NASA/TM—2009-216096

Analysis of Electromagnetic Wave Propagation in a Magnetized Re-Entry Plasma Sheath Via the Kinetic Equation

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December 2009

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This report contains preliminary findings,
subject to revision as analysis proceeds.

Level of Review: This material has been technically reviewed by technical management.

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Abstract

Based on a theoretical model of the propagation of electromagnetic waves through a hypersonically induced plasma, it has been demonstrated that the classical radiofrequency communications blackout that is experienced during atmospheric reentry can be mitigated through the appropriate control of an external magnetic field of nominal magnitude. The model is based on the kinetic equation treatment of Vlasov and involves an analytical solution for the electric and magnetic fields within the plasma allowing for a description of the attendant transmission, reflection and absorption coefficients. The ability to transmit through the magnetized plasma is due to the 'magnetic windows' that are created within the plasma via the well-known 'whistler modes' of propagation. The case of 2 GHz transmission through a re-entry plasma is considered. The coefficients are found to be highly sensitive to the prevailing electron density and will thus require a dynamic control mechanism to vary the magnetic field as the plasma evolves through the re-entry phase.

1.0 Introduction

Electromagnetic wave (EM) propagation through a flowing plasma layer to maintain communications and navigation to hypersonic and space re-entry vehicles has been a problem for over 45 years. The classical radio frequency (RF) blackout results when the associated EM wave is reflected and absorbed by the free electrons that make-up the plasma sheath which envelops the re-entering vehicle. The plasma is the result of extreme heating of air by the strong shock wave that is created by the leading edges of the vehicle [Bletzinger, et al., 2005 and references therein]. Such a plasma has associated with it a characteristic frequency, the 'plasma frequency', below which RF transmission is reflected and absorbed; RF radiation above this frequency will easily pass through the plasma. Since this plasma frequency is proportional to the electron concentration within the plasma, this frequency will vary according to the altitude and shape of the vehicle, as well as the angle of attack. Thus, for a given communications scenario, a particular frequency is assigned; As the vehicle begins its re-entry into the earth's atmosphere, shock waves are created and a plasma begins to be formed. At a particular electron concentration, the plasma frequency will exceed that of the communications link and transmission to and from the vehicle will cease. As the vehicle begins to decelerate, the electron density and thus the plasma frequency decrease and RF communication to and from the vehicle once again becomes possible.

Several approaches have been advanced in order to mitigate RF blackout. The obvious ones are to use communications frequencies higher than that of the plasma frequency that will develop during re-entry. However, as mentioned above, a specific maximum plasma frequency is dependent upon several parameters and becomes very difficult to establish. A high power can be used to overcome the reflection and absorption of the plasma. This, however, severely complicates the design and operation of a communications system, especially the equipment on the re-entering vehicle. Another method is to inject electrophilic substances (quenchants) into the plasma flow field of the sheath so as to de-ionize the plasma (at least, severely reduce the electron concentration) and lower the plasma frequency below that of the communications link. An application of this method occurred during the re-entry of Gemini 3 in 1965 [Schroeder and Russo, 1968] where it was demonstrated that plasma reflection and absorption can be significantly reduced to allow for re-entry communications. Another technique that can be employed is

that of establishing ‘magnetic windows’ within the plasma. Here, a static magnetic field is applied to the plasma to essentially establish ‘whistler modes’ of propagation [Usui, et al., 2000]. Early experiments using this concept were performed in 1964 [Russo and Hughes, 1964]. Here, a magnetic field of 750 Gauss was used in a ground experiment employing a plasma from a solid rocket motor. It was shown that a signal improvement of about 20 dB can be realized. However, it was recently argued [Sharkey, 2003] that magnetic fields on the order if 10^4 Gauss are needed to penetrate a re-entry plasma.

The ‘magnetic windows’ concept is very attractive in that, unlike the use of electrophilic substances where a supply of such material is required throughout the blackout period (which, for some planetary re-entry scenarios, can have a duration of 10 to 20 minutes), the magnitude of the static magnetic field can be adjusted as conditions require to maintain a window in the otherwise RF opaque plasma. The electrophilic technique is an ‘active’ mitigation method whereas the magnetic windows technique is a ‘passive’ method. If it can be demonstrated that magnetic fields of nominal strength can be used to elicit a magnetic window within a plasma with re-entry parameters, and, additionally, it can be shown how the magnitude of this magnetic field needs to be adjusted to maintain a window as the plasma properties evolve, then the use of an external magnetic field as a mitigation technique will be a viable one.

This prescription can only be accomplished by realistically modeling the plasma propagation environment. It is the purpose of this report to analytically study such a situation. A mathematical model will is constructed, based on the Vlasov equations, i.e., the Maxwell equations supplemented with the kinetic (Boltzmann) equation describing the electron distribution, that will capture the propagation process through a flowing plasma immersed in an external magnetic field. The model will then be solved in the approximation of weak spatial dispersion; the case of strong spatial dispersion will be treated in a forthcoming report. It is shown that an applied magnetic field of nominal strength can alter the plasma and create “magnetic windows” through which electromagnetic radiation can propagate.

2.0 The Initial Equations

2.1 Incorporating the Maxwell Equations with Kinetic Theory—The Vlasov Equations

The basic starting point for modeling electromagnetic wave propagation through a flowing hypersonic plasma are, of course, the Maxwell Equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.2}$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{2.3}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \tag{2.4}$$

The expressions relating the charge density ρ and current density $\vec{j}$ within the plasma are given in terms of the statistical distribution functions $f_s(\vec{r}, \vec{\upsilon}, t)$ governing the charge carriers of the species s at position $\vec{r}$ and velocity $\vec{\upsilon}$ at time t

$$\rho = \sum_s e_s n_s \int_{-\infty}^{\infty} f_s(\vec{r}, \vec{\upsilon}, t) d^3\upsilon \tag{2.5}$$

$$\vec{j} = \sum_s e_s n_s \int_{-\infty}^{\infty} \vec{\upsilon} f_s(\vec{r}, \vec{\upsilon}, t) d^3\upsilon \tag{2.6}$$

Here, s denotes either electron or ion charge carriers where $n_s = n_{ion}$ is the number of positively charged ions, $e_s = e_{ion} = +e$ or $n_s = n_e$ is the number of negatively charged electrons, $e_s = e_e = -e$. The statistical distribution function for species s $f_s(\vec{r}, \vec{\upsilon}, t)$ is given by the Boltzmann equation incorporating collisions via the Krook model [Tanenbaum, 1967; Bhatnagar, et al., 1954] for collisions, viz,

$$\frac{\partial f_s}{\partial t} + \vec{\upsilon} \cdot \vec{\nabla}_{\vec{r}} f_s + \vec{F}_s \cdot \vec{\nabla}_{\vec{\upsilon}} f = -\nu_s (f_s - f_{s0}) \tag{2.7}$$

where ν_s is the effective collision frequency and $f_{s0}(\vec{r}, \vec{\upsilon})$ is the initial equilibrium distribution of the *s*-th species. The model for the effective collision frequency used here, along with the basis of the Krook model, is discussed in Appendix A. The force entering Equation (2.7) is given by the Lorentz force involving both the electric $\vec{E}$ and magnetic $\vec{B}$ fields of the wave as well as an externally applied magnetic field
$\vec{B}_0$,

$$\vec{F}_s = \frac{e_s}{m_s}\left(\vec{E} + \frac{\vec{\upsilon}}{c} \times \left(\vec{B} + \vec{B}_0\right)\right) \tag{2.8}$$

Collectively, the Maxwell equations, Equations (2.1) to (2.4), the Equations (2.5) and (2.6), and the Boltzmann equation, Equation (2.7) with Equation (2.8) are known as the Vlasov equations.

These coupled integral and differential equations must now be applied to the situation of a hypersonically flowing plasma upon which is incident a plane electromagnetic wave. The situation is depicted in Figure 1. The tangential velocity $\vec{V}_T$ is due to the plasma flowing along the hypersonic vehicle surface. The velocity $\vec{V}_L$ is due to the motion of the hypersonic vehicle toward the observer. The plasma is taken to be of infinite extent along the *x* and *y* axes. An incident plane electromagnetic field impinges on the moving plasma layer from the transmitter fixed on the earth and travels along the *z*-axis in the *–z* direction. The constant homogeneous magnetic field emanates from an antenna on the surface of the vehicle along the *z*-axis in the *+z* direction. The surface in the $x - y$ plane is taken to be along the solid surface of the vehicle and that at the front of the layer of thickness *L* is taken to be open to the atmosphere. The values of the $\vec{E}$ and $\vec{B}$ fields of the electromagnetic wave are reckoned with respect to the observer in the rest frame with respect to the moving plasma layer. Given these conditions, one can write for the functional dependence of the field and current density,

$$\vec{E} = \vec{E}(z,t) = E_x(z,t)\hat{x} + E_y(z,t)\hat{y} \tag{2.9a}$$

$$\vec{B} = \vec{B}(z,t) = B_x(z,t)\hat{x} + B_y(z,t)\hat{y} + B_0\hat{z} \tag{2.9b}$$

$$\vec{j} = \vec{j}(z,t) = j_x(z,t)\hat{x} + j_y(z,t)\hat{y} \tag{2.9c}$$

The problem is most easily dealt with by transforming the fields into the reference frame moving with the plasma.

2.2 Transformation of the Fields Into the Reference Frame Moving with the Plasma

The transformation of the fields $\vec{E}$ and $\vec{B}$ in the reference frame of the observer (transmitter) to those $\vec{E}'$ and $\vec{B}'$ that are seen in the frame moving with the plasma layer is given by the well-known relations [Jackson, 1975]

$$\vec{E}' = \gamma\left(\vec{E} + \vec{\beta} \times \vec{B}\right) - \frac{\gamma^2}{\gamma + 1}\vec{\beta}\left(\vec{\beta} \cdot \vec{E}\right) \tag{2.10}$$

$$\vec{B}' = \gamma\left(\vec{B}' - \vec{\beta} \times \vec{E}\right) - \frac{\gamma^2}{\gamma + 1}\vec{\beta}\left(\vec{\beta} \cdot \vec{B}\right) \tag{2.11}$$

where

$$\vec{\beta} \equiv \frac{V_T \hat{x} + V_L \hat{z}}{c}, \gamma \equiv \sqrt{\left(1 - \beta^2\right)} \tag{2.12}$$

and c is the velocity of light. Also, for the current density

$$\vec{j}' = \gamma\left(\vec{j} - \vec{\beta}\rho\right) \tag{2.13}$$

The invariance of the Maxwell equations has that, in the frame moving with the plasma,

$$\vec{\nabla}' \times \vec{E}' + \frac{1}{c}\frac{\partial \vec{B}'}{\partial t'} = 0 \tag{2.14}$$

$$\vec{\nabla}' \times \vec{B}' - \frac{1}{c}\frac{\partial \vec{E}'}{\partial t'} = \frac{4\pi}{c}\vec{j}' \tag{2.15}$$

where $\vec{\nabla}'$ and t' are the appropriately transformed divergence and time. Consider now the extreme case in which a hypersonic velocity of Mach 35 is realized, i.e., $V_T = V_L \approx 1.2 \times 10^4$ m/s. This yields $\beta \approx 4.0 \times 10^{-5} << 1$. In this case, the transformed equations Equations (2.13) and (2.14) essentially reduce to those in the rest frame, i.e., Equations (2.3) and (2.4). Similarly, for the homogeneous applied magnetic field, $\vec{B}_0' \approx \vec{B}_0$. Hence, to within a first order approximation, Equations (2.1) to (2.4) can be employed in the reference frame moving with the plasma. One can now incorporate these relations in the evaluation of Equation (2.7).

One should also address the Doppler shifts that will be incurred in the transformation to the reference frame moving with the plasma. This will not be addressed at this point as it is the goal of this work to establish the possibility of the magnetic windows concept. A more careful study with will appear in a forthcoming work will incorporate Doppler effects.

The final goal in the analysis that follows is the evaluation of the reflection, transmission, and absorption coefficients of the plane wave interacting with the plasma layer; the 'flow chart' of this process is depicted in Figure 2. The scattering process, as well as its solution, goes as follows. The electric and magnetic fields of the plane electromagnetic wave induces a Lorentz force on the plasma electrons. The induced force, in turn, produces a variation in the electron distribution function governing the position and momentum of the electrons. This gives rise to an induced current within the plasma that produces an associated variation of the fields within the plasma which, once again, induces an additional Lorentz force. This non-linear process finally results in the establishment of field distributions at the

boundaries of the plasma that determines the corresponding reflection, transmission, and absorption coefficients of the incident field.

3.0 Application of the Foregoing to the Boltzmann Equation

3.1 Reduction of the Equation

Since the current density is only a function of the longitudinal coordinate z, so too will be the distribution function of Equation (2.7)

$$f_s(\vec{r},\vec{\upsilon},t) = f_s(z,\vec{\upsilon},t) \tag{3.1}$$

The associated equilibrium distribution function is taken to be given by the Maxwell distribution

$$f_{s0}(\vec{\upsilon}) = f_{s0}(|\vec{\upsilon}|) = \left(\frac{1}{\pi}\right)^{3/2}\left(\frac{1}{\upsilon_{Ts}}\right)^{3}\exp\left(-\frac{\upsilon^2}{\upsilon_{Ts}^2}\right) \tag{3.2}$$

where the thermal velocity of the s-th species is given by

$$\upsilon_{Ts} \equiv \sqrt{\frac{2k_B T_s}{m_s}} \tag{3.3}$$

Here, k_B is Boltzmann's constant, T_s is the absolute temperature of the *s*-th species of charges with mass m_s. The distribution function $f_s(z,\vec{\upsilon},t)$ will be taken to be related to that of Equation (3.2) by a small perturbation $\phi_s(z,\vec{\upsilon},t)$, viz,

$$f_s(z,\vec{\upsilon},t) = f_{s0}(|\vec{\upsilon}|) + \phi(z,\vec{\upsilon},t), \phi(z,\vec{\upsilon},t) << f_{s0}(|\vec{\upsilon}|) \tag{3.4}$$

In order for $\phi_s(z,\vec{\upsilon},t)$ to be treated as a perturbation, the fields of the incident wave must also be treated as a first order perturbation; writing the temporal dependence of the fields of the incident electromagnetic wave as

$$E_x(z,t) = E_x(z)\exp(-i\omega t), \qquad \text{etc.}$$

one has from Equations (2.9a) and (2.9b)

$$\vec{E} = \left(E_x(z)\hat{x} + E_y(z)\right)\exp(-i\omega t) \tag{3.5}$$

$$\vec{B} = \left(B_x(z)\hat{x} + B_y(z)\hat{y}\right)\exp(-i\omega t) + B_0(z) \tag{3.6}$$

in which one must now require for a perturbation solution, $E_x, E_y, B_x, B_y << B_0$. Finally, since the time-harmonic fields of Equations (3.5) and (3.6) are taken to be the source of perturbation of the distribution function given by Equation (3.4), one can further write

$$\phi_s(z,\vec{\upsilon},t) = \phi_s(z,\vec{\upsilon})\exp(-i\omega t) \tag{3.7}$$

Substituting Equations (3.4) to (3.7) into Equation (2.7) and dropping quantities that are second order perturbations with respect to the fields and to $\phi_s(z,\vec{\upsilon})$ yields

$$\begin{aligned}
&-i(\omega+i\nu_s)\phi_s(z,\vec{\upsilon})\exp(-i\omega t)+\upsilon_z\frac{\partial\phi_s(z,\vec{\upsilon})}{\partial z}\exp(-i\omega t)\\
&\quad+\left(\frac{e_s}{m_s}\right)\left[\vec{E}(z)\cdot\vec{\nabla}_{\vec{\upsilon}}f_{s0}(|\vec{\upsilon}|)+\frac{\vec{\upsilon}}{c}\times\vec{B}(z)\cdot\vec{\nabla}_{\vec{\upsilon}}f_{s0}(|\vec{\upsilon}|)\right]\exp(-i\omega t)\\
&\quad+\left(\frac{e_s}{m_s}\right)\left[\frac{\vec{\upsilon}}{c}\times B_0\hat{z}\cdot\vec{\nabla}_{\vec{\upsilon}}f_{s0}(|\vec{\upsilon}|)+\frac{\vec{\upsilon}}{c}\times B_0\hat{z}\cdot\vec{\nabla}_{\vec{\upsilon}}\phi_s(z,\vec{\upsilon})\exp(-i\omega t)\right]=0
\end{aligned}\tag{3.8}$$

However, remembering the identity $(\vec{\upsilon}\times\vec{B})\cdot\vec{\upsilon}=0$, one has that

$$\frac{\vec{\upsilon}}{c}\times\vec{B}(z)\cdot\vec{\nabla}_{\vec{\upsilon}}f_{s0}(|\vec{\upsilon}|)=\frac{\vec{\upsilon}}{c}\times\vec{B}(z)\cdot\frac{\vec{\upsilon}}{\upsilon}\frac{\partial f_{s0}}{\partial\upsilon}=0\tag{3.9}$$

and similarly for the term $(\vec{\upsilon}/c)\times B_0\hat{z}\cdot\vec{\nabla}_{\vec{\upsilon}}f_{s0}(|\vec{\upsilon}|)$. (The term $(\vec{\upsilon}/c)\times B_0\hat{z}\cdot\vec{\nabla}_{\vec{\upsilon}}\phi_s(z,\vec{\upsilon})\neq 0$ since $\phi_s(z,\vec{\upsilon})$ is not a function of the scalar υ.) Hence, Equation (3.8) reduces to

$$\begin{aligned}
&-i(\omega+i\nu_s)\phi_s(z,\vec{\upsilon})+\upsilon_z\frac{\partial\phi_s(z,\vec{\upsilon})}{\partial z}\exp(-i\omega t)\\
&\quad+\left(\frac{e_s}{m_s}\right)\vec{E}(z)\cdot\vec{\nabla}_{\vec{\upsilon}}\phi_s(z,\vec{\upsilon})+\left(\frac{e_s}{m_s}\right)\left(\frac{\vec{\upsilon}}{c}\times B_0\hat{z}\right)\cdot\vec{\nabla}_{\vec{\upsilon}}\phi_s(z,\vec{\upsilon})=0
\end{aligned}\tag{3.10}$$

Defining the cyclotron frequency

$$\omega_{cs}\equiv\frac{e_sB_0}{m_sc}\tag{3.11}$$

and expanding $\vec{\upsilon}\times\hat{z}\cdot\vec{\nabla}_{\vec{\upsilon}}\phi_s=(\upsilon_y\,\partial/\partial\upsilon_x-\upsilon_x\,\partial/\partial\upsilon_y)\phi_s$ as well as $\vec{E}\cdot\vec{\nabla}_{\vec{\upsilon}}f_{s0}=(E_x\,\partial/\partial\upsilon_x+E_y\,\partial/\partial\upsilon_y)f_{s0}$, Equation (3.10) becomes

$$\upsilon_z\frac{\partial\phi_s}{\partial z}-i(\omega+i\nu_s)\phi_s+\omega_{cs}\left(\upsilon_y\frac{\partial}{\partial\upsilon_x}-\upsilon_x\frac{\partial}{\partial\upsilon_y}\right)\phi_s+\left(\frac{e_s}{m_s}\right)\left(E_x\frac{\partial}{\partial\upsilon_x}+E_y\frac{\partial}{\partial\upsilon_y}\right)f_{s0}=0\tag{3.12}$$

This differential equation is most easily dealt with in plane-polar coordinates $\upsilon_\perp$ and θ defined by $\upsilon_x=\upsilon_\perp\cos\theta,\upsilon_y=\upsilon_\perp\sin\theta$. In this instance, one has

$$\frac{\partial}{\partial\upsilon_x}=\cos\theta\frac{\partial}{\partial\upsilon_\perp}-\frac{\sin\theta}{\upsilon_\perp}\frac{\partial}{\partial\theta},\frac{\partial}{\partial\upsilon_y}=\sin\theta\frac{\partial}{\partial\upsilon_\perp}+\frac{\cos\theta}{\upsilon_\perp}\frac{\partial}{\partial\theta}\tag{3.13}$$

allowing Equation (3.12) to be written as (noting that f_{s0} is isotropic and independent of the polar angle θ)

$$\upsilon_z\frac{\partial\phi_s}{\partial z}-i(\omega+i\nu_s)\phi_s-\omega_{cs}\frac{\partial\phi_s}{\partial\theta}+\left(\frac{e_s}{m_s}\right)(E_x\cos\theta+E_y\sin\theta)\frac{\partial f_{s0}}{\partial\upsilon_\perp}=0\tag{3.14}$$

Finally, using the auxiliary fields [Gross, 1951; Bell and Buneman, 1964] defined by

$$E_L \equiv E_x - iE_y, E_R \equiv E_x + iE_y \tag{3.15}$$

gives

$$E_x \cos\theta + E_y \sin\theta = \frac{1}{2}\left(E_L \exp(i\theta) + E_R \exp(-i\theta)\right) \tag{3.16}$$

Substituting this into Equation (3.14) and rearranging terms yields

$$\omega_{cs}\frac{\partial\phi_s(z,\vec{\upsilon})}{\partial\theta} - \upsilon_z\frac{\partial\phi_s(z,\vec{\upsilon})}{\partial z} + i(\omega + i\nu_s)\phi_s(z,\vec{\upsilon}) = \left(\frac{e_s}{2m_s}\right)\left(E_L \exp(i\theta) + E_R \exp(-i\theta)\right)\frac{\partial f_{s0}}{\partial\upsilon_\perp} = 0 \tag{3.17}$$

This form of the differential equation can now be solved using the method of characteristics. Writing the equation as a relation along its characteristic curve, one has

$$\frac{d\phi_s}{dt} = \frac{\partial\phi_s}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\phi_s}{\partial z}\frac{\partial z}{\partial t} = \omega_{cs}\frac{\partial\phi_s}{\partial\theta} - \upsilon_z\frac{\partial\phi_s}{\partial z}$$

Therefore,

$$\theta = \omega_{cs}t + \theta', z = -\upsilon_z t \tag{3.18}$$

which combines to give $\theta = -(\omega_{cs}/\upsilon_z)z + \theta'$. Changing back to the variables θ and z, $d/dt = -\upsilon_z(d/dz)$ and Equation (3.17) becomes the first order differential equation

$$\begin{aligned} -\upsilon_z\frac{d\phi_s(z,\vec{\upsilon})}{dz} &+ i(\omega + i\nu_s)\phi_s(z,\vec{\upsilon}) \\ &= \left(\frac{e_s}{2m_s}\right)\left(E_L \exp\left(-i(\omega_{cs}/\upsilon_z)z + i\theta'\right) + E_R \exp\left(i(\omega_{cs}/\upsilon_z)z - i\theta'\right)\right)\frac{\partial f_{s0}}{\partial\upsilon_\perp} = 0 \end{aligned} \tag{3.19}$$

This equation must now be solved across the plasma layer of thickness L, $0 \le z \le L$.

3.2 Solution of the Differential Equation Across the Plasma Layer

Consider the Fourier series of the function $\phi_s(z,\vec{\upsilon})$ over the interval $0 \le z \le L$;

$$\phi_s(z,\vec{\upsilon}) = \sum_{l=-\infty}^{\infty}\tilde{\phi}_s(\kappa_l,\vec{\upsilon})\exp(i\kappa_l z), \quad \kappa_l \equiv \frac{l\pi}{L} \tag{3.20}$$

$$\tilde{\phi}_s(\kappa_l,\vec{\upsilon}) = \frac{1}{L}\int_0^L \phi_s(z,\vec{\upsilon})\exp(-i\kappa_l z)\,dz \tag{3.21}$$

Thus, multiplying Equation (3.19) by $\exp(-i\kappa_l z)$ and integrating over z, using Equation (3.21) gives

$$\tilde{\phi}_s\left(\kappa_l,\vec{\upsilon}\right)\left[i\kappa_l-\frac{i\left(\omega+i\nu_s\right)}{\upsilon_z}\right]L=\phi_s\left(0,\vec{\upsilon}\right)-\phi_s\left(L,\vec{\upsilon}\right)(-1)^l- \\ -\left(\frac{L}{\upsilon_z}\right)\left(\frac{e_s}{2m_s}\right)\frac{\partial f_{s0}}{\partial\upsilon_\perp}\left[\tilde{E}_L{}'\left(\kappa_l+\frac{\omega_{cs}}{\upsilon_z}\right)+\tilde{E}_R{}'\left(\kappa_l-\frac{\omega_{cs}}{\upsilon_z}\right)\right] \tag{3.22}$$

where $\exp\left(-i\kappa_l L\right)=\exp\left(-l\pi\right)=\left(-1\right)^l$ and $\tilde{E}_L{}'\left(\kappa\right)\equiv\tilde{E}_L\left(\kappa\right)\exp\left(i\theta\right)$, $\tilde{E}_R{}'\left(\kappa\right)\equiv\tilde{E}_R\left(\kappa\right)\exp\left(-i\theta\right)$. At this point, contact must be made to the current density expression given by Equation (2.6); converting to plane polar coordinates,

$$\vec{j}\left(z\right)=\sum_s e_s n_s\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\upsilon_\perp\left(\hat{x}\cos\theta+\hat{y}\sin\theta\right)f_s\left(z,\upsilon_\perp,\upsilon_z,t\right)\upsilon_\perp d\upsilon_\perp d\theta d\upsilon_z \tag{3.23}$$

As before, considering the identity

$$\hat{x}\cos\theta+\hat{y}\sin\theta=\frac{1}{2}\left[\left(\hat{x}-i\hat{y}\right)\exp\left(i\theta\right)+\left(\hat{x}+i\hat{y}\right)\exp\left(-i\theta\right)\right]$$

and using it in Equation (3.23) and taking the scalar product with the particular vector $\hat{x}+i\hat{y}$ yields,

$$j_R\left(z,t\right)\equiv j_x\left(z,t\right)+ij_y\left(z,t\right)=\sum_s e_s n_s\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\upsilon_\perp^2\exp\left(i\theta\right)f_s\left(z,\upsilon_\perp,\upsilon_z,t\right)d\upsilon_\perp d\theta d\upsilon_z \tag{3.24}$$

Thus, the current density has been converted to a complex scalar current density in the transverse plane. Substituting now Equations (3.4) and (3.7) into Equation (3.24), and using Equation (3.2), one as that the $f_{s0}\left(\left|\vec{\upsilon}\right|\right)$ term vanishes upon performing the υ_z integration. Hence, what survives is the result

$$j_R\left(z\right)=\sum_s e_s n_s\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\upsilon_\perp^2\exp\left(i\theta\right)\phi_s\left(z,\upsilon_\perp,\upsilon_z\right)d\upsilon_\perp d\theta d\upsilon_z \tag{3.25}$$

where $j_R\left(z,t\right)=j_R\left(z\right)\exp\left(-i\omega t\right)$. Thus, the current density associated with the polarized wave fields E_R and E_L of Equation (3.15) are also polarized; however E_R and E_L remain uncoupled. Therefore, one can use either E_R or E_L separately in Equation (3.22). Selecting E_R, i.e., the whistler mode $E_R=E_x+iE_y$, Equation (3.22) becomes, upon using the definition $\kappa_l'\equiv\kappa_l-\omega_{cs}/\upsilon_z$,

$$\tilde{\phi}_s\left(\kappa_l',\upsilon_\perp,\upsilon_z\right)=\left[\left(\kappa_l'-\Omega_s\right)iL\right]^{-1}\left\{\Delta_s\left(\upsilon_\perp,\upsilon_z\right)-L\left(\frac{1}{\upsilon_z}\right)\left(\frac{e_s}{2m_s}\right)\left(\frac{\partial f_{s0}}{\partial\upsilon_\perp}\right)\tilde{E}_R'\left(\kappa_l'\right)\right\} \tag{3.26}$$

where

$$\Omega_s\equiv\frac{\omega-\omega_{cs}+i\nu_s}{\upsilon_z} \tag{3.27}$$

and

$$\Delta_s(\upsilon_\perp,\upsilon_z)\equiv\phi_s(0,\upsilon_\perp,\upsilon_z)-\phi_s(L,\upsilon_\perp,\upsilon_z)(-1)^l \tag{3.28}$$

Now, rewriting Equation (3.25) by limiting the υ_z integration in velocity space to the interval $0\le\upsilon_z<\infty$ gives

$$j_R(z)=\sum_s e_s n_s\int_0^\infty\int_0^{2\pi}\int_0^\infty \upsilon_\perp^2\exp(i\theta)\Phi_s(z,\upsilon_\perp,\upsilon_z)\,d\upsilon_\perp d\theta d\upsilon_z \tag{3.29}$$

where

$$\Phi_s(z,\upsilon_\perp,\upsilon_z)\equiv\phi_s(z,\upsilon_\perp,\upsilon_z)+\phi_s(z,\upsilon_\perp,-\upsilon_z) \tag{3.30}$$

The Fourier transform of Equation (3.29) is, using the form of Equation (3.21),

$$\tilde{j}_R(\kappa_l)=\sum_s e_s n_s\int_0^\infty\int_0^{2\pi}\int_0^\infty \upsilon_\perp^2\exp(i\theta)\tilde{\Phi}_s(\kappa_l,\upsilon_\perp,\upsilon_z)\,d\upsilon_\perp d\theta d\upsilon_z \tag{3.31}$$

where

$$\tilde{\Phi}_s(\kappa_l,\upsilon_\perp,\upsilon_z)\equiv\tilde{\phi}_s(\kappa_l,\upsilon_\perp,\upsilon_z)+\tilde{\phi}_s(\kappa_l,\upsilon_\perp,-\upsilon_z) \tag{3.32}$$

Substituting Equation (3.26) into Equation (3.32) yields, after expanding terms and simplifying,

$$\begin{aligned}\tilde{\Phi}_s(\kappa_l,\upsilon_\perp,\upsilon_z)=\frac{i}{\Omega_s^2-\kappa_l^2}\Bigg\{&\frac{\kappa_l}{L}\left(\Delta_s(\upsilon_\perp,\upsilon_z)+\Delta_s(\upsilon_\perp,-\upsilon_z)\right)\\&-2\frac{\Omega_s}{\upsilon_z}K_s\tilde{E}_R'(\kappa_l)+\frac{\Omega_s}{L}\left(\Delta_s(\upsilon_\perp,\upsilon_z)-\Delta_s(\upsilon_\perp,-\upsilon_z)\right)\Bigg\}\end{aligned} \tag{3.33}$$

where

$$K_s\equiv\frac{e_s}{2m_s}\left(\frac{\partial f_{s0}}{\partial\upsilon_\perp}\right) \tag{3.34}$$

Using Equation (3.33) in Equation (3.31) will give a relation for the current density. However, values for the parameters $\Delta_s(\upsilon_\perp,\upsilon_z)$ must first be determined. By their definition of Equation (3.28), they are functions of the boundary values of the functions ϕ_s.

3.3 Incorporation of Boundary Conditions

By the definition of the problem, the plasma at the surface $z=0$ is against the solid body of the hypersonic vehicle. Thus, reflective boundary conditions prevail from which one can write

$$\phi_s(0,\upsilon_\perp,\upsilon_z)=\phi_s(0,\upsilon_\perp,-\upsilon_z) \tag{3.35}$$

However, at the boundary $z=L$, the plasma is open to the atmosphere so diffusive conditions prevail which give

$$\phi_s\left(L,\upsilon_\perp,-\upsilon_z\right)=0 \tag{3.36}$$

With these assignments, Equation (3.33) becomes

$$\begin{aligned}\tilde{\Phi}_s\left(\kappa_l,\upsilon_\perp,\upsilon_z\right)=\frac{i}{\Omega_s^2-\kappa_l^2}\Bigg\{&\frac{\kappa_l}{L}\Big(2\phi_s\left(0,\upsilon_\perp,\upsilon_z\right)-\phi_s\left(L,\upsilon_\perp,\upsilon_z\right)(-1)^l\Big)\\&-2\frac{\Omega_s}{\upsilon_z}K_s\tilde{E}_R'\left(\kappa_l\right)-\frac{\Omega_s}{L}\phi_s\left(L,\upsilon_\perp,\upsilon_z\right)(-1)^l\Bigg\}\end{aligned} \tag{3.37}$$

At this point, the values of the distribution functions at the boundaries, $\phi_s\left(0,\upsilon_\perp,\upsilon_z\right)$ and $\phi_s\left(L,\upsilon_\perp,\upsilon_z\right)$ still remain unknown. In order to determine these values, a self-constancy condition can be applied. Hence, using Equations (3.35) and (3.36) with Equation (3.32) gives

$$\Phi_s\left(0,\upsilon_\perp,\upsilon_z\right)=2\phi_s\left(0,\upsilon_\perp,\upsilon_z\right) \tag{3.38}$$

and

$$\Phi_s\left(L,\upsilon_\perp,\upsilon_z\right)=\phi_s\left(L,\upsilon_\perp,\upsilon_z\right) \tag{3.39}$$

But by Equation (3.20),

$$\Phi_s\left(0,\upsilon_\perp,\upsilon_z\right)=\sum_{l=-\infty}^{\infty}\tilde{\Phi}_s\left(\kappa_l,\upsilon_\perp,\upsilon_z\right) \tag{3.40}$$

and

$$\Phi_s\left(L,\upsilon_\perp,\upsilon_z\right)=\sum_{l=-\infty}^{\infty}\tilde{\Phi}_s\left(\kappa_l,\upsilon_\perp,\upsilon_z\right)(-1)^l \tag{3.41}$$

Hence, form these relations, one has

$$\sum_{l=-\infty}^{\infty}\tilde{\Phi}_s\left(\kappa_l,\upsilon_\perp,\upsilon_z\right)=2\phi_s\left(0,\upsilon_\perp,\upsilon_z\right) \tag{3.42}$$

and

$$\sum_{l=-\infty}^{\infty}\tilde{\Phi}_s\left(\kappa_l,\upsilon_\perp,\upsilon_z\right)(-1)^l=\phi_s\left(L,\upsilon_\perp,\upsilon_z\right) \tag{3.43}$$

Equation (3.37) can now be used on the left sides of Equations (3.42) and (3.43) to give two equations in the two unknowns $\phi_s\left(0,\upsilon_\perp,\upsilon_z\right)$ and $\phi_s\left(L,\upsilon_\perp,\upsilon_z\right)$. To this end, substituting Equation (3.37) into Equation (3.42) and solving for $\phi_s\left(0,\upsilon_\perp,\upsilon_z\right)$,

$$\phi_s\left(0,\upsilon_\perp,\upsilon_z\right)=-\sum_{l=-\infty}^{\infty}\left(\frac{i}{\Omega_s^2-\kappa_l^2}\right)\left\{\left(\frac{\Omega_s}{\upsilon_z}\right)K_s\tilde{E}_R'\left(\kappa_l\right)+\left(\frac{\Omega_s}{2L}\right)\phi_s\left(L,\upsilon_\perp,\upsilon_z\right)(-1)^l\right\} \tag{3.44}$$

where the terms that contain the factors κ_l vanish, i.e.,

$$\sum_{l=-\infty}^{\infty} \frac{i\kappa_l}{\Omega_s^2 - \kappa_l^2} = 0, \quad \sum_{l=-\infty}^{\infty} \frac{i\kappa_l (-1)^l}{\Omega_s^2 - \kappa_l^2} = 0 \tag{3.45}$$

Similarly, using Equation (3.37) in Equation (3.43) and solving for $\phi_s(L, \upsilon_\perp, \upsilon_z)$ yields

$$\phi_s(L, \upsilon_\perp, \upsilon_z) = -\sum_{l=-\infty}^{\infty} \left(\frac{i}{\Omega_s^2 - \kappa_l^2} \right) \left\{ \left(2\frac{\Omega_s}{\upsilon_z} \right) K_s (-1)^l \tilde{E}_R'(\kappa_l) + \left(\frac{\Omega_s}{L} \right) \phi_s(L, \upsilon_\perp, \upsilon_z) \right\} \tag{3.46}$$

At this point, it will facilitate further evaluations to simplify Equations (3.44) and (3.46) by analytically performing summations over the parameters *l* where possible. Hence, using the summations

$$\sum_{l=-\infty}^{\infty} \left(\frac{i\Omega_s}{\Omega_s^2 - \kappa_l^2} \right) = iL\cot(\Omega_s L), \qquad \sum_{l=-\infty}^{\infty} \left(\frac{i\Omega_s}{\Omega_s^2 - \kappa_l^2} \right) (-1)^l = iL\csc(\Omega_s L) \tag{3.47}$$

allow Equations (3.44) and (3.46) to be written,

$$\phi_s(0, \upsilon_\perp, \upsilon_z) = -\sum_{l=-\infty}^{\infty} \left(\frac{i\Omega_s}{\Omega_s^2 - \kappa_l^2} \right) \left(\frac{1}{\upsilon_z} \right) K_s \tilde{E}_R'(\kappa_l) - \frac{i}{2} \phi_s(L, \upsilon_\perp, \upsilon_z) \csc(\Omega_s L) \tag{3.48}$$

and

$$\phi_s(L, \upsilon_\perp, \upsilon_z) = -\sum_{l=-\infty}^{\infty} 2 \left(\frac{i\Omega_s}{\Omega_s^2 - \kappa_l^2} \right) \left(\frac{1}{\upsilon_z} \right) K_s (-1)^l \tilde{E}_R'(\kappa_l) - i\phi_s(L, \upsilon_\perp, \upsilon_z) \cot(\Omega_s L) \tag{3.49}$$

Finally, the solution of these relations for $\phi_s(0, \upsilon_\perp, \upsilon_z)$ and $\phi_s(L, \upsilon_\perp, \upsilon_z)$ is straightforward; after using some trigonometric identities, integral

$$\phi_s(L, \upsilon_\perp, \upsilon_z) = \left(\exp(2i\Omega_s L) - 1 \right) \sum_{l=-\infty}^{\infty} C_s(l) (-1)^l \tilde{E}_R'(\kappa_l) \tag{3.50}$$

$$\phi_s(0, \upsilon_\perp, \upsilon_z) = \sum_{l=-\infty}^{\infty} C_s(l) \tilde{E}_R'(\kappa_l) \left((-1)^l \exp(2i\Omega_s L) - 1 \right) \tag{3.51}$$

where

$$C_s(l) \equiv \left(\frac{i\Omega_s}{\Omega_s^2 - \kappa_l^2} \right) \left(\frac{1}{\upsilon_z} \right) K_s \tag{3.52}$$

Equations (3.50) and (3.51) give the sought-after expressions for the boundary values of ϕ_s needed in the evaluation of Equation (3.37).

3.4 Calculation of the Current Density

Using Equation (3.2) in Equation (3.34) and evaluating Equation (3.52), one has

$$C_s(l) = -\left(\frac{e_s}{m_s}\right)\left(\frac{1}{\upsilon_{Ts}}\right)^2\left(\frac{\upsilon_\perp}{\upsilon_z}\right)\left(\frac{i\Omega_s}{\Omega_s^2-\kappa_l^2}\right) f_{s0}\left(|\vec{\upsilon}|\right) \tag{3.53}$$

One can now substitute Equations (3.50) and (3.51) into Equation (3.37) and this intermediate result into Equation (3.31), upon remembering the definition $\tilde{E}_R'(\kappa_l) \equiv \tilde{E}_R(\kappa_l)\exp(-i\theta)$, to finally obtain the Fourier Transform of the current density within the plasma layer, viz,

$$\begin{aligned}\tilde{j}_R(\kappa_l) = -\sum_s e_s n_s\left(\frac{e_s}{m_s}\right)\left(\frac{1}{\upsilon_{Ts}}\right)^2 \int_0^\infty \int_0^{2\pi} \int_0^\infty \upsilon_\perp^3 f_{s0}\left(|\vec{\upsilon}|\right)\Bigg\{ &-2i\left(\frac{\Omega_s}{\Omega_s^2-\kappa_l^2}\right)\left(\frac{1}{\upsilon_z}\right)\tilde{E}_R(\kappa_l) \\ &+2i\left(\frac{\kappa_l}{\Omega_s^2-\kappa_l^2}\right)\left(\frac{1}{L}\right)\sum_{l'=-\infty}^{\infty}\left(\frac{i\Omega_s}{\Omega_s^2-\kappa_{l'}^2}\right)\left(\frac{1}{\upsilon_z}\right)\tilde{E}_R(\kappa_{l'})\left[(-1)^{l'}\exp(i\Omega_s L)-1\right] \\ &-\left(\frac{i}{\Omega_s-\kappa_l}\right)\left(\frac{1}{L}\right)\left(\exp(2i\Omega_s L)-1\right)(-1)^l\sum_{l'=-\infty}^{\infty}\left(\frac{i\Omega_s}{\Omega_s^2-\kappa_{l'}^2}\right)\left(\frac{1}{\upsilon_z}\right)(-1)^{l'}\tilde{E}_R(\kappa_{l'})\Bigg\} d\upsilon_\perp d\theta d\upsilon_z\end{aligned} \tag{3.54}$$

Using Equation (3.2) once again and performing the $\upsilon_\perp$ and θ integrations gives

$$\begin{aligned}\tilde{j}_R(\kappa_l) = \sum_s e_s n_s\left(\frac{1}{m_s}\right)\left(\frac{1}{\pi}\right)^{1/2}\left(\frac{1}{\upsilon_{Ts}}\right)\int_0^\infty \exp\left(-\frac{\upsilon_z^2}{\upsilon_{Ts}^2}\right)\Bigg\{ &2i\left(\frac{\Omega_s}{\Omega_s^2-\kappa_l^2}\right)\left(\frac{1}{\upsilon_z}\right)\tilde{E}_R(\kappa_l) \\ &+2\left(\frac{\kappa_l}{\Omega_s^2-\kappa_l^2}\right)\left(\frac{1}{L}\right)\sum_{l'=-\infty}^{\infty}\left(\frac{\Omega_s}{\Omega_s^2-\kappa_{l'}^2}\right)\left(\frac{1}{\upsilon_z}\right)\tilde{E}_R(\kappa_{l'})\left[(-1)^{l'}\exp(i\Omega_s L)-1\right] \\ &-\left(\frac{1}{\Omega_s-\kappa_l}\right)\left(\frac{1}{L}\right)\left(\exp(2i\Omega_s L)-1\right)(-1)^l\sum_{l'=-\infty}^{\infty}\left(\frac{i\Omega_s}{\Omega_s^2-\kappa_{l'}^2}\right)\left(\frac{1}{\upsilon_z}\right)(-1)^{l'}\tilde{E}_R(\kappa_{l'})\Bigg\} d\upsilon_z\end{aligned} \tag{3.55}$$

At this point, it facilitates further calculation to change variables using the prescription

$$w_s \equiv \frac{\upsilon_z}{\upsilon_{Ts}} \tag{3.56}$$

which gives

$$\Omega_s = \Omega_s(w_s) = \frac{\omega - \omega_{cs} + i\nu_s}{w_s\upsilon_{Ts}} \tag{3.57}$$

Incorporating Equations (3.56) and (3.57) into Equation (3.55) yields

$$\tilde{j}_R(\kappa_l) = \sigma(\kappa_l)\tilde{E}_R(\kappa_l) - \sum_{l'=-\infty}^{\infty}\sigma(\kappa_l,\kappa_{l'})\tilde{E}_R(\kappa_{l'}) \tag{3.58}$$

where

$$\sigma(\kappa_l) \equiv \sum_s 2ie_s^2 n_s \left(\frac{1}{m_s}\right)\left(\frac{1}{\pi}\right)^{1/2}\left(\frac{1}{\upsilon_{Ts}}\right)\int_0^\infty \frac{\Omega_s(w_s)}{\Omega_s^2(w_s)-\kappa_l^2}\left(\frac{1}{w_s}\right)\exp\left(-w_s^2\right)dw_s \tag{3.59}$$

is the single mode conductivity and

$$\begin{aligned}\sigma(\kappa_l,\kappa_{l'}) \equiv \sum_s 2e_s^2 n_s \left(\frac{1}{m_s}\right)\left(\frac{1}{\pi}\right)^{1/2}\left(\frac{1}{\upsilon_{Ts}}\right)\left(\frac{1}{L}\right)\int_0^\infty \Bigg\{ & \frac{\kappa_l \Omega_s(w_s)}{\left(\Omega_s^2(w_s)-\kappa_l^2\right)\left(\Omega_s^2(w_s)-\kappa_{l'}^2\right)} \\ & \cdot\left(\frac{1}{w_s}\right)\left(1-(-1)^{l'}\exp\left(i\Omega_s(w_s)L\right)\right)-\left(1-\exp\left(2i\Omega_s(w_s)L\right)\right) \\ & \cdot\frac{\Omega_s(w_s)}{\left(\Omega_s(w_s)-\kappa_l\right)\left(\Omega_s^2(w_s)-\kappa_{l'}^2\right)}\left(\frac{1}{w_s}\right)(-1)^{l+l'}\Bigg\}\exp\left(-w_s^2\right)dw_s\end{aligned} \tag{3.60}$$

is the multi-mode conductivity. Thus, the current density anywhere within the magnetized plasma layer due to the incident electromagnetic wave is given by

$$j_R(z) = \sum_{l=-\infty}^{\infty}\sigma(\kappa_l)\tilde{E}_R(\kappa_l)\exp(i\kappa_l z) - \sum_{l=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty}\sigma(\kappa_l,\kappa_{l'})\tilde{E}_R(\kappa_{l'})\exp(i\kappa_l z) \tag{3.61}$$

At this point, contact must be made with the current density entering into Maxwell's equations.

3.5 Connecting the Current Density in the Plasma Layer with the Maxwell Equations

One first needs to relate the auxiliary complex electric field $E_R(z) = E_x(z) + iE_y(z)$ of the incident wave to the governing Maxwell Equations. Returning to Equations (2.3) and (2.4) and employing Equations (3.5) and (3.6) gives in terms of the component fields

$$-\frac{\partial E_y}{\partial z}\hat{x} + \frac{\partial E_x}{\partial z}\hat{y} = ik\left(B_x\hat{x} + B_y\hat{y}\right) \tag{3.62}$$

and

$$-\frac{\partial B_y}{\partial z}\hat{x} + \frac{\partial B_x}{\partial z}\hat{y} = -ik\left(E_x\hat{x} + E_y\hat{y}\right) + \frac{4\pi}{c}\left(j_x\hat{x} + j_y\hat{y}\right) \tag{3.63}$$

Multiplying these relations by the auxiliary vector $\hat{x} + i\hat{y}$, as used earlier, gives, respectively,

$$\frac{\partial E_R}{\partial z} - kB_R = 0 \tag{3.64}$$

and

$$\frac{\partial B_R}{\partial z} + kE_R = -i\frac{4\pi}{c}j_R \tag{3.65}$$

where $B_R(z) \equiv B_x(z) + iB_y(z)$ and $j_R(z) \equiv j_x(z) + ij_y(z)$. From these two relations, the corresponding wave equation for E_R can be derived in the usual way,

$$\frac{\partial^2 E_R}{\partial z^2} + k^2 E_R = -ik\frac{4\pi}{c} j_R \tag{3.66}$$

As the equations remain amenable in the Fourier domain, the transform of Equation (3.66) is now needed to be used with Equation (3.58): thus, applying

$$\tilde{E}_R(\kappa_l) = \frac{1}{L}\int_0^L E_R(z)\exp(-i\kappa_l z)\,dz \tag{3.67}$$

by multiplying Equation (3.66) by $\exp(-i\kappa_l z)$, integrating by parts, and using Equation (3.64) yields

$$k\left(\frac{B_R(L)(-1)^l - B_R(0)}{L}\right) + i\kappa_l\left(\frac{E_R(L)(-1)^l - E_R(0)}{L}\right) - \kappa_l^2\tilde{E}_R(\kappa_l) + k^2\tilde{E}_R(\kappa_l) = -ik\frac{4\pi}{c}\tilde{j}_R(\kappa_l) \tag{3.68}$$

Finally, substituting Equation (3.58) into Equation (3.68) and rearranging terms gives

$$\begin{aligned}\left(\kappa_l^2 - k^2 - ik\frac{4\pi}{c}\sigma(\kappa_l)\right)\tilde{E}_R(\kappa_l) = k&\left(\frac{B_R(L)(-1)^l - B_R(0)}{L}\right)\\ &+ i\kappa_l\left(\frac{E_R(L)(-1)^l - E_R(0)}{L}\right) - ik\frac{4\pi}{c}\sum_{l'=-\infty}^{\infty}\sigma(\kappa_l,\kappa_{l'})\tilde{E}_R(\kappa_{l'})\end{aligned} \tag{3.69}$$

This equation connects the fields on the surface boundaries of the plasma layer to the Fourier transform of the fields within the plasma. This relation can be put into a more familiar form simply by dividing through by k^2 using Equation (3.57) and rearranging some factors to obtain

$$\begin{aligned}\left(\frac{\kappa_l^2}{k^2} - \tilde{\varepsilon}(\kappa_l)\right)\tilde{E}_R(\kappa_l) + &\sum_{l'=-\infty}^{\infty} S(\kappa_l,\kappa_{l'})\tilde{E}_R(\kappa_{l'})\\ &= i\frac{\kappa_l}{k}\left(\frac{E_R(L)(-1)^l - E_R(0)}{kL}\right) + \frac{B_R(L)(-1)^l - B_R(0)}{kL}\end{aligned} \tag{3.70}$$

with

$$\tilde{\varepsilon}(\kappa_l) \equiv 1 - \frac{2}{\sqrt{\pi}}\sum_s \frac{\omega_{ps}^2}{\omega}\int_0^\infty \frac{(\omega - \omega_{cs} + i\nu_s)\exp(-w_s^2)}{(\omega - \omega_{cs} + i\nu_s)^2 - \upsilon_{Ts}^2\kappa_l^2 w_s^2}\,dw_s \tag{3.71}$$

and

$$S\left(\kappa_l,\kappa_{l'}\right)\equiv i\frac{2}{\sqrt{\pi}}\sum_s\left(\frac{\omega_{ps}^2}{\omega L}\right)\int_0^\infty\frac{\omega-\omega_{cs}+i\nu_s}{\left(\omega-\omega_{cs}+i\nu_s\right)^2-\upsilon_{Ts}^2\kappa_{l'}^2w_s^2}\left\{\frac{\kappa_l\upsilon_{Ts}^2w_s^2}{\left(\omega-\omega_{cs}+i\nu_s\right)^2-\upsilon_{Ts}^2\kappa_l^2w_s^2}\right.$$
$$\cdot\left[1-\left(-1\right)^{l'}\exp\left(i\frac{\omega-\omega_{cs}+i\nu_s}{\upsilon_{Ts}w_s}L\right)\right] \tag{3.72}$$
$$\left.-\frac{\upsilon_{Ts}w_s\left(-1\right)^{l+l'}}{\omega-\omega_{cs}+i\nu_s-\upsilon_{Ts}\kappa_l w_s}\left[1-\exp\left(2i\frac{\omega-\omega_{cs}+i\nu_s}{\upsilon_{Ts}w_s}L\right)\right]\right\}\exp\left(-w_s^2\right)dw_s$$

where the plasma frequency ω_{ps} of the *s*-th charge species is given by

$$\omega_{ps}^2\equiv\frac{4\pi e_s^2n_s}{m_s} \tag{3.73}$$

Equation (3.70) forms the set of equations for the transform of the electric field within the plasma layer that will be considered in what is to follow.

Unfortunately, the evaluation of the integrals indicated in Equations (3.71) and (3.72) cannot be completed in its entirety. Although Equation (3.71) does indeed possess an analytical solution, its form makes impossible the analytical evaluation of the electric field from the transformed field $\tilde{E}_R\left(\kappa_l\right)$ in Equation (3.70). However, in the limits of weak spatial dispersion, the equations lend themselves to analytical evaluation. Of course, it must be established that the re-entry plasma can be treated as one possessing weak spatial dispersion. Again, it must be kept in mind that what is needed here are expressions for the reflection, transmission, and absorption coefficients at the $z=0$ and $z=L$ boundaries of the plasma layer. Thus, the transformation of the fields $\tilde{E}_R\left(\kappa_l\right)$ on the left side of Equation (3.70) must be evaluated at these surfaces. This is the goal of the following development.

4.0 Evaluation in the Limit of Weak Spatial Dispersion

4.1 The Limit of Weak Spatial Dispersion

Equation (3.71) can be rewritten as

$$\tilde{\varepsilon}\left(\kappa_l\right)=1-\frac{2}{\sqrt{\pi}}\sum_s\frac{\omega_{ps}^2}{\omega}\int_0^\infty\frac{\exp\left(-w_s^2\right)\Big/\left(\omega-\omega_{cs}+i\nu_s\right)}{1-\upsilon_{Ts}^2\kappa_l^2w_s^2\Big/\left(\omega-\omega_{cs}+i\nu_s\right)^2}dw_s \tag{4.1}$$

Letting $\left(\omega-\omega_{cs}+i\nu_s\right)\sim\left(\omega-\omega_{cs}\right)^2+\nu_s^2$, consider the terms within the denominator of Equation (4.1) in the case where

$$\frac{\upsilon_{Ts}\kappa_l w_s}{\left(\left(\omega-\omega_{cs}\right)^2+\nu_s^2\right)^{1/2}}<<1$$

which can be re-expressed as

$$\frac{\upsilon_{Ts}}{L\left(\left(\omega-\omega_{cs}\right)^2+\nu_s^2\right)^{1/2}}<\frac{1}{lw_s} \tag{4.2}$$

Now, the integral over the variable w_s is rapidly attenuated due to the presence of the exponential function. Hence, only small values of this variable need be considered. Take the value of this parameter to be on the order of unity. There then remains the range of values of the mode number *l*. In order to incorporate the long range effects within the plasma (for spatial dispersion effects), small values of this parameter need be used; thus, take $l=1$. Hence, the condition of Equation (4.2) can finally be written as

$$\frac{\upsilon_{Ts}}{L\left(\left(\omega-\omega_{cs}\right)^2+\nu_s^2\right)^{1/2}}<1 \tag{4.3}$$

At resonance where $\omega\sim\omega_{cs}$, this condition becomes

$$\upsilon_{Ts}<L\nu_s \tag{4.4}$$

which defines the region of weak spatial dispersion for this problem. The range over which this inequality holds must now be examined.

4.2 Weak Dispersion and a Re-entry Plasma

Since both the thermal velocity and collision frequency are functions of plasma temperature, it will be instructive to determine over what temperature ranges Equation (4.4) holds for nominal thicknesses and electron number densities. Figures 3 and 4 show the region of applicability of the constraint of Equation (4.4) for a one-component electron plasma of thickness $L=30\,\text{cm}$. As seen from these plots, the applicability region becomes smaller over a range of temperatures as the electron concentration decreases. Thus, for $n_e>10^{10}/\text{cm}^3$, the approximation of weak spatial dispersion can be expected to hold for nominal re-entry temperatures that are typically $T_e>3000\,\text{K}$. The case in which $n_e\le 10^{10}/\text{cm}^3$ requires one to consider the case of strong spatial dispersion.

In the case of weak spatial dispersion for this one-component plasma ($s\equiv e$), Equation (4.1) reduces to

$$\tilde{\varepsilon}\left(\kappa_l\right)\approx 1-\frac{2}{\sqrt{\pi}}\frac{\omega_{pe}^2}{\omega}\int_0^\infty\frac{\exp\left(-w_e^2\right)}{\left(\omega-\omega_{ce}+i\nu_e\right)}dw_e=1-\frac{1}{\omega}\frac{\omega_{pe}^2}{\left(\omega-\omega_{ce}+i\nu_e\right)} \tag{4.5}$$

The analysis of Equation (3.72) proceeds along the same lines where, it must be added, that the exponential factors oscillate away to zero. Also, with Equation (4.4) prevailing, the terms in Equation (3.72) can be neglected in this approximation. In this event, Equation (3.70) can be written, using Equation (4.5),

$$\left(\kappa_l^2-k^2+k^2A(\omega)\right)\tilde{E}_R\left(\kappa_l\right)=i\frac{\kappa_l}{L}\left(E_R\left(L\right)\left(-1\right)^l-E_R\left(0\right)\right)+\frac{k}{L}\left(B_R\left(L\right)\left(-1\right)^l-B_R\left(0\right)\right) \tag{4.6}$$

where

$$A(\omega) \equiv \frac{1}{\omega} \frac{\omega_{pe}^2}{(\omega - \omega_{ce} + i\nu_e)} \tag{4.7}$$

From Equation (4.6), expressions for the boundary values $E_R(0)$ and $E_R(L)$ can be obtained from which incident, reflected, and transmitted fields can be related finally giving rise to the associated coefficients. Thus, defining

$$D_l(\omega) \equiv \kappa_l^2 - k^2 + k^2 A(\omega) \tag{4.8}$$

one has

$$E_R(0) = \sum_{l=-\infty}^{\infty} \tilde{E}_R(\kappa_l) = \sum_{l=-\infty}^{\infty} \frac{1}{D_l(\omega)} \left(\frac{k}{L}\right) \left(B_R(L)(-1)^l - B_R(0)\right) \tag{4.9}$$

and

$$E_R(L) = \sum_{l=-\infty}^{\infty} \tilde{E}_R(\kappa_l)(-1)^l = \sum_{l=-\infty}^{\infty} \frac{1}{D_l(\omega)} \left(\frac{k}{L}\right) \left(B_R(L) - B_R(0)(-1)^l\right) \tag{4.10}$$

where, as noted earlier, the term with the coefficient κ_l sums to zero.

At this point, one needs to relate the magnetic fields $B_L(0)$ and $B_R(L)$ to the prevailing electric fields on the boundaries so as to obtain from Equations (4.9) and (4.10) two equations in the two unknowns $E_R(0)$ and $E_R(L)$. From these, the reflection and transmission coefficients will be obtained. To this end, from Equation (3.64),

$$B_R(z) = \frac{1}{k} \frac{\partial E_R(z)}{\partial z} \tag{4.11}$$

As for the fields above the plasma layer at $z = L$, one can write

$$E_R(z) = E_0 \exp\left(ik(L - z)\right) + E_{\text{Refl}} \exp\left(-ik(L - z)\right) \tag{4.12}$$

where E_0 is the incident electric field of the wave upon the plasma layer and E_{Refl} is the field reflected from the layer. Using Equation (4.12) in Equation (4.11) gives

$$B_R(z) = -i\left[E_0 \exp\left(ik(L - z)\right) - E_{\text{Refl}} \exp\left(-ik(L - z)\right)\right] \tag{4.13}$$

Hence, on the boundary $z = L$,

$$E_R(L) = E_0 + E_{\text{Refl}}, \qquad B_R(L) = -i(E_0 - E_{\text{Refl}}) \tag{4.14}$$

Similarly, for the transmitted field into the region $z < 0$,

$$E_R(z) = E_{\text{Trans}} \exp\left(ik(L - z)\right) \tag{4.15}$$

giving at the boundary $z = 0$

$$E_R(0) = E_{\text{Trans}} \exp(ikL), \qquad B_R(0) = -iE_{\text{Trans}} \exp(ikL) \tag{4.16}$$

Substituting Equations (4.14) and (4.16) into Equations (4.9) and (4.10), defining

$$r \equiv \frac{E_{\text{Refl}}}{E_0}, \qquad t \equiv \frac{E_{\text{Trans}}}{E_0} \tag{4.17}$$

as well as

$$S_1(\omega) \equiv \left(\frac{k}{L}\right) \sum_{l=-\infty}^{\infty} \frac{(-1)^l}{D_l(\omega)} = -\frac{\csc\left(kL\sqrt{1-A(\omega)}\right)}{\sqrt{1-A(\omega)}} \tag{4.18}$$

$$S_2(\omega) \equiv \left(\frac{k}{L}\right) \sum_{l=-\infty}^{\infty} \frac{1}{D_l(\omega)} = -\frac{\cot\left(kL\sqrt{1-A(\omega)}\right)}{\sqrt{1-A(\omega)}} \tag{4.19}$$

yields

$$t \exp(ikL) = -iS_1(\omega)(1-r) + iS_2(\omega) t \exp(ikL) \tag{4.20}$$

$$1 + r = -iS_2(\omega)(1-r) + iS_1(\omega) t \exp(ikL) \tag{4.21}$$

The solutions of these simultaneous equations for *r* and *t* will then yield the related coefficients of reflection $R \equiv |r|^2$ and $T \equiv |t|^2$ as well as the related coefficient of absorption $A \equiv 1 - |r|^2 - |t|^2$. Equations (4.20) and (4.21) give, in this case of weak spatial dispersion,

$$r = \frac{1 - S_1^2(\omega) + S_2^2(\omega)}{\left(i + S_2(\omega)\right)^2 - S_1^2(\omega)} \tag{4.22}$$

and

$$t = \frac{2iS_1(\omega)\exp(-ikL)}{\left(i + S_2(\omega)\right)^2 - S_1^2(\omega)} \tag{4.23}$$

Figures 5 to 7 display the results of the calculation of these relations for three electron concentrations for a plasma of thickness $L = 30\,\text{cm}$ and $T_e = 3000\,\text{K}$ [Sharkey, 2004] upon which an electromagnetic wave of frequency $f = 2.0\,\text{GHz}$ is incident. As Figs. 5 and 6 show, the reflection coefficient (*R*) remains at 1.0 and the transmission coefficient (*T*) remains at 0.0 until the magnitude of the applied magnetic field goes above 500 Gauss. The absorption coefficient (*A*) also begins to rise (clearly shown in Fig. 6). After this point, the coefficients rapidly oscillate and tend to their limits $R \to 0.0$, $T \to 1.0$, and $A \to 0.0$. For $n_e = 1.0 \times 10^{11}/\text{cm}^3$, these limits are quickly approached. At $n_e = 1.0 \times 10^{12}/\text{cm}^3$, oscillations of the values of *R* and *T* remain beyond the magnitude of 10,000 Gauss although the values are clearly separated. However, Fig. 7 for the case of $n_e = 1.0 \times 10^{13}/\text{cm}^3$ shows that the oscillations remain mixed together where the values of these coefficients exchange local maxima and minima.

5.0 Conclusion

It is thus demonstrated that a magnetic field of nominal magnitude applied to the plasma makes the plasma transparent to frequencies smaller than the plasma frequency thus substantiating the magnetic windows concept. Magnetic fields on the order of 4 kGauss will render a plasma transparent at 2 GHz with an electron concentration of 10^{12} electrons/cm^3. As the plasma density evolves throughout re-entry, the reflection, transmission, and absorption coefficients can oscillate over large ranges for a fixed applied magnetic field. Hence, a variable magnetic field controlled, e.g, by the value of received power, must be considered. Also, the case of strong spatial dispersion as well as the situation intermediate to weak and strong spatial dispersion must yet be considered.

This example of a homogeneous plasma in a homogeneous magnetic field is certainly an idealization of what would exist in reality. However, these results provide impetus for further work with more realistic situations. Although Halbach magnets can be used to generate the applied magnetic field, some spatial variation of the field should be introduced. The variations inherent in the plasma thickness, temperature, and density must be addressed in the specification and design of a closed loop control system to adjust the external magnetic field to maintain transparency. Finally, account of the other components of the plasma must be made. In fact, with the surface ablation that occurs during re-entry, one should consider a dusty plasma, the kinetics of which can be vastly different from the simple one-component electron plasma considered here.

Appendix A.—A Model of the Effective Collision Frequency

For purposes of the discussion given here, consider a two component plasma made up of electrons and ions in an electric field $\vec{E}$. The Boltzmann equation including the associated collision integral for an electron in a two-component plasma is given by

$$\frac{\partial f}{\partial t} + \vec{\upsilon} \cdot \vec{\nabla}_{\vec{r}} f + \frac{e}{m} \vec{E} \cdot \vec{\nabla}_{\vec{\upsilon}} f + S_{\text{coll}} = 0 \tag{A1.1}$$

where the complete collision integral is

$$S_{\text{coll}} \equiv \int\int q(u,\theta) u \left[f(\vec{\upsilon}) F(\vec{\upsilon}_1) - f(\vec{\upsilon}') F(\vec{\upsilon}_1') \right] d^3\upsilon d\Omega \tag{A1.2}$$

In the case where the collision is limited to the scattering of an electron from an ion, $q(u,\theta)$ is the differential cross-section of electron scattering where θ is the scattering angle, $u \equiv |\vec{\upsilon} - \vec{\upsilon}_1|$ is the relative velocity of the electron with respect to the ion, i.e., $\vec{\upsilon}$ and $\vec{\upsilon}_1$ are the velocities, respectively, of the electron and ion after collision whereas $\vec{\upsilon}'$ and $\vec{\upsilon}_1'$ are those before the collision. Additionally, $F(\vec{\upsilon}_1)$ is the velocity distribution of the ions which is usually taken to be given by the Maxwell distribution of the form of Equation (3.2), and $d\Omega = \sin\theta d\theta d\phi$ is the differential scattering angle.

Simplifications will now be introduced using what is known about the electrons in the plasma. First, it is assumed that the thermal velocity of the electrons and ions are much greater than the associated directed velocities in the electric field. In this isotropic plasma case (i.e., where the magnetic field $\vec{B} = 0$), the spatial gradient of the distribution *f* is directed along the *z* axis, parallel to $\vec{E}$. Hence, one can treat the "directional" part of *f* as a perturbation in velocity space. The distribution can thus be expanded into zero-order spherical polynomials, i.e., Legendre polynomials $P_l(\cos\vartheta)$ where ϑ is the angle between $\vec{E}$ and $\vec{\upsilon}$, viz.,

$$f = f(\vec{r}, \vec{\upsilon}, t) = \sum_{l=0}^{\infty} P_l(\cos\vartheta) f_l(\vec{r}, \vec{\upsilon}, t) \tag{A1.3}$$

Writing

$$\vec{E} \cdot \vec{\nabla}_{\vec{\upsilon}} f = E \cos\vartheta \frac{\partial f}{\partial \upsilon} + \frac{E \sin^2\vartheta}{\upsilon} \frac{\partial f}{\partial(\cos\vartheta)} \tag{A1.4}$$

and using Equations (A1.3) and (A1.4) in Equation (A1.1) give the following system of equations for f_l

$$\begin{gathered}
\frac{\partial f_0}{\partial t} + \frac{\upsilon}{3}\frac{\partial f_1}{\partial t} + \frac{eE}{3m\upsilon^2}\frac{\partial}{\partial \upsilon}\left(\upsilon^2 f_1\right) + S_0 = 0 \\
\frac{\partial f_1}{\partial t} + \upsilon\left(\frac{\partial f_0}{\partial z} + \frac{2}{5}\frac{\partial f_2}{\partial z}\right) + \frac{eE}{m}\left(\frac{\partial f_0}{\partial \upsilon} + \frac{2}{5\upsilon^3}\frac{\partial}{\partial \upsilon}\left(\upsilon^3 f_2\right)\right) + S_1 = 0 \\
\frac{\partial f_2}{\partial t} + \upsilon\left(\frac{2}{3}\frac{\partial f_1}{\partial z} + \frac{7}{3}\frac{\partial f_3}{\partial z}\right) + \frac{eE}{m}\left(\frac{2}{3}\upsilon\frac{\partial}{\partial \upsilon}\left(\frac{1}{\upsilon}\right) f_1 + \frac{3}{7\upsilon^4}\frac{\partial}{\partial \upsilon}\left(\upsilon^4 f_3\right)\right) + S_2 = 0 \\
\vdots
\end{gathered} \tag{A1.5}$$

where

$$
\begin{aligned}
S_l &\equiv \frac{2l+1}{4\pi}\int P_l\left(\cos\vartheta\right) S d\Theta \\
&= \frac{2l+1}{4\pi}\iiint q(u,\theta) u P_l\left(\cos\vartheta\right)\left[\sum_{l'=0}^{\infty} f_{l'} P_{l'}\left(\cos\vartheta'\right) F - F'\sum_{l'=0}^{\infty} f_{l'}' P_{l'}\left(\cos\vartheta''\right)\right] d^3\upsilon_1 d\Omega d\Theta \\
&= \iint q(u,\theta) u\left(F f_l - F' f_l' P_l\left(\cos\theta\right)\right) d^3\upsilon d\Omega
\end{aligned}
\tag{A1.6}
$$

In arriving at the result of Equation (A1.6), the integration over $d\Theta = \sin\vartheta d\vartheta d\varphi$ made use of the fact that

$$\cos\vartheta'' = \cos\theta\cos\vartheta' + \sin\theta\sin\vartheta'\cos\varphi$$

and the addition theorem for Legendre polynomials was employed.

The chain of equations of Equation (A1.5) can be terminated at the second one if the perturbation component f_2 can be neglected compared to the fundamental component f_0, i.e.,

$$\frac{\partial f_0}{\partial\upsilon} >> \frac{1}{\upsilon^3}\frac{\partial}{\partial\upsilon}\left(\upsilon^3 f_2\right) \tag{A1.7}$$

Before Equation (A1.7) can be established, one now needs to consider the expressions for the scattering perturbations S_0, S_1, and S_2. To this end, considerations will only be made of elastic collisions; in this case, Equation (A1.6) gives for S_1

$$S_1 = \iint q(u,\theta) u\left(f_1(\vec{\upsilon}) F(\vec{\upsilon}_1) - P_1\left(\cos\theta\right) f_1(\vec{\upsilon}') F(\vec{\upsilon}_1')\right) d^3\upsilon_1 d\Omega \tag{A1.8}$$

Assuming that the energy of the electron changes only slightly after collision with the ion, one has that $|\vec{\upsilon}'| \approx |\vec{\upsilon}|$ and $|\vec{\upsilon}_1'| \approx |\vec{\upsilon}_1|$ where $|\vec{\upsilon}| >> |\vec{\upsilon}_1|$. In this instance, Equation (A1.8) yields

$$S_1 = f_1(\vec{\upsilon})\iint q(u,\theta)\upsilon F(\vec{\upsilon}_1)\left(1-\cos\theta\right) d^3\upsilon_1 d\Omega = \nu_1(\upsilon) f_1(\upsilon) \tag{A1.9}$$

where the velocity dependent collision frequency is defined by

$$\nu_1(u) = N_{\text{ion}}\upsilon\int q(\upsilon,\theta)\left(1-\cos\theta\right) d\Omega, \quad N_{\text{ion}} \equiv \int F(\upsilon_1) d^3\upsilon_1 \tag{A1.10}$$

Equation (A1.9) begins to form the basis of the Krook model introduced in Equation (2.7) where a velocity independent collision frequency ν_s was used. It is the central purpose of this Appendix to obtain an expression of this velocity independent collision frequency from collision theory. Before this can be accomplished, however, it remains to establish the prevailing conditions that allow the termination of the system of equations of Equation (A1.5).

One can similarly show that

$$
\begin{aligned}
S_2 &= \int\int q(u,\theta) u\left(f_2(\vec{\upsilon}) F(\vec{\upsilon}_1) - P_2(\cos\theta)\, f_2(\vec{\upsilon}') F(\vec{\upsilon}_1') \right) d^3\upsilon_1 d\Omega \\
&= f_2(\vec{\upsilon}) \int\int q(u,\theta) \upsilon F(\vec{\upsilon}_1) \left(1 - \frac{3\cos 2\theta + 1}{4} \right) d^3\upsilon_1 d\Omega \\
&= \nu_2(\upsilon)\, f_2(\upsilon)
\end{aligned}
\tag{A1.11}
$$

where

$$
\nu_2(u) = N_{\text{ion}} \upsilon \int q(\upsilon,\theta) \left(1 - \frac{3\cos 2\theta + 1}{4} \right) d\Omega, \quad N_{\text{ion}} \equiv \int F(\upsilon_1) d^3\upsilon_1
\tag{A1.12}
$$

Thus, the quantities $\nu_1(\upsilon)$ and $\nu_2(\upsilon)$ are of the same order of magnitude and, for purposes of the discussion to follow, $\nu(\upsilon) \equiv \nu_1(\upsilon) \approx \nu_2(\upsilon)$. Finally, an expression for S_0 needs to be secured. The calculation is trivial upon using Equation (A1.6) and noting the fact that energy exchange during collisions does not occur with the approximations used above; hence, to first order, $S_0 = 0$. Finally, specializing to the special case of spatial homogeneity, $\partial/\partial z = 0$ in Equations (A1.5). Given the developments above, these relations reduce to

$$
\frac{\partial f_0}{\partial t} + \frac{eE}{3m\upsilon^2} \frac{\partial}{\partial \upsilon} \left(\upsilon^2 f_1 \right) = 0
\tag{A1.13}
$$

$$
\frac{\partial f_1}{\partial t} + \frac{eE}{m} \frac{\partial f_0}{\partial \upsilon} + \nu(\upsilon)\, f_1 = 0
\tag{A1.14}
$$

$$
\frac{\partial f_2}{\partial t} + \frac{eE}{m} \left(\frac{2}{3} \upsilon \frac{\partial}{\partial \upsilon} \left(\frac{1}{\upsilon} \right) f_1 + \frac{3}{7\upsilon^4} \frac{\partial}{\partial \upsilon} \left(\upsilon^4 f_3 \right) \right) + \nu(\upsilon)\, f_2 = 0
\tag{A1.15}
$$

Consider now the steady state situation where $f_1, f_2 \sim \exp(-i\omega t)$ and hence $\partial f_1/\partial t \sim -i\omega f_1$ and $\partial f_2/\partial t \sim -i\omega f_2$. Using these in Equations (A1.14) and (A1.15), dropping the second term within the parentheses of Equation (A1.15) and solving for $|f_2|$ gives

$$
|f_2| \sim \frac{e^2 E^2}{m^2 \left(\omega^2 + \nu^2 \right)} \upsilon \left| \frac{\partial}{\partial \upsilon} \left(\frac{1}{\upsilon} \frac{\partial f_0}{\partial \upsilon} \right) \right|
\tag{A1.16}
$$

Hence, the condition of Equation (A1.7) becomes

$$
f_0 >> \frac{e^2 E^2}{m^2 \left(\omega^2 + \nu^2 \right)} \left(\frac{1}{\upsilon^2} \right) \left| \frac{\partial}{\partial \upsilon} \left(\upsilon^2 \frac{\partial f_0}{\partial \upsilon} \right) \right|
\tag{A1.17}
$$

If Equation (A1.17) holds, the original system of equations of Equation (A1.5) reduce to two:

$$
\frac{\partial f_0}{\partial t} + \frac{eE}{3m\upsilon^2} \frac{\partial}{\partial \upsilon} \left(\upsilon^2 f_1 \right) = 0
\tag{A1.18}
$$

$$\frac{\partial f_1}{\partial t}+\frac{eE}{m}\frac{\partial f_0}{\partial \upsilon}+\nu(\upsilon)\,f_1=0 \tag{A1.19}$$

Finally, if the electric field is weak enough where the second term of Equation (A1.18) can be taken as a first order perturbation, the symmetric portion of the distribution *f* retains its Maxwellian nature and the solution to the problem reduces to (remembering Equation (A1.3))

$$\begin{aligned} f &= f_0 + f_1\cos\theta \\ \frac{\partial f_1}{\partial t}+\frac{eE}{m_e}\frac{\partial f_0}{\partial \upsilon}+\nu(\upsilon)\,f_1&=0 \end{aligned} \tag{A1.20}$$

In what is to follow, it is found advantageous to give the perturbation f_1 a vector character by writing Equation (A1.20) as

$$f = f_0+\frac{\vec{f}_1\cdot\vec{\upsilon}}{\upsilon} \tag{A1.21}$$

$$\frac{\partial \vec{f}_1}{\partial t}+\frac{e\vec{E}}{m_e}\frac{\partial f_0}{\partial \upsilon}+\nu(\upsilon)\,\vec{f}_1=0 \tag{A1.22}$$

where $\vec{f}_1$ is given the direction of $\vec{E}$.

The total current induced in the plasma by the free electrons is given by

$$\vec{j}=en_e\int \vec{\upsilon}\,f(\vec{\upsilon})d^3\upsilon \tag{A1.23}$$

Using Equation (A1.21) and converting the υ-integration into one in spherical coordinates gives, noting the isotropy of f_0,

$$\vec{j}=2\pi en_e\int_0^\infty\int_0^{2\pi}\upsilon^3\vec{f}_1\sin\theta\,d\theta\,d\upsilon \tag{A1.24}$$

Now, from Equation (A1.22), one has in the steady state

$$\vec{f}_1=-\frac{\frac{e\vec{E}}{m_e}\frac{\partial f_0}{\partial \upsilon}}{\nu(\upsilon)-i\omega} \tag{A1.25}$$

Using Equation (3.2) in this result and substituting into Equation (A1.24) yields

$$\vec{j}=\frac{8e^2n_e\vec{E}}{3\sqrt{\pi}m_e}\left(\int_0^\infty\frac{\nu(x)x^4\exp(-x^2)}{\omega^2+\nu^2(x)}dx+i\omega\int_0^\infty\frac{x^4\exp(-x^2)}{\omega^2+\nu^2(x)}dx\right) \tag{A1.26}$$

where $x\equiv\upsilon\sqrt{m_e/2k_BT_e}$. In the event that the collision frequency is independent of the velocity, i.e., $\nu(x)=\nu=\text{const.}$, the integrals in Equation (A1.26) can be performed and give the result

$$\vec{j} = \frac{e^2 n_e \vec{E}}{m_e}\left(\frac{\nu}{\omega^2+\nu^2} + i\omega\frac{1}{\omega^2+\nu^2}\right) \tag{A1.27}$$

This result is identical to the one that is obtained by using elementary considerations. That is, using the concept of an effective collision frequency ν_{eff}, one simply uses Newton's Law of motion for an electron in an electric field and writes

$$m_e \frac{d\vec{\upsilon}}{dt} = e\vec{E} - m_e \nu_{\text{eff}} \vec{\upsilon} \tag{A1.28}$$

Here, the effective collision frequency enters as a friction term. Again, considering the steady state and solving for the velocity, one gets for the associated current

$$\vec{j} = e n_e \vec{\upsilon} = \frac{e^2 n_e \vec{E}}{m_e}\left(\frac{\nu_{\text{eff}}}{\omega^2+\nu_{\text{eff}}^2} + i\omega\frac{1}{\omega^2+\nu_{\text{eff}}^2}\right) \tag{A1.29}$$

In order to reconcile these two approaches, one considers the limiting case where $\omega^2 >> \nu_{\text{eff}}^2, \nu^2(x)$. Taking Equations (A1.26) and (A1.29) in this limit, equating the two results and solving for ν_{eff} gives in terms of the velocity dependent collision frequency

$$\nu_{\text{eff}} = \frac{8}{3\sqrt{\pi}}\int_0^\infty \nu(x)\, x^4 \exp\left(-x^2\right) dx \tag{A1.30}$$

It is now necessary to find an expression for $\nu(x)$ defined by Equation (A1.10), given the scattering situation assumed here. That is, elastic scattering of a fast moving electron from an essentially stationary heavy ion. In this Coulomb scattering case, one must use the Rutherford scattering formula for the cross-section required in Equation (A1.10), viz.,

$$q(\upsilon,\theta) = \left(\frac{e^2}{2m_e\upsilon^2}\right)^2 \frac{1}{\sin^4\left(\theta/2\right)} \tag{A1.31}$$

Substituting this expression into Equation (A1.10) and performing the azimuthal integration,

$$\begin{aligned} \nu(\upsilon) &= 2\pi n_{\text{ion}}\upsilon\left(\frac{e^2}{2m_e\upsilon^2}\right)^2 \int_{\theta_{\min}}^\infty \frac{1-\cos\theta}{\sin^4\left(\theta/2\right)}\sin\theta d\theta \\ &= 2\pi n_{\text{ion}}\left(\frac{e^4}{m_e^2\upsilon^3}\right)\ln\left(1+\cot^2\left(\frac{\theta_{\min}}{2}\right)\right) \end{aligned} \tag{A1.32}$$

where $\theta_{\min}$ is the minimum scattering angle which is related to the maximum value of the impact parameter $b_{\max}$,

$$\tan\left(\theta_{\min}/2\right) = \frac{e^2}{m_e\upsilon^2 b_{\max}} \tag{A1.33}$$

Here, the impact parameter is determined by the fact that in a plasma where the interaction between the ions and electrons is through a Coulomb field only, distances on the order of the Debye radius r_D cannot be exceeded since this is the maximum distance at which substantial interaction occurs between the electron and ion; at distances greater than this, the field of the ion decreases exponentially. Hence, one can write

$$b_{\max} = r_D = \left(\frac{k_B^2 T_{\text{ion}} T_e}{4\pi e^2 n_{\text{ion}} \left(k_B T_{\text{ion}} + k_B T_e \right)} \right)^{1/2} = \left(\frac{k_B T_e}{8\pi e^2 n_{\text{ion}}} \right)^{1/2} \tag{A1.34}$$

where the last result issues from the assumption that $T_{\text{ion}} \approx T_e$. For a typical re-entry plasma, $r_D \sim 10^{-4}$ cm. Given the relative values of the parameters involved, one can write from Equation (A1.33),

$$\theta_{\min} = 2\tan^{-1}\left(\frac{e^2}{m_e \upsilon^2 r_D} \right) \approx \frac{2e^2}{m_e \upsilon^2 r_D} \tag{A1.35}$$

Substituting this result into Equation (A1.32) and series expanding the $\cot^2$ function finally gives

$$\nu(\upsilon) = 2\pi n_{\text{ion}} \left(\frac{e^4}{m_e^2 \upsilon^3} \right) \ln\left(1 + \frac{r_D^2 m_e^2 \upsilon^4}{e^4} \right) \tag{A1.36}$$

Finally, applying the substitution $\upsilon = x\sqrt{2k_B T_e / m_e}$ in Equation (A1.36) and using this result in Equation (A1.30) yields

$$\begin{aligned} \nu_{\text{eff}} &= \frac{8}{3\sqrt{\pi}} \left(2\pi n_{\text{ion}} \right) \left(\frac{e^4}{m_e^2} \right) \left(\frac{m_e}{2k_B T_e} \right)^{3/2} \int_0^\infty x \ln\left(1 + A^2 x^4 \right) \exp\left(-x^2 \right) dx \\ &= \frac{8}{3\sqrt{\pi}} \left(\frac{2\pi n_{\text{ion}} e^4}{m_e^2} \right) \left(\frac{m_e}{2k_B T_e} \right)^{3/2} \left[\frac{1}{2} \sin\left(\frac{1}{A} \right) \left(\pi - 2\,\text{si}\left(\frac{1}{A} \right) \right) - \cos\left(\frac{1}{A} \right) \text{ci}\left(\frac{1}{A} \right) \right] \end{aligned} \tag{A1.37}$$

where $A \equiv 2k_B T_e r_D / e^2$ and $\text{ci}(\cdots)$ and $\text{si}(\cdots)$ are the cosine and sine integrals, respectively. For a typical re-entry plasma, $A \sim 10^2 >> 1$. Thus, expanding the functions within the brackets of Equation (A1.37) in an ascending series, the sine terms are negligible and one is left with $\cos(1/A)\,\text{ci}(1/A) \approx -\ln A + \gamma$ where γ is Euler's constant, $\gamma \approx 0.577$. Hence, Equation (A1.37) reduces to

$$\nu_{\text{eff}} = \frac{8}{3\sqrt{\pi}} \left(\frac{2\pi n_{\text{ion}} e^4}{m_e^2} \right) \left(\frac{m_e}{2k_B T_e} \right)^{3/2} \ln\left(\frac{1.12 k_B T_e r_D}{e^2} \right) \tag{A1.38}$$

This is the expression for the effective collision frequency used in this work. To make contact with the notation used in the text, one has $\nu_s = \nu_{\text{eff}}$ where $s \equiv e$. It must be remembered that this formulation only accounts for elastic collisions between electrons and heavy ions. Of course, other scattering processes can occur such as elastic and inelastic collisions with molecules, collisions with dust grains in which charge transfer can also be attendant with impact, etc.

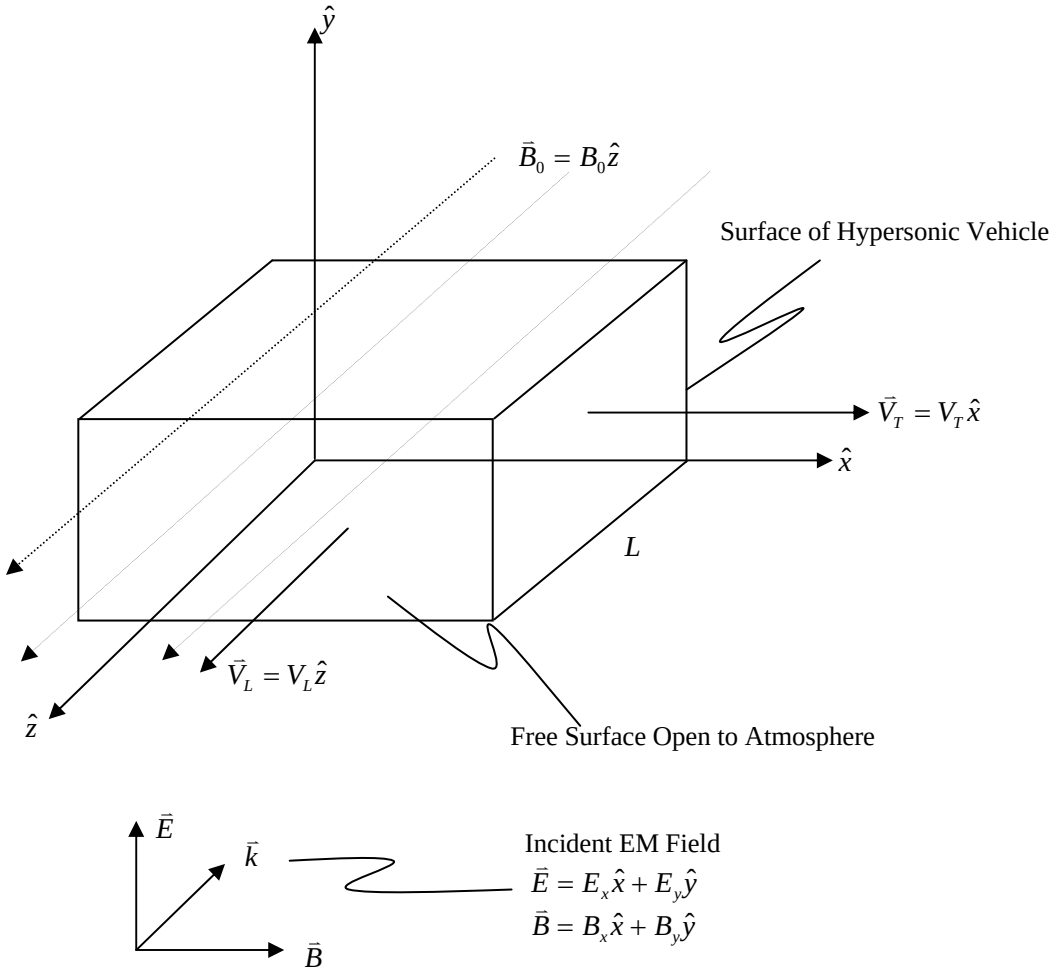


Figure 1.—The flowing plasma and its various fields and velocities.

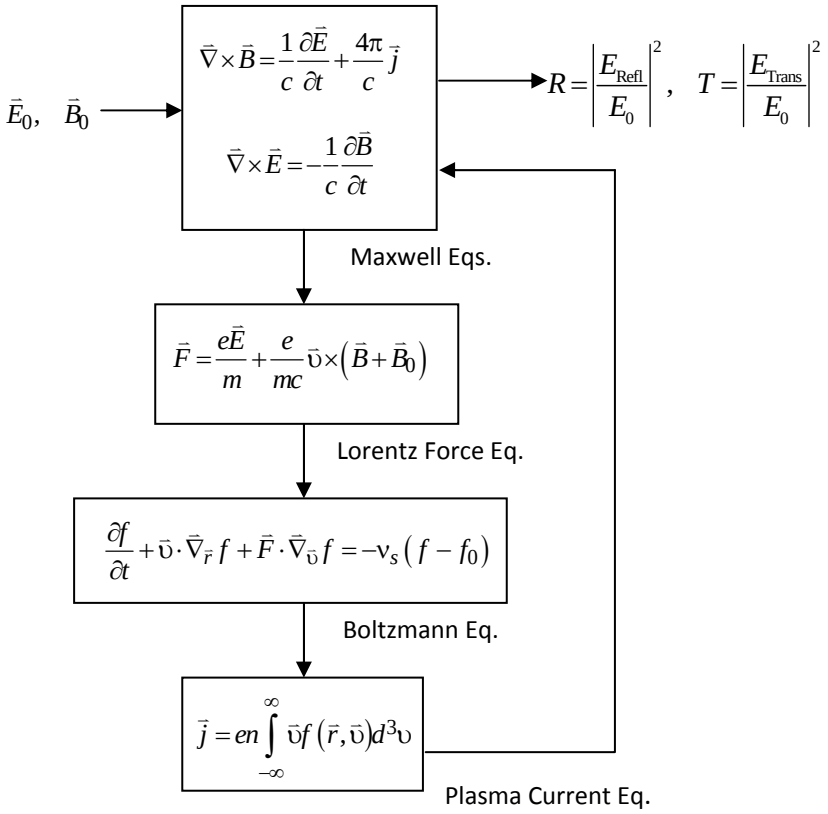


Figure 2.—Electromagnetic and kinetic equations for a plasma in an external magnetic field and an incident electromagnetic wave.

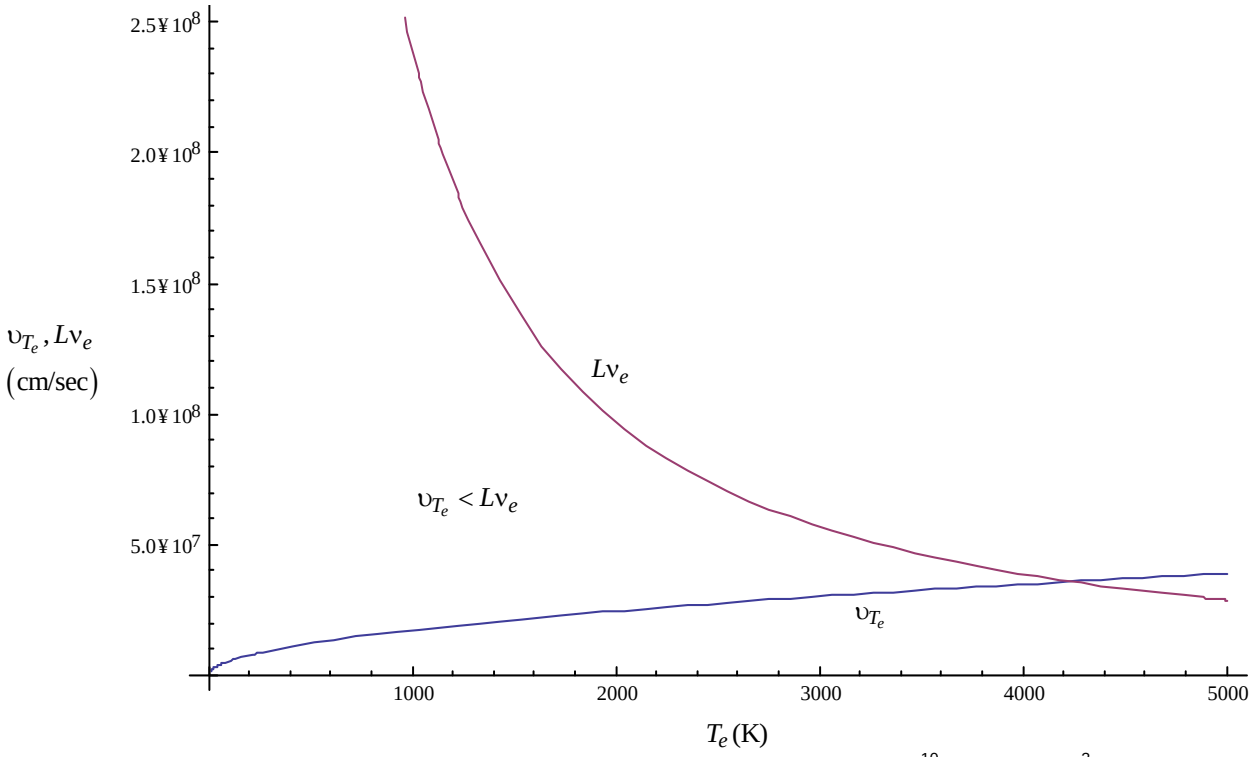


Figure 3.—Region of applicability for weak spatial dispersion approximation for n_e=10^{10} electrons/cm^{3}and L=30 cm.

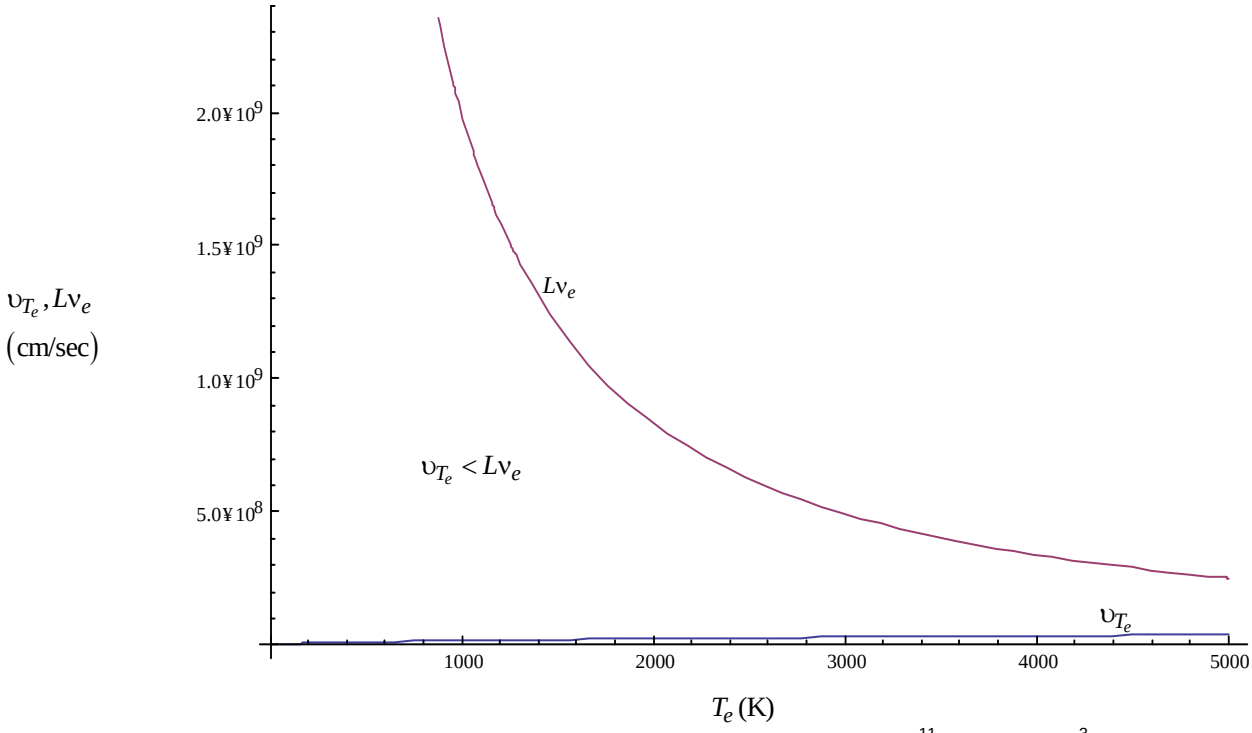
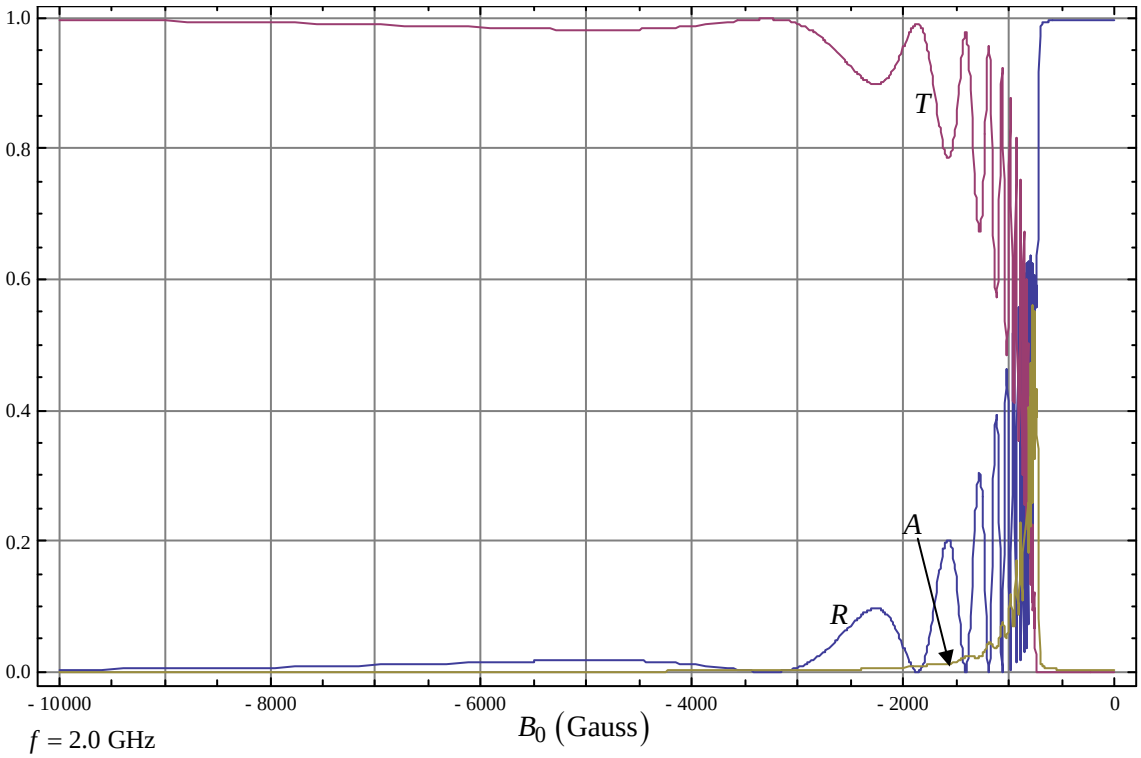


Figure 4.—Region of applicability for weak spatial dispersion approximation for n_e=10^{11} electrons/cm^3 and L=30 cm.

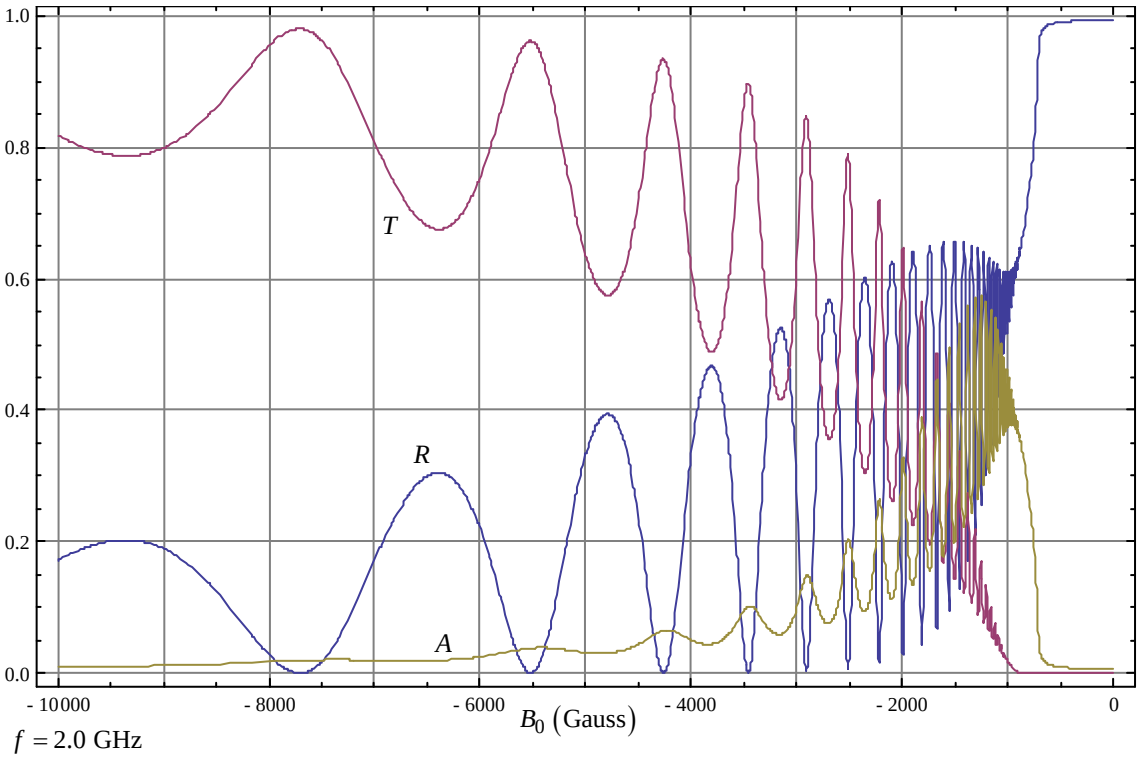


f = 2.0 GHz

T_e = 3000 K

L = 30 cm

Figure 5.—Reflection, transmission, and absorption coefficients versus applied magnetic field strength for n_e=10^{11} electrons/cm^3.



$f = 2.0$ GHz

$T_e = 3000$ K

$L = 30$ cm

Figure 6.—Reflection, transmission, and absorption coefficients versus applied magnetic field strength for n_e=10^{12} electrons/cm^3.

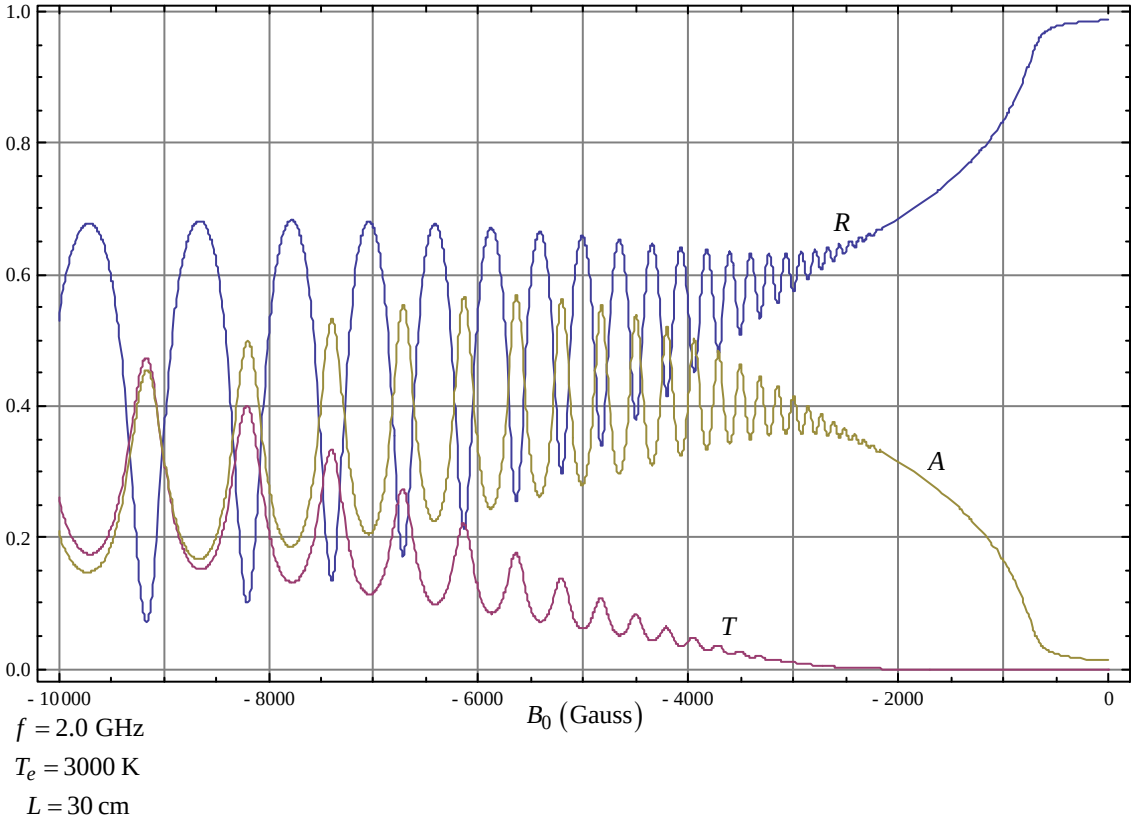


Figure 7.—Reflection, Transmission, and Absorption Coefficients Versus Applied Magnetic Field Strength for n_e=10^{13} electrons/cm^3.

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REPORT DOCUMENTATION PAGE	Form Approved OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
01-12-2009	Technical Memorandum	

4. TITLE AND SUBTITLE
Analysis of Electromagnetic Wave Propagation in a Magnetized Re-Entry Plasma Sheath Via the Kinetic Equation

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Manning, Robert, M.

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER
WBS 599489.02.07.03.04.02.02

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
John H. Glenn Research Center at Lewis Field
Cleveland, Ohio 44135-3191

8. PERFORMING ORGANIZATION REPORT NUMBER
E-17149

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSORING/MONITOR'S ACRONYM(S)
NASA

11. SPONSORING/MONITORING REPORT NUMBER
NASA/TM-2009-216096

12. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified-Unlimited
Subject Categories: 04 and 75
Available electronically at http://gltrs.grc.nasa.gov
This publication is available from the NASA Center for AeroSpace Information, 443-757-5802

13. SUPPLEMENTARY NOTES

14. ABSTRACT
Based on a theoretical model of the propagation of electromagnetic waves through a hypersonically induced plasma, it has been demonstrated that the classical radiofrequency communications blackout that is experienced during atmospheric reentry can be mitigated through the appropriate control of an external magnetic field of nominal magnitude. The model is based on the kinetic equation treatment of Vlasov and involves an analytical solution for the electric and magnetic fields within the plasma allowing for a description of the attendant transmission, reflection and absorption coefficients. The ability to transmit through the magnetized plasma is due to the 'magnetic windows' that are created within the plasma via the well-known 'whistler modes' of propagation. The case of 2 GHz transmission through a re-entry plasma is considered. The coefficients are found to be highly sensitive to the prevailing electron density and will thus require a dynamic control mechanism to vary the magnetic field as the plasma evolves through the re-entry phase.

15. SUBJECT TERMS
Radio transmission; Wave propagation; Plasmas; Spacecraft communication; Blackout

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			STI Help Desk (email:help@sti.nasa.gov)
U	U	U	UU	38	19b. TELEPHONE NUMBER *(include area code)* 443-757-5802