma_{os} \tag{8}$$

Equation (8) is the basis of a LSBF evaluation of the Weibull parameters. From the inert data, a set of n linear equations is obtained. In matrix notation $\{Y\} = [A]\,\{X\}$ where the j^{th} term of the column vector $\{Y\}$ is $\ln\,[\ln\,(1 - P_{fj})^{-1}/A_{Tj}]$ and vector $\{X\}$ is $\begin{Bmatrix} m_s \\ -m_s \ln \sigma_{os} \end{Bmatrix}$. The matrix [A] is the same as that in equation (3), except that the subscript v is replaced by s. The solution is obtained by the same method as that for the volume flaw solution. Assume σ_{us}, and solve for m_s and σ_{os}. With these values, compute the failure stresses, $\sigma_{fj_{max,comp}}$ for all n specimens. The computed failure stress for the j^{th} specimen is

$$\sigma_{fj_{max,comp}} = \sigma_{os}\left[\frac{\ln(1 - P_{fj})^{-1}}{A_{Tj}}\right]^{\frac{1}{m_s}} + \sigma_{us} \tag{9}$$

Evaluate equation (6), the sum of the residuals squared. Continue the process for another value of σ_{us}. Compute the sum of the residuals squared. Continue until the parameters $(m_s, \sigma_{os}, \sigma_{us})$ produce the minimum value of the sum by equation (6).

Pure Bending (Fig. 1(b)), Volume Flaws

Substituting the expressions $dV = L_2 b_j\, dy$ and $\sigma_{fj}\,(x,y,z) = 2\,\sigma_{fj_{max}}\, y/W_j$ into equation (1) results in

$$(1 - P_{fj})^{-1} = \exp\left[\left(\frac{1}{\sigma_{ov}}\right)^{m_v} b_j L_2 \int_{\delta_{1j}}^{\frac{W_j}{2}} \left(\frac{2\sigma_{fj_{max}}\, y}{W_j} - \sigma_{uv}\right)^{m_v} dy\right]$$

$$= \exp\left[\left(\frac{1}{\sigma_{ov}}\right)^{m_v} \frac{V_{Tj}}{1 + m_v} \frac{\left(\sigma_{fj_{max}} - \sigma_{uv}\right)^{1+m_v}}{\sigma_{fj_{max}}}\right] \tag{10}$$

where $\delta_{1j} = \sigma_{uv} W_j/(2\,\sigma_{fj_{max}})$ and $V_{Tj} = L_2 b_j W_j/2$. The following expression is derived from equation (10):

$$\ln\left[\frac{\ln(1 - P_{fj})^{-1}}{V_{Tj}}\right] + \ln \sigma_{fj_{max}} = (1 + m_v)\ln\left(\sigma_{fj_{max}} - \sigma_{uv}\right)$$

$$+ \ln\left[\frac{1}{(1 + m_v)\sigma_{ov}^{m_v}}\right] \tag{11}$$

The j^{th} term of the column vector $\{Y\}$ is $\ln\,[\ln\,(1 - P_{fj})^{-1}/V_{Tj}] + \ln\,\sigma_{fj_{max}}$ and the vector $\{X\}$ is $\begin{Bmatrix} (1 + m_v) \\ \ln\left[(1 + m_v)\sigma_{ov}^{m_v}\right]^{-1} \end{Bmatrix}$

The matrix [A] is the same as in equation (3). Assume σ_{uv}, and from equation (11), solve the system of n linear equations in a LSBF sense, where j varies from 1 to n. In matrix notation, $\{Y\} = [A]\,\{X\}$, and the solution to this set of equations is obtained from equation (4). The final solution is the set of parameters associated with the minimum sum of the squares of the residuals defined by equation (6). They are obtained by the following procedure. Assume σ_{uv}, and solve for m_v and σ_{ov}. With these values compute the predicted failure stresses $\sigma_{fj_{max}}$ at all (n) P_{fj} data points. A simple method is to assume $\sigma_{fj_{max,assumed}}$, and solve for $\sigma_{fj_{max,comp}}$ where

$$\sigma_{fj_{max,comp}} = \left[\frac{\sigma_{fj_{max,assumed}}\,\sigma_{ov}^{m_v}\,(1 + m_v)\ln(1 - P_{fj})^{-1}}{V_{Tj}}\right]^{\frac{1}{1+m_v}} + \sigma_{uv} \tag{12}$$

The next assumed value is $\sigma_{fj_{max,assumed}} = 0.5\left[\sigma_{fj_{max,comp}} + \sigma_{fj_{max,previous\ assumed}}\right]$. Repeat this process until $\sigma_{fj_{max,comp}}$ is within some specified tolerance of $\sigma_{fj_{max,assumed}}$. Then compute the sum of the residuals squared by means of equation (6). Repeat all of the previous steps until the minimum value of the sum of the residuals squared is obtained. The parameters $(m_v, \sigma_{ov}, \sigma_{uv})$ associated with this minimum are the solution.

Pure Bending (Fig. 1(b)), Surface Flaws

From equation (1)

$$(1 - P_{fj})^{-1} = \exp\left[\left(\frac{1}{\sigma_{os}}\right)^{m_s} \int_{A_{Tj}} \left(\sigma_{fj}(x,y,z) - \sigma_{us}\right)^{m_s} dA_j\right] \tag{13}$$

where A_{Tj} is the tensile surface area of specimen j. Therefore, considering both the side and bottom surfaces of the specimen yields

$$\left(1-P_{fj}\right)^{-1} = \exp\left[\left(\frac{1}{\sigma_{os}}\right)^{m_s}\left(2L_2\int_{\delta_{1j}}^{\frac{W_j}{2}}\left(\sigma_{fj}(x,y,z)-\sigma_{us}\right)^{m_s} \times dy + b_jL_2\left(\sigma_{fj_{max}}-\sigma_{us}\right)^{m_s}\right)\right] \quad (14)$$

where

$$\sigma_{fj}(x,y,z) = \frac{2y\sigma_{fj_{max}}}{W_j} \quad \text{and} \quad \delta_{1j} = \frac{\sigma_{us}W_j}{2\sigma_{fj_{max}}}$$

Thus

$$\ln\left[\frac{\ln\left(1-P_{fj}\right)^{-1}}{L_2W_j}\right] - \ln\left[\left(1-\frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)+\frac{b_j}{W_j}(1+m_s)\right] = m_s\ln\left(\sigma_{fj_{max}}-\sigma_{us}\right) - \ln\left[\sigma_{os}^{m_s}(1+m_s)\right] \quad (15)$$

Solve in a LSBF sense the set of linear equations obtained by means of equation (15) and denoted in matrix form by $\{Y\} = [A]\{X\}$. The j[th] term of the column vector $\{Y\}$ is $\ln[\ln(1-P_{fj})^{-1}/(L_2W_j)] - \ln[(1-\sigma_{us}/\sigma_{fj_{max}}) + (1+m_s)b_j/W_j]$ and the vector $\{X\}$ is $\left\{\begin{matrix} m_s \\ -\ln\left[\sigma_{os}^{m_s}(1+m_s)\right]\end{matrix}\right\}$. The LSBF parameters are obtained in the following way: Assume σ_{us}, and keep this value fixed. To evaluate the vector $\{Y\}$, assume a value for m_s ($m_{s,assumed}$) based on the two parameter solution. Evaluate the column vector $\{Y\}$ and the matrix $[A]$. Solve for the vector $\{X\}$ by equation (4). Compare the computed value of the Weibull modulus $m_{s,comp}$ with $m_{s,assumed}$. Repeat this process until both values, $m_{s,comp}$ and $m_{s,assumed}$, are within some specified tolerance. With these parameters ($m_s, \sigma_{os}, \sigma_{us}$) and P_{fj}, compute the n values of $\sigma_{fj_{max,comp}}$ where

$$\sigma_{fj_{max,comp}} = \sigma_{os}\left[\frac{\left(1+m_s\right)\left(1-P_{fj}\right)^{-1}}{L_jW_j\left(1-\frac{\sigma_{us}}{\sigma_{fj_{max,assumed}}}\right)+b_jL_j}\right]^{\frac{1}{m_s}} + \sigma_{us} \quad (16)$$

Assume a value for $\sigma_{fj_{max,assumed}}$, and iterate until $\sigma_{fj_{max,comp}}$ is within some specified tolerance of $\sigma_{fj_{max,assumed}}$. Obtain the sum of the residuals squared by equation (6). Repeat the process. The parameters that produce the minimum value of the sum of the residuals squared are the solution.

Four Point Bend Specimen Fig. 1(c)), Volume Flaws

Substituting the inner span and outer span stress distributions $\sigma_{fj}(x,y,z) = 2\sigma_{fj_{max}}y/W_j$ and $\sigma_{fj_{max}}(x,y,z) = 4\sigma_{fj_{max}}Xy/(L_1W_j)$ into equation (1) results in

$$P_{fj} = 1-\exp\left\{-\frac{L_1b_jW_j}{2(1+m_v)}\left(\frac{\sigma_{fj_{max}}}{\sigma_{ov}}\right)^{m_v}\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1+m_v} \times dy + \frac{L_2}{L_1}\left(1-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1-m_v}\right]\right\} \quad (17)$$

where $\delta_{1j} = \sigma_{uv}W_j/(2\sigma_{fj_{max}})$. From equation (17) we obtain

$$\ln\left[\frac{\ln\left(1-P_{fj}\right)^{-1}}{\frac{L_1W_jb_j}{2}}\right] - \ln\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1+m_v}dy + \frac{L_2}{L_1}\left(1-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1+m_v}\right] = m_v\ln\sigma_{fj_{max}} - \ln\left[\sigma_{ov}^{m_v}(1+m_v)\right] \quad (18)$$

Solve the set of linear equations obtained by equation (18), denoted in matrix form by $\{Y\} = [A]\{X\}$. For constant values L_1 and L_2, the j[th] term of column vector $\{Y\}$ is

$$y_j = \ln\left[\frac{\ln\left(1-P_{fj}\right)^{-1}}{\frac{L_1W_jb_j}{2}}\right] - \ln\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1+m_v}dy + \frac{L_2}{L_1} \times \left(1-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1+m_v}\right].$$

The j^{th} row of matrix [A] is $\left[\ln\sigma_{fj_{max}} \quad 1.0\right]$ and {X} = $\left\{\begin{matrix} m_v \\ -\ln\left[\sigma_{ov}^{m_v}(1+m_v)\right]\end{matrix}\right\}$. Assume a value for the threshold stress σ_{uv} and an appropriate value for the Weibull modulus $m_{v,assumed}$ (based on the two parameter solution). Evaluate column vector {Y} and matrix [A]. Solve for solution vector {X} by equation (4). With σ_{uv} fixed, solve for $m_{v,comp}$. Iterate until $m_{v,comp}$ is within a given tolerance of $m_{v,assumed}$. To compute the sum of the squares of the residuals by equation (6), we obtain $\sigma_{fj_{max,comp}}$ from the following equation:

$$\sigma_{fj_{max,comp}} = \frac{\sigma_{ov}}{V_{Tj}^{1/m_v}}\left[\frac{(1+m_v)\ln(1-P_{fj})^{-1}}{\displaystyle\int_{\delta_{1j}}^{W_j/2}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{uv}}{\sigma_{fj_{max,assumed}}}\right)^{1+m_v}dy+\frac{L_2}{L_1}\left(1-\frac{\sigma_{uv}}{\sigma_{fj_{max,assumed}}}\right)^{1+m_v}}\right]^{\frac{1}{m_v}} \tag{19}$$

Assume a value of $\sigma_{fj_{max,assumed}}$, and iterate until $\sigma_{fj_{max,comp}}$ is within some specified tolerance of $\sigma_{fj_{max,assumed}}$. Evaluate the sum of the residuals squared by equation (6). Repeat the process. The parameters that produce the minimum value of the sum of the residuals squared are the solution.

If all failures occur within the inner span and the tensile stress distribution outside the inner span is neglected (L_1 = 0.0), equation (17) becomes the pure bend solution, that is,

$$P_{fj} = 1-\exp\left[-\frac{L_2 b_j W_j}{2(1+m_v)}\left(\frac{\sigma_{fj_{max}}}{\sigma_{ov}}\right)^{m_v}\left(1-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1+m_v}\right]$$

Therefore,

$$\ln\left[\frac{\ln(1-P_{fj})^{-1}}{\frac{L_2 b_j W_j}{2}}\right]+\ln\sigma_{fj_{max}} = (1+m_v)\ln\left(\sigma_{fj_{max}}-\sigma_{uv}\right) - \ln\left[(1+m_v)\sigma_{ov}^{m_v}\right] \tag{20}$$

Equation (20) is the same as equation (11).

Three Point Bend (Fig. 1(d)), Volume Flaws

Substituting $\sigma_{fj_{max}}(x,y,z) = 4\sigma_{fj_{max}}/(L_1 W_j)$ into equation (1) results in

$$\ln(1-P_{fj})^{-1} = 2b_j\int_{\delta_{1j}}^{\frac{W_j}{2}}\int_{\delta_{2j}}^{\frac{L_1}{2}}\left(\frac{\frac{4\sigma_{fj_{max}}xy}{L_1W_j}-\sigma_{uv}}{\sigma_{ov}}\right)^{m_v}dx\,dy \tag{21}$$

where $\delta_{1j} = \sigma_{uv}W_j/(2\sigma_{fj_{max}})$ and $\delta_{2j} = L_1W_j\sigma_{uv}/(4\sigma_{fj_{max}}y)$. With $V_{Tj} = L_1 b_j W_j/2$, integration of equation (21) results in

$$\ln\left[\frac{\ln(1-P_{fj})^{-1}}{V_{Tj}}\right]-\ln\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1+m_v}dy\right] = m_v\ln\sigma_{fj_{max}} - \ln\left[(1+m_v)\sigma_{ov}^{m_v}\right] \tag{22}$$

Equation (22) is the limit case of equation (18) with L_2 = 0.0. Solve the set of linear equations denoted in matrix notation by {Y} = [A] {X} by the LSBF method (eq. (4)). The j^{th} value of column vector {Y} is

$$y_j = \ln\left[\frac{\ln(1-P_{fj})^{-1}}{V_{Tj}}\right]-\ln\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{uv}}{\sigma_{fj_{max}}}\right)^{1+m_v}dy\right]$$

and the vector is $\{X\} = \left\{\begin{matrix} m_v \\ -\ln\left[(1+m_v)\sigma_{ov}^{m_v}\right]\end{matrix}\right\}$. To obtain an initial value of m_v, let σ_{uv} = 0 and solve for the uniaxial Weibull two parameter distribution satisfying equation (4). Starting with this computed value of m_v as $m_{v,assumed}$ and a fixed value of σ_{uv}, evaluate the integral (eq. (22)) in column vector {Y}. The integral is evaluated numerically between the lower limit δ_{1j} and upper limit $W_j/2$. Obtain from solution vector $\{X\}$, $m_{v,comp}$. A solution is obtained when $m_{v,comp}$ is within some specified tolerance of $m_{v,assumed}$. When this does not occur, the next choice for $m_{v,assumed}$ is 0.5 ($m_{v,comp}$ + $m_{v,previous\ assumed}$). Iterate until $m_{v,assumed}$ is within some specified tolerance of $m_{v,comp}$. To compute the sum of the residuals squared by equation (6), we evaluate $\sigma_{fj_{max,comp}}$ in the following way: For the n data values $(P_{fj}, \sigma_{fj_{max}})$ where j = 1, n, assume $\sigma_{fj_{max,assumed}} = \sigma_{fj_{max}}$. The lower integration limit is $\delta_{1j} = \sigma_{uv} W_j/(2\sigma_{fj_{max,assumed}})$. With this limit, solve for $\sigma_{fj_{max,comp}}$.

$$\sigma_{fj_{max,comp}} = \sigma_{ov} V_{Tj}^{-\frac{1}{m_v}} \left[\frac{(1+m_v)\ln(1-P_{fj})^{-1}}{\int_{\delta_{1j}}^{\frac{W_j}{2}} \frac{1}{y}\left(\frac{2y}{W_j} - \frac{\sigma_{uv}}{\sigma_{fj_{max,assumed}}}\right)^{1+m_v} dy} \right]^{\frac{1}{m_v}} \tag{23}$$

Assume a new value for $\sigma_{fj_{max,assumed}} = 0.5$ ($\sigma_{fj_{max,previous\ assumed}}$ + $\sigma_{fj_{max,comp}}$). Repeat the process, integrating over the new limit δ_{1j} until the previous assumed value is within some specified tolerance of the computed new value ($\sigma_{fj_{max,comp}} \simeq \sigma_{fj_{max,assumed}}$). Compute the sum of the residuals squared by equation (6). Repeat the process assuming a new value for σ_{uv}, and continue until the minimum sum of the residuals squared by equation (6) is obtained. When this occurs, the values of m_v, σ_{uv}, and σ_{ov} are the three material parameters.

Four Point Bend Specimen (Fig. 1(c)), Surface Flaws

Substitute into equation (7) the inner span side surface uniaxial tensile stress distribution $\sigma_{fj}(x,y,z) = 2\sigma_{fj_{max}} y/W_j$, the bottom surface tensile stress distribution $\sigma_{fj}(x,y,z) = \sigma_{fj_{max}}$ and the outer span uniaxial tensile surface stress distributions $\sigma_{fj}(x,y,z) = 4\sigma_{fj_{max}} Xy/(L_1 W_j)$ and $\sigma_{fj}(x,y,z) = 2\sigma_{fj_{max}} x/L_1$. Normalizing the area with respect to $L_1 W_j$ results in the following equation:

$$P_{fj} = 1 - \exp\left\{ -\left(\frac{\sigma_{fj_{max}}}{\sigma_{os}}\right)^{m_s} \frac{L_1 W_j}{(1+m_s)} \left[\int_{\delta_{1j}}^{\frac{W_j}{2}} \frac{1}{y}\left(\frac{2y}{W_j} - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} \times dy + \frac{L_2 W_j + L_1 b_j}{L_1 W_j}\left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} + \frac{(1+m_s)L_2 b_j}{L_1 W_j} \times \left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} \right] \right\} \tag{24}$$

where $\delta_{1j} = W_j \sigma_{us}/(2\sigma_{fj_{max}})$. Thus

$$\ln\left[\frac{\ln(1-P_{fj})^{-1}}{L_1 W_j}\right] - \ln\left[\int_{\delta_{1j}}^{\frac{W_j}{2}} \frac{1}{y}\left(\frac{2y}{W_j} - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} dy + \frac{L_2 W_j + L_1 b_j}{L_1 W_j}\left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} + \frac{(1+m_s)L_2 b_j}{L_1 W_j} \times \left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{m_s} \right] = m_s \ln \sigma_{fj_{max}} - \ln\left[\sigma_{os}^{m_s}(1+m_s)\right] \tag{25}$$

Solve the set of linear equations obtained from equation (25) (denoted by $\{Y\} = [A]\{X\}$, in matrix form) by the LSBF method (eq. (4)). The j^{th} value of column vector $\{Y\}$ is

$$y_j = \ln\left[\frac{\ln(1-P_{fj})^{-1}}{L_1 W_j}\right] - \ln\left[\int_{\delta_{1j}}^{\frac{W_j}{2}} \frac{1}{y}\left(\frac{2y}{W_j} - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} dy + \frac{L_2 W_j + L_1 b_j}{L_1 W_j}\left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} + \frac{(1+m_s)L_2 b_j}{L_1 W_j} \times \left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{m_s} \right]$$

and vector$\{X\} = \left\{ \begin{matrix} m_s \\ -\ln\left[\sigma_{os}^{m_s}(1+m_s)\right] \end{matrix} \right\}$. Row j of matrix [A] is $[\sigma_{fj_{max}} \;\; 1.0]$. Assume a value for the threshold stress σ_{us} and an appropriate value for the Weibull modulus, $m_{s,assumed}$ (based on the two parameter solution). Evaluate column vector $\{Y\}$ and matrix [A]. The vector $\{X\}$ is evaluated by equation (4). With σ_{us} fixed, solve for $m_{s,comp}$. When $m_{s,comp}$ is within a given tolerance of $m_{s,assumed}$, a solution results. To compute the sum of the squares of the residuals by equation (6), the values of $\sigma_{fj_{max,comp}}$ are obtained from the following equation:

$$\sigma_{fj_{max,comp}} = \frac{\sigma_{os}}{\left(L_1 W_j\right)^{\frac{1}{m_s}}} \times \left[\frac{(1+m_s)\ln\left(1-P_{fj}\right)^{-1}}{\int_{\delta_{1j}}^{\frac{W_j}{2}} \frac{1}{y}\left(\frac{2y}{W_j} - \frac{\sigma_{us}}{\sigma_{fj_{max,assumed}}}\right)^{1+m_s} dy + \frac{L_2 W_j + L_1 b_j}{L_1 W_j}\left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max,assumed}}}\right)^{1+m_s} + \frac{(1+m_s) L_2 b_j}{L_1 W_j}\left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max,assumed}}}\right)^{m_s}} \right]^{\frac{1}{m_s}} \tag{26}$$

To determine $\sigma_{fj_{max,comp}}$, the process is the same as that outlined for the four point bend, volume flawed specimen. Likewise, the evaluation of the material parameters is the same as that outlined for the four point bend, volume flawed specimen.

Three Point Bend Specimen (Fig. 1(d)), Surface Flaws

From equation (1)

$$\ln\left(1-P_{fj}\right)^{-1} = \left(\frac{1}{\sigma_{os}}\right)^{m_s} \left[4\int_{\delta_{1j}}^{\frac{W_j}{2}} \int_{\delta_{2j}(y)}^{\frac{L_j}{2}} \left(\frac{4\sigma_{fj_{max}} xy}{L_j W_j} - \sigma_{us}\right)^{m_s} dx\, dy + 2b_j \int_{\delta_{3j}}^{\frac{L_j}{2}} \left(\frac{2x\sigma_{fj_{max}}}{L_j} - \sigma_{us}\right)^{m_s} dx \right] \tag{27}$$

with $\delta_{1j} = \sigma_{us} W_j/(2\sigma_{fj_{max}})$. $\delta_{2j}\, y = \sigma_{us} L_j W_j/(4\sigma_{fj_{max}})$, and $\delta_{3j} = \sigma_{us} L_j/2\sigma_{fj_{max}}$. Thus

$$\ln\left[\frac{\ln\left(1-P_{fj}\right)^{-1}}{L_j W_j}\right] - \ln\left[\int_{\delta_{1j}}^{\frac{W_j}{2}} \frac{1}{y}\left(\frac{2y}{W_j} - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} dy + \frac{b_j}{W_j} \times \left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} \right] = m_s \ln \sigma_{fj_{max}} - \ln\left[(1+m_s)\sigma_{os}^{m_s}\right] \tag{28}$$

In matrix form $\{Y\} = [A]\,\{X\}$ and the jth term of column vector $\{Y\}$ is

$$y_j = \ln\left[\frac{\ln\left(1-P_{fj}\right)^{-1}}{L_j W_j}\right] - \ln\left[\int_{\delta_{1j}}^{\frac{W_j}{2}} \frac{1}{y}\left(\frac{2y}{W_j} - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} dy + \frac{b_j}{W_j}\left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s} \right]$$

and $\{X\} = \left\{ \begin{matrix} m_s \\ -\ln\left[\sigma_{os}^{m_s}(1+m_s)\right] \end{matrix} \right\}$. The solution to this set of linear equations denoted symbolically by $\{Y\} = [A]\,\{X\}$ is solved by equation (4). Starting with $\sigma_{us} = 0.0$, solve for m_s and the scale factor σ_{os}. Next, assume a value for σ_{us}. The integrand is a function of m_s. Starting with an assumed value of m_s, iterate until $m_{s,assumed}$ is within some specified limit of

$m_{s,comp}$. Then evaluate the scale factor. To find the value of $\sigma_{fj_{max,comp}}$ associated with the probability of failure P_{fj}, and computed material parameters ($m_s, \sigma_{os}, \sigma_{us}$), satisfy the following condition:

$$\sigma_{fj_{max,comp}}\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{us}}{\sigma_{fj_{max,comp}}}\right)^{1+m_s}dy + \frac{b_j}{W_j}\left(1-\frac{\sigma_{us}}{\sigma_{fj_{max,comp}}}\right)^{1+m_s}\right]^{\frac{1}{m_s}} = \left[\left(\frac{1+m_s}{L_jW_j}\right)\sigma_{os}^{m_s}\ln\left(1-P_{fj}\right)^{-1}\right]^{\frac{1}{m_s}} \quad (29)$$

Assume values for $\sigma_{fj_{max,comp}}$ and iterate until the left side is equal to the right side constant. Compute the sum of the residuals squared via equation (6). Repeat the process assuming a new value for σ_{us}, and continue until the minimum value of the sum of the residuals squared is obtained. When this occurs, the parameters ($m_s, \sigma_{us}, \sigma_{os}$) are the solution.

Three Parameter Specimen Uniaxial Weibull Model

This report deals with material properties that are independent of the component geometry. However, a simple model is often used to obtain the inert strength probabilistic distribution of a given component (refs. 2 and 3). The characteristic strength parameter σ_θ in this model is component dependent and is not a material property. For completeness, this model is briefly mentioned. This model equation for volume flaws is formulated as

$$P_{fj} = 1-\exp\left[-\left(\frac{\sigma_{fj_{max}}-\sigma_{uv}}{\sigma_{\theta v}}\right)^{m_v}\right] \quad (30)$$

For surface flaws, subscript v is replaced by subscript s. Since the characteristic strength is not a material property, this model has its limitations. It is commonly used and is mentioned for completeness.

Experimental Applications

The examples in this section make use of some of the developed equations. Inert failure data are analyzed from the following Modulus of Rupture (MOR) bar data:

(1) Four point bend room temperature failure data of sintered silicon nitride, table I (ref. 1). All failures were due to volume flaws and occurred within the inner span. These data were also used for four point bend surface flaw analysis to illustrate the application of the developed equations.

(2) Three point bend transverse annealed silicon carbide data at 1300 °C , table II (Private communication from Sung Choi and Jonathan Salem, NASA Lewis Research Center). All failures were caused by surface flaws.

(3) Three point bend longitudinal annealed silicon carbide data at 1300 °C, table III (Private communication from Sung Choi and Jonathan Salem, NASA Lewis Research Center). All failures were caused by surface flaws.

To develop confidence in the method developed in this report, comparisons were made with the pure bend results from reference 1. The equations for the four point bend and three point bend specimens were then developed and programmed.

Sintered Silicon Nitride (Pure Bend Analysis, Volume Flaws, Table I)

Monolithic silicon nitride data (SNW-1000, GTE Wesco Division, table I) obtained from reference 1 are used to compare the results of various LSBF techniques. All of the data in table I contain failures that occurred within the inner span. In reference 1, pure bend loading (fig. 1(b)) was therefore assumed applicable to these data. The three material parameters were computed using Cooper's method (ref. 4), a modified LSBF approach (ref. 5), and the method developed herein. Table IV summarizes the results of the three techniques used in the analysis of these data and the results obtained herein of the two parameter model (σ_{uv} = 0). Figure 2 is a plot of the data points and the cumulative Weibull two parameter distribution curve. Figure 3 is a plot of the data points and the cumulative three parameter Weibull distribution curve.

Silicon Carbide (Three Point Bend Surface Flaws, Tables II and III)

Table V summarizes the results obtained for the two and three parameter uniaxial Weibull models. The cumulative distribution curves for three point bend data (Private communication from Sung Choi and Jonathan Salem, NASA Lewis Research Center) for transverse and longitudinal annealed silicon carbide and the data points are plotted in figures 4 to 7. The two parameter distribution curves are plotted in figures 4 and 6. The three parameter distribution curves are plotted in figures 5 and 7.

Sintered Silicon Nitride (Four Point Bend Analysis, Volume Flaws, Table I)

Table VI summarizes the results for the two and three parameter uniaxial Weibull models. Figure 8 is a plot of the two parameter cumulative distribution curve and data points. Figure 9 is a plot of the three parameter cumulative distribution curve and data points.

Sintered Silicon Nitride (Four Point Bend Analysis, Surface Flaws, Table I)

The four point bend volume flaw data are also used for a four point bend surface flaw analysis to illustrate the application of the developed equations. It is realized that these data are not representative of the physical problem. Table VII summarizes the results for the two and three parameter uniaxial Weibull models applied to these data. Figure 10 is a plot of the two parameter cumulative distribution curve and data points. Figure 11 is a plot of the three parameter cumulative distribution curve and data points.

Discussion and Conclusions

Solutions are obtained from inert failure data based on the minimizing of the sum of the residuals squared as a necessary and sufficient condition. There are programs to evaluate the three parameters fitted to the specimen uniaxial Weibull model (eq. (30), refs. 2 and 3). The characteristic strength, $\sigma_{\theta v}$, a parameter in this model, is not a material property but component dependent. The results obtained using this model are only applicable to that specific component made from the same test material.

In this report, material property parameter estimation methods are developed based on the uniaxial Weibull model (eq. (1)). The parameters so obtained are applicable to any component made from the same test data material. For the sintered silicon nitride four point bend inert volume flaw failure data (table I), Cooper's method (ref. 4), a modified LSBF method (refs. 1 and 3), and the approach developed herein were used to minimize the sum of the squares of the residuals based on the pure bend solution (all failures occurred within the inner span). Comparing the results reveals that the largest variation of the sum of the squares of the residuals (table IV) was less than 3 percent. Further comparison of the results of the three methods indicated the approach used herein was slightly less conservative in the low probability of failure regions ($P_f < 0.05$) and slightly more conservative in the upper region ($P_f > 0.25$). Figures 2 and 3 are plots of the data points and the computed cumulative distribution curves for the two and three parameter uniaxial Weibull models based on the pure bend solution.

The results for the silicon carbide three point bend surface flaw data (tables II and III) from longitudinal and transverse annealed specimens are summarized in table V. Comparing the two and three parameter models reveals that there are large differences in the Weibull modulii and scale factors. The cumulative distribution curves are plotted in figures 4 to 7. Superimposing the two and three parameter curves reveals small but significant differences because most designs are based on the very low probability of failure region.

The four point bend solutions to the two and three parameter uniaxial Weibull models applied to the data in table I are summarized in table VI. The cumulative distribution curves are plotted in figures 8 and 9. Figures 2 and 3 are the cumulative distribution curves for the pure bend solution. A comparison of figure 2 with figure 8 and figure 3 with figure 9 indicates the four point bend results for both cases are slightly more conservative in the lower probability of failure region and slightly less conservative in the higher probability of failure region.

Four point bend volume flaw data in table I are used for a four point bend, surface flaw analysis to illustrate the application of the developed equations. It is realized that these data are not representative of the physical problem. The results are plotted in figures 10 and 11 and summarized in table VII.

Justification for applying the three parameter model rather than the two parameter model will depend on which model better predicts the behavior of a component with its observed performance.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, March 1996

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TABLE I.—FOUR POINT BEND SILICON NITRIDE VOLUME FLAW INERT FAILURE DATA (fig. 1(a))
[L_1 = 20.8 mm, L_2 = 19.6 mm, b = 4.0 mm, W = 3.1 mm.]

Specimen number	Failure strength, MPa
1	613.9
2	623.4
3	639.3
4	642.1
5	653.8
6	662.4
7	669.5
8	672.8
9	681.3
10	682.0
11	699.0
12	714.5
13	717.4
14	725.5
15	741.6
16	744.9
17	751.0
18	761.7
19	763.9
20	774.2
21	791.6
22	795.2
23	829.8
24	838.4
25	856.4
26	868.3
27	882.9

TABLE II.—THREE POINT BEND SILICON CARBIDE SURFACE FLAW TRANSVERSE ANNEALED INERT FAILURE DATA (fig. 1(c))
[Span = L_1 = 19.936 mm.]

Specimen number	Thickness, b_j, mm	Depth, W_j, mm	Failure load, kg
1	2.991	1.873	14.25
2	2.999	1.875	15.00
3	2.999	1.873	16.20
4	2.999	1.877	14.85
5	3.000	1.874	15.08
6	2.995	1.871	13.13
7	2.996	1.875	14.18
8	2.998	1.874	15.00
9	2.997	1.879	14.78
10	2.999	1.875	11.63
11	2.997	1.874	12.15
12	2.998	1.877	14.33
13	2.998	1.877	14.33
14	2.999	1.879	13.23
15	2.995	1.876	12.83
16	2.994	1.874	15.75
17	2.998	1.881	16.23
18	2.997	1.876	12.83
19	2.994	1.877	12.75
20	2.997	1.876	13.05
21	2.999	1.879	16.05
22	3.001	1.875	14.85
23	3.000	1.877	16.23
24	3.000	1.875	13.20
25	2.993	1.871	17.63
26	2.993	1.879	12.30
27	2.995	1.879	16.05
28	2.996	1.876	15.08
29	2.994	1.877	13.05
30	2.996	1.877	10.05
31	2.996	1.872	12.75
32	2.995	1.874	17.70
33	2.996	1.876	15.30
34	2.994	1.878	11.55

TABLE III.—THREE POINT BEND SILICON CARBIDE SURFACE FLAW LONGITUDINAL ANNEALED INERT FAILURE DATA (fig. 1(c))
[Span, L_1 = 19.936 mm.]

Specimen number	Thickness, b_j, mm	Depth, W_j, mm	Failure load, kg
1	2.999	1.866	12.98
2	2.998	1.866	14.55
3	3.001	1.867	17.18
4	2.998	1.868	15.53
5	2.998	1.869	14.10
6	3.000	1.863	15.68
7	3.000	1.871	15.98
8	3.003	1.870	16.95
9	3.001	1.863	17.25
10	3.002	1.863	12.00
11	2.991	1.866	15.75
12	2.994	1.864	15.23
13	2.992	1.866	14.78
14	2.993	1.866	14.25
15	2.995	1.868	14.93
16	2.996	1.869	12.15
17	2.996	1.870	15.38
18	2.996	1.871	14.78
19	2.997	1.871	14.70
20	2.997	1.872	12.90
21	3.002	1.873	9.98
22	3.001	1.874	10.95
23	3.000	1.867	15.53
24	3.002	1.871	14.40
25	3.003	1.864	13.50
26	3.000	1.864	12.53
27	3.000	1.864	12.90
28	3.000	1.865	13.95
29	3.002	1.864	14.70
30	3.002	1.863	12.45
31	2.992	1.859	12.75
32	2.993	1.868	13.95
33	2.992	1.868	15.53
34	2.994	1.869	17.55
35	2.975	1.870	12.08

TABLE IV.—WEIBULL PARAMETERS OBTAINED FROM SILICON NITRIDE (SNW-1000) FOUR POINT BEND VOLUME FLAW INERT FAILURE DATA (TABLE I)

[Data in table I are analyzed as a pure bend solution over the inner span.]

The probability fo failure for a given value of $\sigma_{fj_{max}}$ is defined as $P_{fj} = 1.0 - \exp\left[-\frac{1}{\sigma_{ov}^{m_v}}\left(\frac{V_{Tj}}{1+m_v}\right)\frac{\left(\sigma_{fj_{max}} - \sigma_{uv}\right)^{1+m_v}}{\sigma_{fj_{max}}}\right]$

Pure bend solution (volume flaws, fig. 1(b))	Weibull modulus, m_v	Scale factor, σ_{ov}, MPa-m $^{3/m_v}$	Threshold stress, σ_{uv}, MPa	Sum of residuals squared, $\sum_{j=1}^{27}\left(\sigma_{f_{computed}} - \sigma_{f_{data}}\right)_j^2$
Starlinger, et al. and Cooper - LSBF (ref. 1)	1.625	0.00258276	560.84	2684.4
Starlinger, et al. - Modified LSBF (ref. 1)	1.677	0.00370464	558.08	2664.0
LSBF[a]	1.608	0.00218469	565.195	2743.7
Two parameter LSBF method[b]	11.306	150.1733	0.0	13440

[a]LSBF applied to eq. (11).

[b]σ_{uv} set equal to zero in eq. (11).

TABLE V.—WEIBULL PARAMETERS OBTAINED FROM THREE POINT BEND SILICON CARBIDE SURFACE FLAW INERT FAILURE DATA (TABLES II AND III)

The probability of failure for a given value of $\sigma_{fj_{max}}$ defined as

$$P_{fj} = 1.0 - \exp\left\{-\left(\frac{\sigma_{fj_{max}}}{\sigma_{os}}\right)^{m_s}\frac{L_j W_j}{\left(1+m_s\right)}\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j} - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s}dy + \frac{b_j}{W_j}\left(1 - \frac{\sigma_{us}}{\sigma_{fj_{max}}}\right)^{1+m_s}\right]\right\}$$

where $\delta_{1j} = \frac{\sigma_{us}W_j}{2\sigma_{fj_{max}}}$

LSBF best method (surface flaws, fig. 1(d))	Weibull modulus, m_s	Scale factor, σ_{os}, MPa - m $^{2/m_s}$	Threshold stress, σ_{us}, MPa	Sum of residuals squared, $\sum_{j=1}^{n}\left(\sigma_{f_{computed}} - \sigma_{f_{data}}\right)_j^2$
Transverse-annealed two-parameter model	9.294	114.52	0.0	2665
Transverse-annealed three-parameter model	4.024	11.71	190.0	1942
Longitudinal-annealed two-parameter model	9.161	114.14	0.0	1417
Longitudinal-annealed three-parameter model	5.893	39.78	120.0	1259

TABLE VI. —WEIBULL PARAMETERS OBTAINED FROM SILICON NITRIDE (SNW-1000) FOUR POINT BEND VOLUME FLAW INERT FAILURE DATA (TABLE I)

The probability of failure for a given value of $\sigma_{fj_{max}}$ is defined as

$$P_{fj}=1-\exp-\left\{\left(\frac{\sigma_{fj\max}}{\sigma_{ov}}\right)^{m_v}\frac{L_1 b_j W_j}{2(1+m_v)}\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{uv}}{\sigma_{fj\max}}\right)^{1+m_v}dy+\frac{L_2}{L_1}\left(1-\frac{\sigma_{uv}}{\sigma_{fj\max}}\right)^{1+m_v}\right]\right\}$$

where $\delta_{1j}=\dfrac{\sigma_{uv}W_j}{2\sigma_{fj\max}}$

LSBF method (volume flaws, fig. 1(c))	Weibull modulus, m_v	Scale factor, σ_{ov}, MPa - m $^{3/m_v}$	Threshold stress, σ_{uv}, MPa	Sum of residuals squared, $\sum_{j=1}^{27}\left(\sigma_{fj\max,comp}-\sigma_{fj\max}\right)_j^2$
Weibull two parameter model	10.841	141.713	0.0	11712.5
Weibull three parameter model	1.443	0.0006804	564.0	2038.5

TABLE VII.—WEIBULL PARAMETERS OBTAINED FROM SILICON NITRIDE (SNW-1000) FOUR POINT BEND VOLUME FLAW INERT FAILURE DATA

[For illustrative purposes the data in table I are analyzed as surface flaw inert failure data.]

The probability of failure for a given value of $\sigma_{fj_{max}}$ is defined as

$$P_{fj}=1-\exp\left\{-\left(\frac{\sigma_{fj\max}}{\sigma_{os}}\right)^{m_s}\frac{L_1 W_j}{(1+m_s)}\left[\int_{\delta_{1j}}^{\frac{W_j}{2}}\frac{1}{y}\left(\frac{2y}{W_j}-\frac{\sigma_{us}}{\sigma_{fj\max}}\right)^{1+m_s}dy+\frac{L_2W_j+L_1b_j}{L_1W_j}\left(1-\frac{\sigma_{us}}{\sigma_{fj\max}}\right)^{1+m_s}\right.\right.$$
$$\left.\left.+\left(1+m_s\right)\frac{L_2b_j}{L_1W_j}\left(1-\frac{\sigma_{us}}{\sigma_{fj\max}}\right)^{m_s}\right]\right\}$$

where $\delta_{1j}=\dfrac{\sigma_{us}W_j}{2\sigma_{fj\max}}$

LSBF method (surface flaws, fig. 1(c))	Weibull modulus, m_s	Scale factor, σ_{os}, MPa - m $^{2/m_s}$	Threshold stress, σ_{us}, MPa	Sum of residuals squared, $\sum_{j=1}^{27}\left(\sigma_{fj_{max,comp}}-\sigma_{fj_{max}}\right)_j^2$
Weibull two parameter model	10.84	325.23	0.0	12067.0
Weibull three parameter model	2.00	1.69	575.0	1928.3

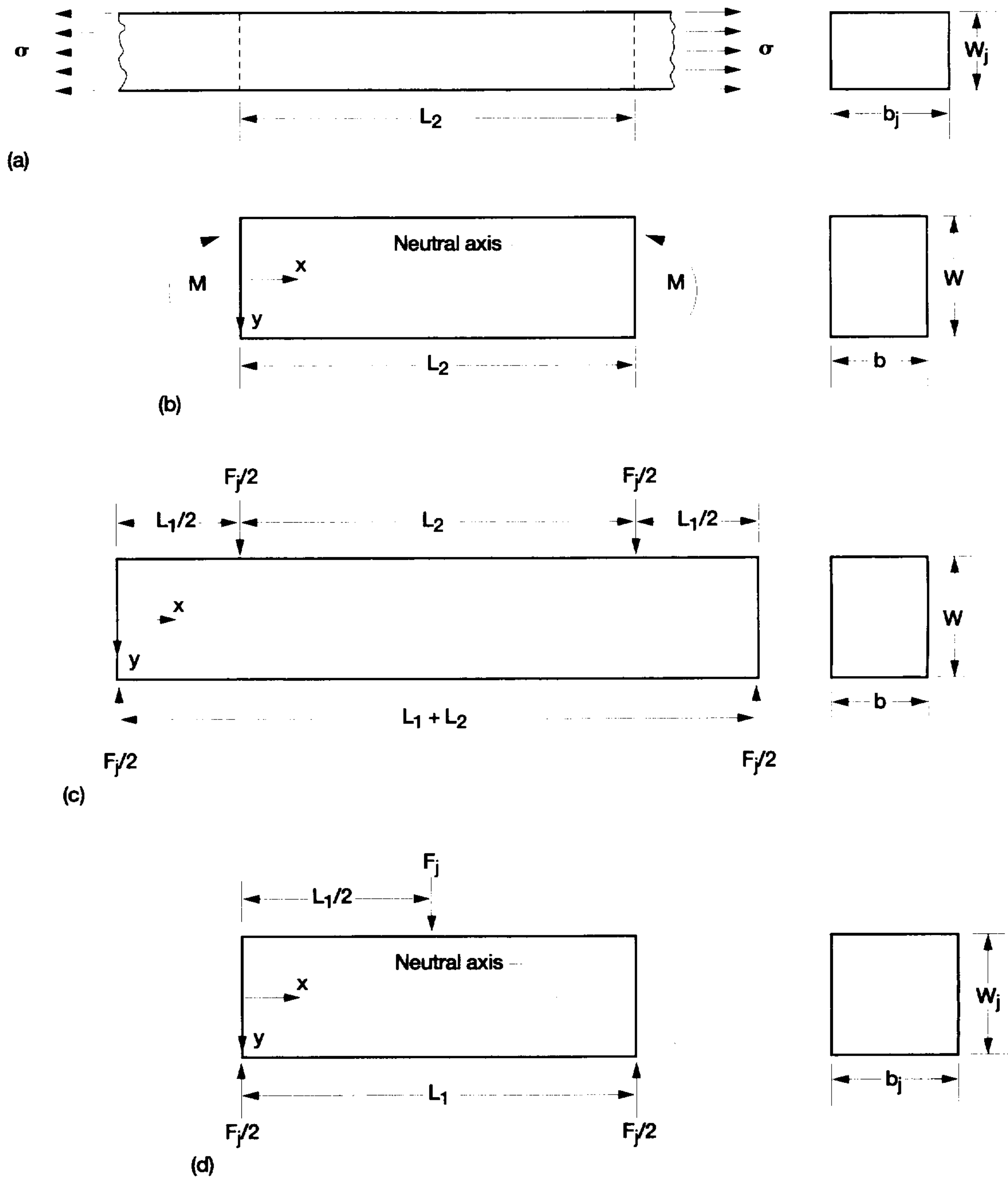

Figure 1.—Specimen loading and geometry. (a) Pure tension. (b) Pure bend silicon nitride (SNW-1000) specimen; L_2 = 19.6 mm, b = 4.0 mm, W = 3.1 mm. (c) Four point bend silicon nitride (SNW-1000) specimen; L_1 = 20.8 mm, L_2 = 19.6 mm, b = 4.0 mm, W = 3.1 mm (table I). (d) Three point bend silicon carbide specimen; L_1 = 19.936 mm (tables II and III contain F_j, b_j, and W_j values).

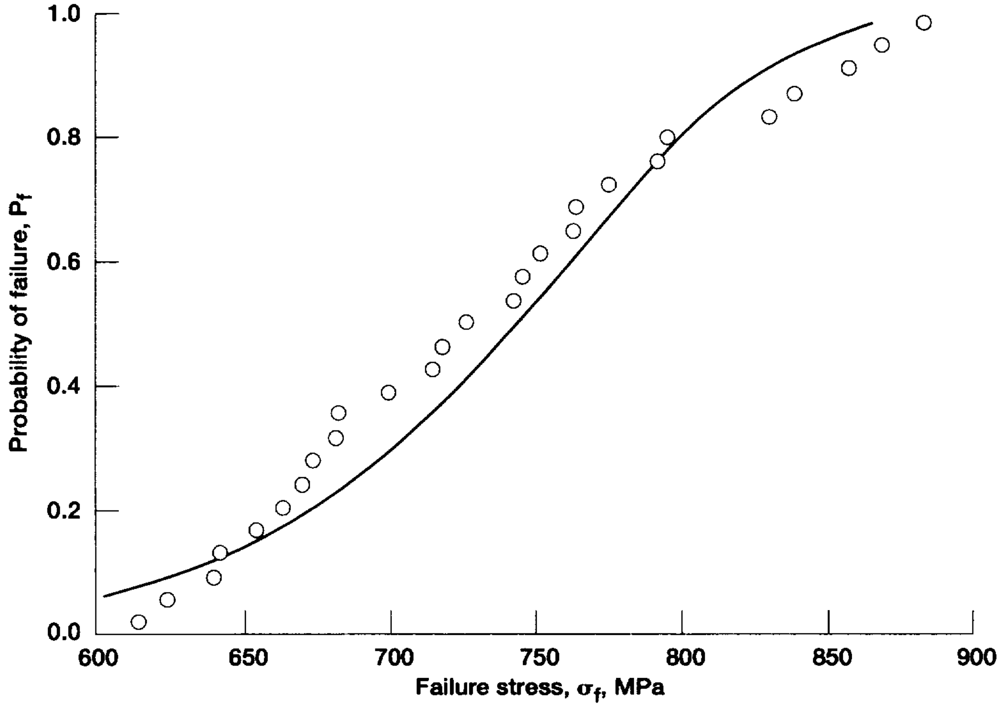

Figure 2.—Distribution curve for two parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to pure bend analysis over inner span (Fig. 1(b), table I). Weibull modulus m_V = 11.306; scale factor σ_{oV} = 150.173 MPa–m^{3/m_V}; threshold stress σ_{uV} = 0.0 MPa; sum of residuals squared = 13440.

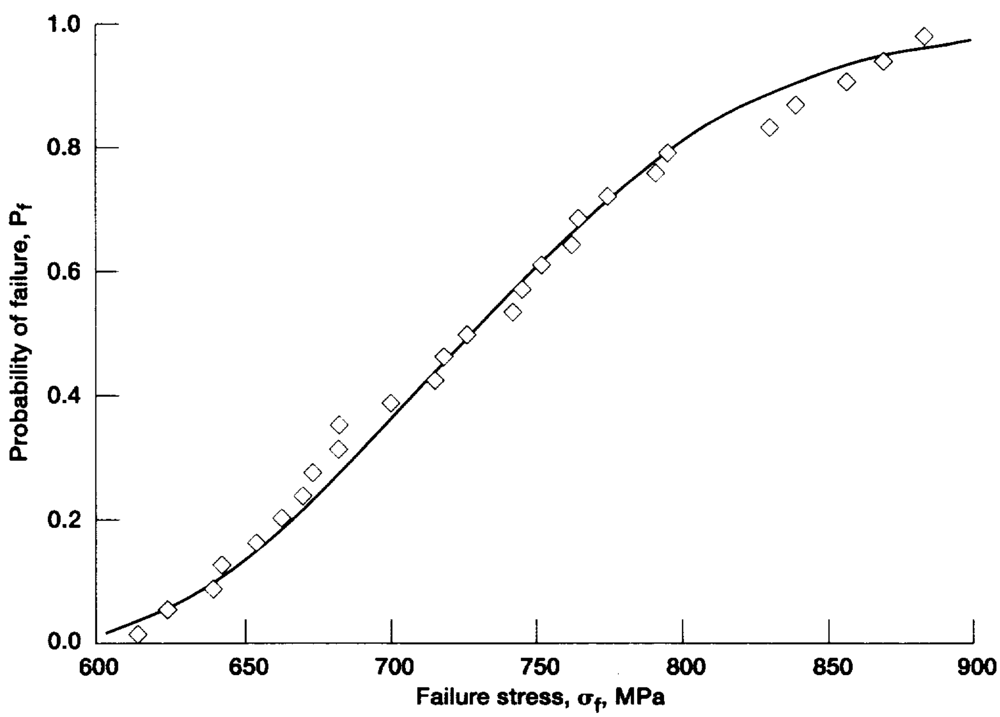

Figure 3.—Distribution curve for three parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to pure bend analysis over inner span (Fig. 1(b), table I). Weibull modulus m_V = 1.608; scale factor σ_{oV} = 0.0021847 MPa–m^{3/m_V}; threshold stress σ_{uV} = 565.2 MPa; sum of residuals squared = 2744.

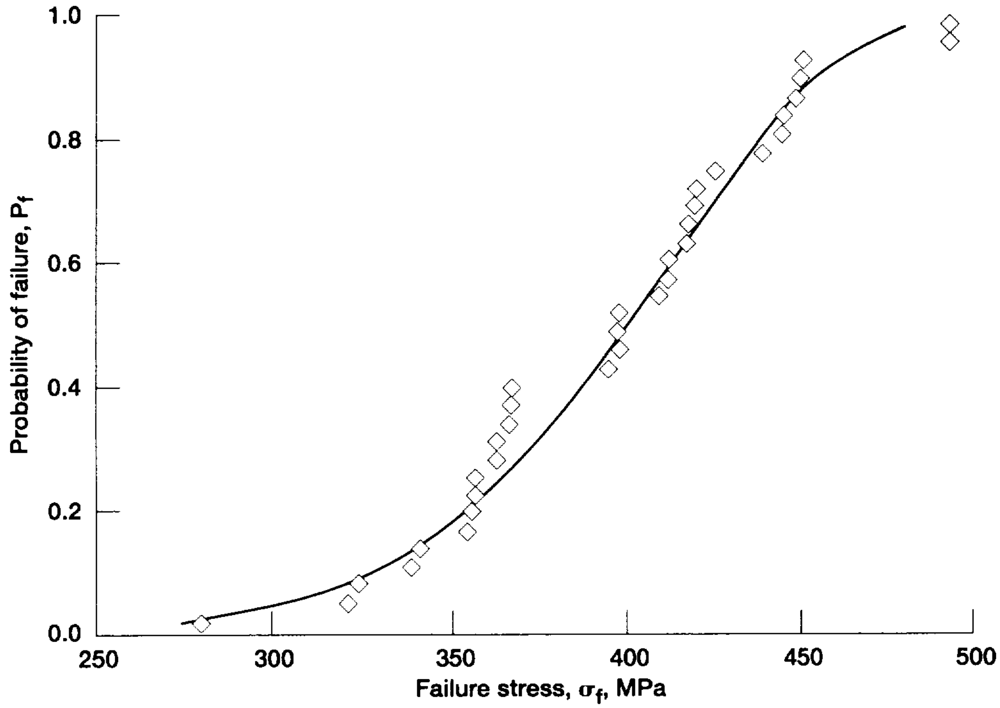

Figure 4.—Distribution curve for two parameter Weibull model from surface flawed inert transverse annealed silicon carbide data via specimens subjected to three point bend analysis (Fig. 1(d), table II). Weibull modulus m_s = 9.294; scale factor σ_{os} = 114.5 MPa–m^{2/m_s}; threshold stress σ_{us} = 0.0 MPa; sum of residuals squared = 2665.

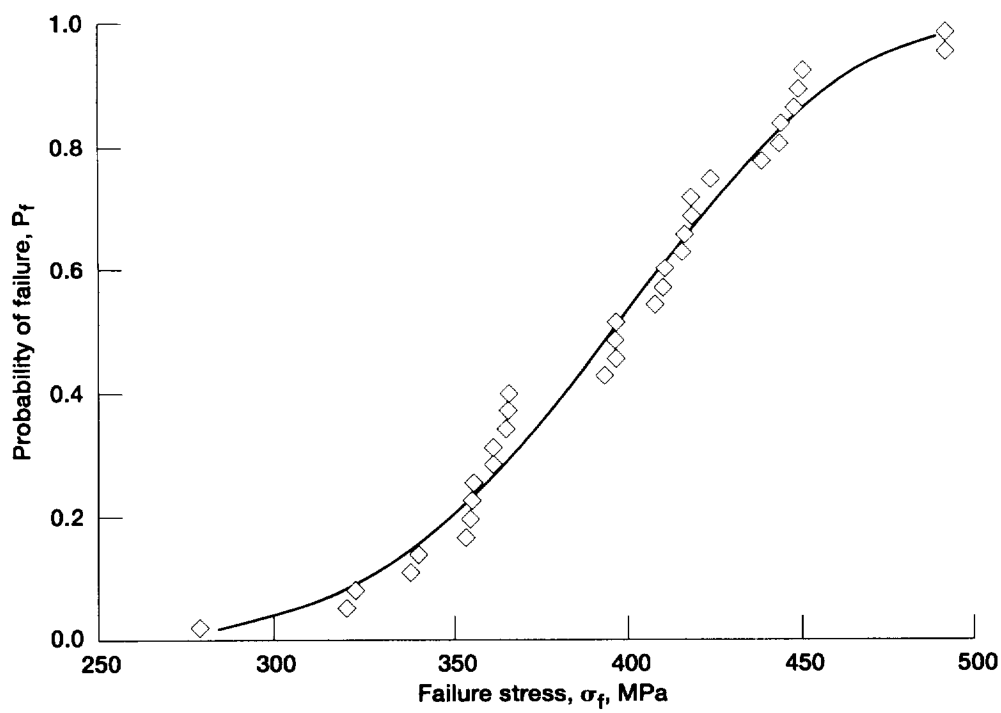

Figure 5.—Distribution curve for three parameter Weibull model from surface flawed inert transverse annealed silicon carbide data via specimens subjected to three point bend analysis (Fig. 1(d), table II). Weibull modulus m_s = 4.024; scale factor σ_{os} = 11.708 MPa–m^{2/m_s}; threshold stress σ_{us} = 190.0 MPa; sum of residuals squared = 1942.

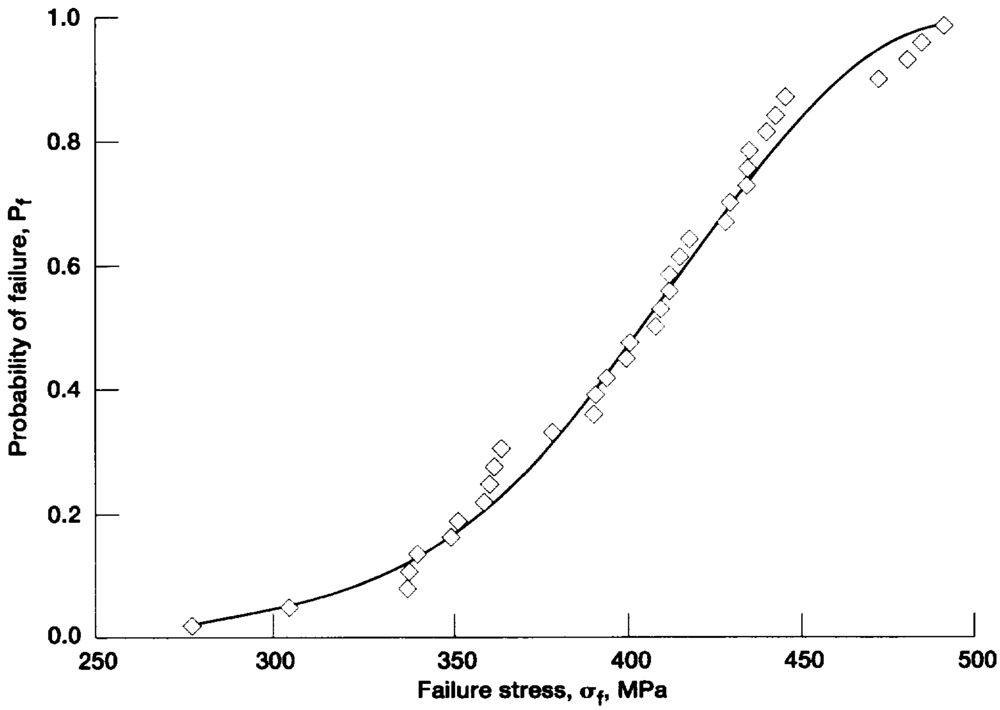

Figure 6.—Distribution curve for two parameter Weibull model from surface flawed inert longitudinal annealed silicon carbide data via specimens subjected to three point bend analysis (Fig. 1(d), table III). Weibull modulus m_s = 9.161; scale factor σ_{os} = 114.14 MPa–m^{2/m_s}; threshold stress σ_{us} = 0.0 MPa; sum of residuals squared = 1417.

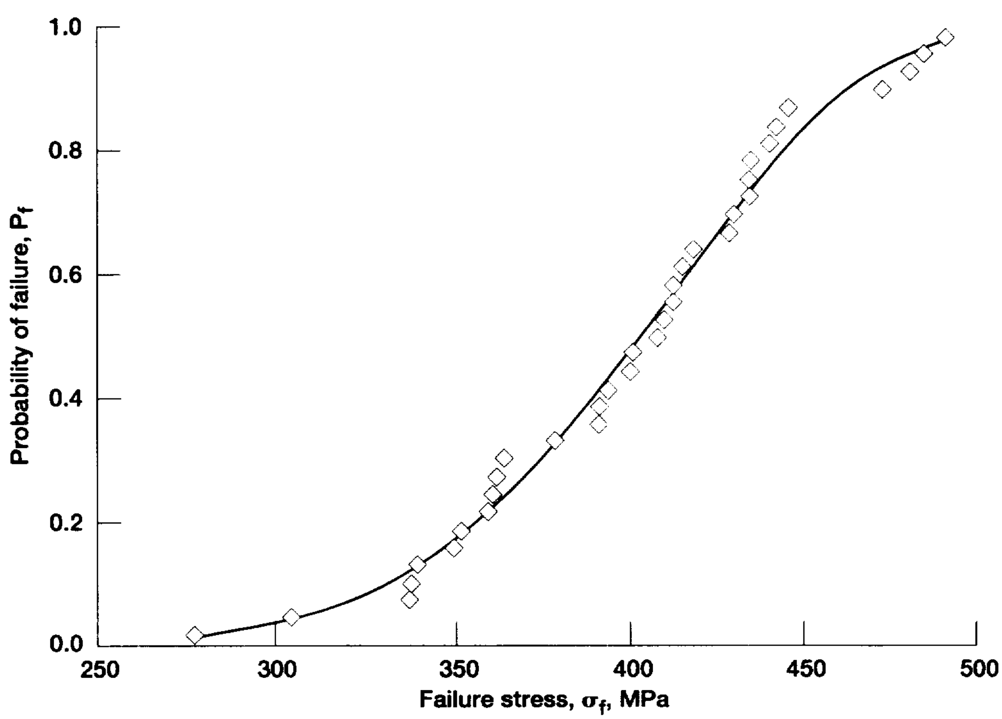

Figure 7.—Distribution curve for three parameter Weibull model from surface flawed inert longitudinal annealed silicon carbide data via specimens subjected to three point bend analysis (Fig. 1(d), table III). Weibull modulus m_s = 5.893; scale factor σ_{os} = 39.78 MPa–m^{2/m_s}; threshold stress σ_{us} = 120.0 MPa; sum of residuals squared = 1259.

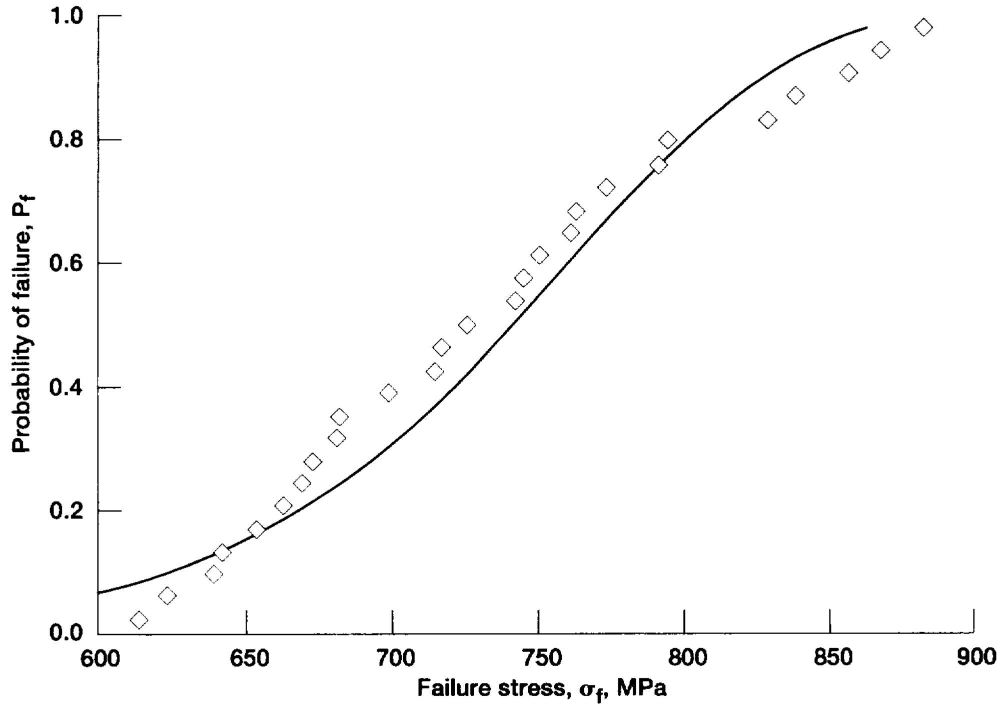

Figure 8.—Distribution curve for two parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to four point bend analysis (Fig. 1(c), table I). Weibull modulus $m_v = 10.84$; scale factor $\sigma_{uv} = 141.7$ MPa–m^{3/m_v}; threshold stress $\sigma_{uv} = 0.0$ MPa; sum of residuals squared = 11713.

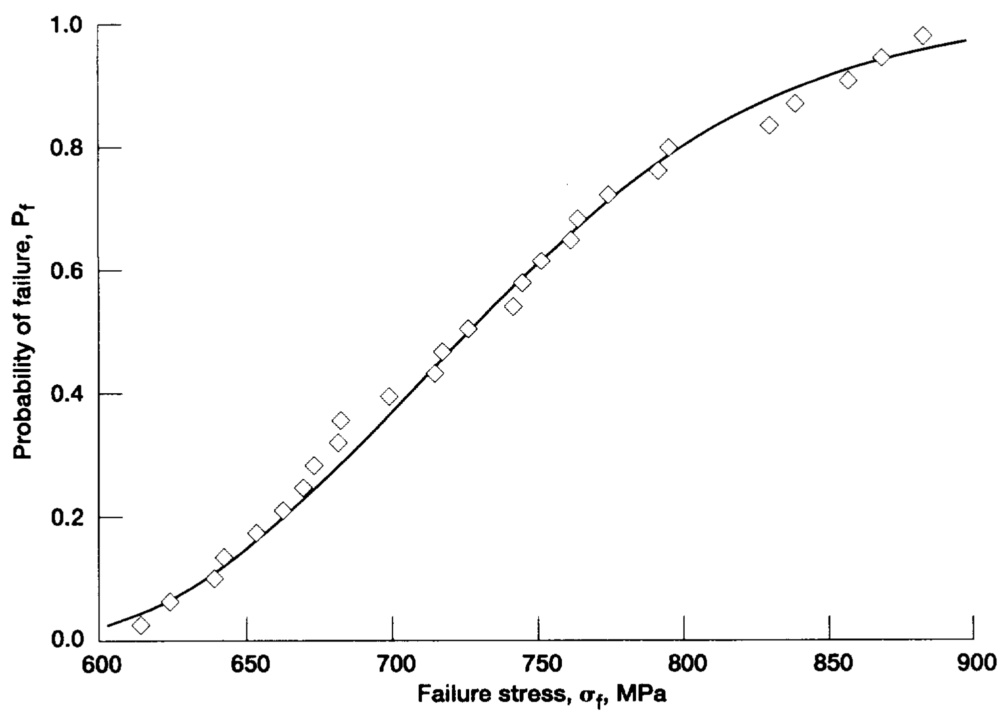

Figure 9.—Distribution curve for three parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to four point bend analysis (Fig. 1(c), table I). Weibull modulus $m_v = 1.443$; scale factor $\sigma_{ov} = 0.000680$ MPa–m^{3/m_v}; threshold stress $\sigma_{uv} = 564.0$ MPa; sum of residuals squared = 2039.

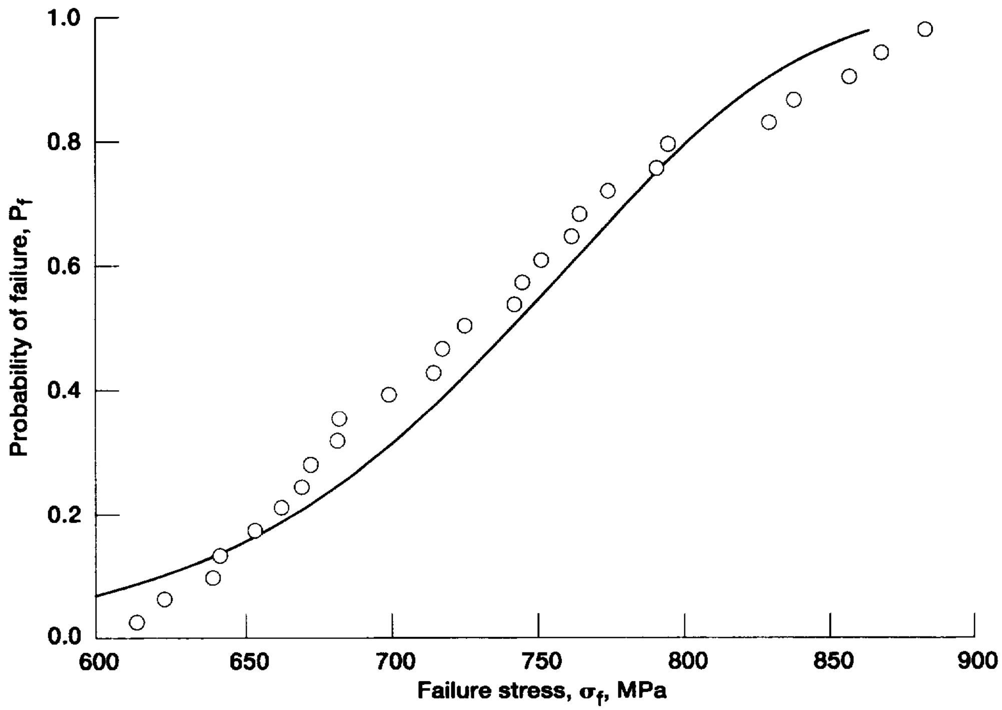

Figure 10.—Distribution curve for two parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to four point bend analysis. As an illustrative example these data were analyzed as if they came from surface flawed specimens (Fig. 1(c), table I). Weibull modulus $m_s = 10.84$; scale factor σ_{os} = 325.2 MPa-m$^{2/m_s}$; threshold stress σ_{us} = 0.0 MPa; sum of residuals squared = 12067.

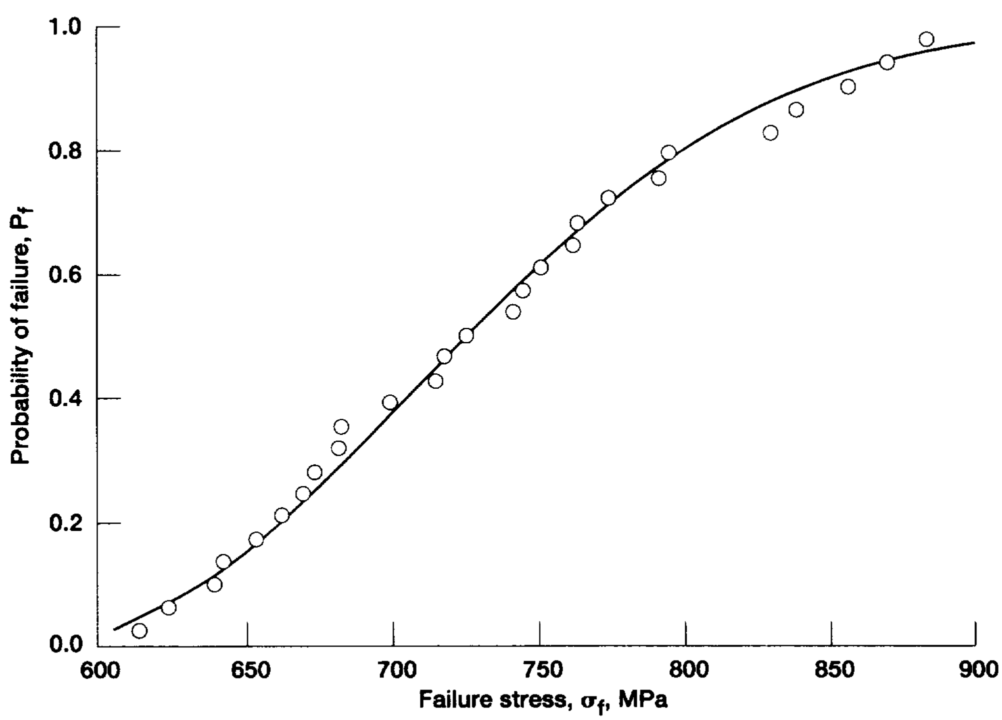

Figure 11.—Distribution curve for three parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to four point bend analysis. As an illustrative example these data were analyzed as if they came from surface flawed specimens (Fig. 1(c), table I). Weibull modulus $m_s = 1.997$; scale factor σ_{os} = 1.6895 MPa-m$^{2/m_s}$; threshold stress σ_{us} = 575.0 MPa; sum of residuals squared = 1928.

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Material characterization parameters obtained from naturally flawed specimens are necessary for reliability evaluation of nondeterministic advanced ceramic structural components. The least squares best fit method is applied to the three paramenter uniaxial Weibull model to obtain the material parameters from experimental tests on volume or surface flawed specimens subjected to pure tension, pure bending, four point or three point loading. Several illustrative example problems are provided.

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