Dispersion of Dissolved or Suspended Materials in Flowing Streams

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By ROBERT E. GLOVER

TRANSPORT OF RADIONUCLIDES BY STREAMS

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ABSTRACT

The findings on stream dispersion described here were obtained by postulating certain possible types of dispersal mechanism, developing the characteristics of these mechanisms analytically, and comparing the results so obtained with data obtained from tests.

The possibility that the dispersal mechanism was a diffusion process was first considered. Analysis of this mechanism led to the conclusion that maximum salinities due to a local introduction of salinity at an upstream point should vary inversely as the square root of the distance from this point. This conclusion was confirmed by tests made on the 400-foot long flume at Allenspark and is in accord with the results of salt velocity tests made on long conduits. It is also in agreement with the results of theoretical work dealing with the combined effect of velocity variations through the cross section of a flowing stream and turbulence (Taylor, G. I., 1954; and Elder, 1959). The dispersing mechanism is therefore identified as being of the diffusion type.

A second possibility considered was that dispersion in the downstream direction was predominantly due to velocity variations in the cross section of the stream. This postulate yielded a formula which indicated that the maximum salinity observed at a downstream point should vary inversely as the distance from the point of introduction. This variation is not confirmed by the data obtained from tests in spite of the fact that these velocity variations are the most potent single factor in the mechanism of longitudinal dispersion. The importance of lateral mixing due to turbulence as a factor in the longitudinal dispersion mechanism must therefore be recognized. This finding is in accord with theoretical developments.

Comparisons of theoretical and test data show a very satisfactory agreement between theoretical findings and flume-test results, but natural streams yield diffusion coefficients approximately 10 to 50 times as large as those obtained on theoretical grounds.

Lateral-diffusion coefficients are much smaller than those for longitudinal diffusion because turbulence is the primary factor in lateral diffusion. In the same stream, the ratio of the two coefficients is of the order of 300.

INTRODUCTION

Streams have long been used as a means for disposal of wastes. It is a common practice, for example, to discharge sewage into flowing streams where oxidation and bacterial actions can destroy the organic materials it contains. The dangers inherent in such practices have long been realized, but with the growing accumulation of nuclear wastes a new danger from contamination by radioactive material may be added. It is difficult to insure that dangerous contaminations cannot happen in some way because, even though the radioactive material is buried, ground water may carry some of it in solution and discharge it into a stream. It is the purpose of this investigation to study the characteristics of the process of dispersal so that, if it is discovered that dangerous materials have escaped, estimates may be made of their location and concentration.

At the beginning of this investigation certain possible dispersion mechanisms were selected, and the consequences of these were worked out analytically. These mechanisms included the concept of dispersion as a diffusion process and the concept of longitudinal dispersion by variation of the stream velocity through the cross section. Laboratory and field investigations were then made, and the observed and theoretical results were compared to determine whether an explanation of the observed facts could be obtained from either of the postulated mechanisms. It will be seen later that the diffusion mechanism did provide a complete explanation.

The characteristic feature of a diffusion process is a rate of propagation of the material being dispersed that is proportional to the gradient of the concentration of the material. The propagation is in the direction of decreasing concentration. The factor of proportionality is called a diffusion coefficient. In this investigation, diffusion coefficients were determined experimentally. Where appropriate, the diffusion equations presented here contain a term to account for the possibility that the added materials may be carried along bodily by the mean stream velocity.

Recent analytical studies based upon measured or derived velocity variations through an idealized stream cross section and the mechanism of turbulence have shown that longitudinal stream dispersion can be described by a virtual coefficient of diffusion (Taylor, G. I., 1954; and Elder, 1959). Such behavior is in

accord with results obtained from this investigation, and a digest of the work is included. A surprising result comes out of the comparison; very good correlations between theoretical work and test results were obtained from the flume experiments, but dispersion rates that are many times those indicated by the analytical work were obtained for natural streams.

One method of velocity determination utilizes the time interval required for a salt cloud to traverse a known distance between electrode positions. This is known as the salt-velocity method. Effective use has been made of the recorded experience in this field in developing the methods described here. In return, these developments provide a new theoretical basis for salt-velocity practices.

In preparing this presentation, it has been kept in mind that the methods developed here might have to be used at a time of threatened disaster when simple and reliable methods would be much needed, and unnecessary complications would become a dangerous nuisance. An investigation such as this one is definitely in the realm of mathematical physics and requires at many points the use of judgment and the making of decisions so that the end product will be at the same time as reliable and as simple to use as possible. The formulas presented here have been carefully tied in with hydraulics concepts so that the wealth of empirical data accumulated in this field can be brought to bear in case of need. Some effort has also been applied to the task of bringing out relationships that will contribute to a sound and useful understanding of the characteristics of the dispersion mechanism. In places, some concessions have been made in the interest of simplicity. These will be mentioned at the appropriate place in the text.

In an investigation of the type described here, there is always a possibility that the time and funds available may be inadequate to settle all the questions which arise during the progress of the work. It will be pertinent, therefore, to comment briefly upon some factors of this kind which appeared during these studies.

The surprising difference in the dispersive power of streams running in flumes and natural channels indicates that more work is needed to determine factors, such as those contained in table 8, that govern a wider range of conditions.

A persistent phenomenon in dispersion tests on natural streams indicates a loss of tracer with time. The limited data now available indicate that the added materials may upset a previously stable chemical regimen with the result that some of the tracer material is removed by chemical reactions between the tracer material and the materials of the bed and banks of the stream. More study is needed to clarify the nature and behavior of this factor.

Throughout the report the term "salinity" is used to describe not only the dissolved salts that may be present in a stream but also those suspended materials that respond to the diffusion process in the same manner as dissolved salts. Such suspended materials include not only the very fine sediments such as fine clay and colloids, but also include micro-organisms such as plankton, which may collect and carry radioactive material.

For the benefit of those who may have occasion to apply these methods, it will be helpful to identify the formulas to be used in typical situations. Formula 3 applies where a quantity of contaminating material is suddenly released in a cross section of a stream and is dispersed in the direction of flow. If conditions are similar except that the contaminating material is originally distributed through the volume of the stream over a length L, formula 4 applies. When a finite amount of a contaminant is suddenly released at a point in the stream and may be dispersed longitudinally and also transversely and vertically, formulas to apply can be made up of formulas 3, 13, and 14, combined according to the product law described in a later paragraph of this report. If the contaminating material is supplied continuously at a point on the bank of the stream, formula 30 will apply. Diffusion coefficients for use with these formulas can be obtained from table 10. Formulas 13, 14, and 30 are appropriate when the contaminating material is released at the bank of the stream where lateral dispersion can act in only one lateral direction, this being the direction toward the other bank of the stream. If the point of release is away from the bank, where lateral dispersion can move the salinity toward both banks, then one-half of the material introduced can be assumed to move each way.

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NOTATION

The following notation will be used:

A	area (sq ft or ft^2)
A_1	a constant
a	$=\dfrac{K_y x^2 + K_z y^2}{4 K_x K_y}$ (sec)
a_1	radius of a circular pipe (ft)
B	the average width of the stream (ft)
b	$=\dfrac{V^2}{4 K_x}$ (1 per sec)
C_1	Chezy's coefficient ($\sqrt{\text{ft per sec}^2}$)
C'	concentration
C	$=\dfrac{K_x}{V}$
D	the average depth of the stream (ft)
d	diameter of an electrode bar (ft)
e	the base of the natural system of logarithms, 2.71828+ (dimensionless)
E	electrical potential in volts (v). Alternating voltages are given as root mean square values.
$E(u)$	the probability integral corresponding to the argument u (dimensionless)
F_1, f_2, f_3	functions defined in the text
g	acceleration of gravity (ft per sec^2)
I	electrical current (amperes)
J_1	$=BDV\displaystyle\int_0^\infty S dt \quad J_2=BDV\displaystyle\int_0^\infty St dt$
k_1	Von Karman's constant (dimensionless)
K'	conductivity $\dfrac{1}{(\text{Ohms})(\text{feet})}$
K''	a constant defined in the text
K_n	modified Bessel functions of the second kind of order n. (This definition conforms to that of the British Association Mathematical tables, vol. 6.)
K_x, K_y, K_z	diffusion coefficients for the directions x, y, and z, respectively (ft^2 per sec)
L	a distance measured downstream from the point $x=0$ (ft)
L'	the length of an electrode (ft)
n	a dimensionless number
n_1	a factor specifying hydraulic roughness
Q_s	a quantity of salt (lb)
q_s	a quantity of salt introduced per unit of time (lb per sec)

r_1	hydraulic radius (ft)
s_1	slope of a stream (dimensionless)
S	salinity, the amount of salt in unit volume of water (lb per ft^3)
S_m	maximum salinity observed at a station (lb per ft^3)
s'	electrode spacing (ft)
t	time coordinate (sec)
t_2	a time defined by the relation $\left(\dfrac{B}{\sqrt{4K_y t_2}}\right)=2$
t_m	the mean transit time corresponding to the center of gravity of the salinity-time curve obtained at a point on a stream (sec)
u	a generalized variable whose value becomes determinate in any specific case
v_*	$\sqrt{\dfrac{\tau_0}{\rho}}$ the shear velocity (ft per sec)
v	a substitution variable
V	the mean stream velocity (ft per sec)
W_1	$=BD\displaystyle\int_{-\infty}^{+\infty} S dx$
W_n	integrals described in the test
w	weight per unit of volume (lb/ft^3)
x, y, z	coordinates measured, respectively, in the direction of streamflow, across the stream and vertically downward (ft). These quantities are measured from a point on the bank of the stream which is assumed to be the point at which the salinity is introduced. These quantities are considered positive in the directions stated.
α	$=\dfrac{xV}{4K_x}$
β	$=ab$
ς	distance measured upward from the bed of a stream (ft)
η	a substitution variable
ξ	a time variable
π	$=3.14159+$
ρ	$=\dfrac{w}{g}$ the density of the fluid
τ_0	the friction stress exerted by the stream on its bed (lb per ft^2)

Although the notation is expressed here in terms of English units, any consistent system of units may be used with the formulas given here, except for the expressions derived from conventional hydraulic formulas which are traditionally tied to specific unit systems. These formulas are identified by a dagger (†).

DEFINITIONS OF TERMS USED IN THIS REPORT

Density current: A current moving along the bottom or top of a stream because of a difference in densities. Such differences of density may be due to temperature differences or differences of salinity between the water of a stream and waters being discharged into it. Suspended sediment may also cause density differences.

Dispersion: A general term used here to describe a scattering of dissolved or suspended material through the volume of a stream without reference to the nature of the mechanism which brings it about.

Diffusion: A specific term used to indicate a dispersion mechanism of the type commonly associated with diffusion phenomena. These phenomena are expressed by differential equations of a characteristic type. Equations 1, 2, 7, 9, and 12 are of this type. They imply a salinity propagation rate proportional to the salinity gradient.

Diffusion coefficient: A coefficient which represents the rate of salinity propagation across a unit area under a unit salinity gradient. It has the dimensions ft^2 per sec.

Salinity: As used here this term designates any kind of a dissolved or suspended material which may be dispersed through the volume of a stream.

Salt-velocity test: A test made to determine the mean velocity of a stream by observing the progress of a salt cloud injected into it.

Velocity variations: Velocity variations in the cross section of a stream. The term velocity, as used here, indicates a temporal mean at a point. It is such a velocity as would be determined by a current meter.

Suspended material: Any material held continuously in suspension, such as clay particles, plankton organisms, and the like which may carry radioactive material and which move freely with the water.

ppm: Parts per million, by weight.

BASIC DIFFERENTIAL EQUATION

Analytical treatments of diffusion phenomena began with the work of Fourier in the year 1812. He was principally concerned with the conduction of heat in solids, although he recognized that the same basic relationships would apply in other fields also. Since his time, phenomena of the diffusion type have been observed in many other fields of which chemical processes, osmosis, ground-water flow, evaporation, atmospheric pollution, sediment transport, combustion processes, propagation of ocean salinity into estuarine channels, biologic changes, flow of water through unsaturated soils, and the propagation of electricity may be mentioned.

The diffusion of momentum by turbulence is of fundamental importance in streamflow. The processes by which water evaporates from lakes, is picked up into the atmosphere and carried away by the wind, and the transport of sediment by streams are related to the process here under study because turbulence is an important factor in each. From our standpoint, it will be important to realize that in the processes to be studied, the differences from other fields are enough, however, so that developments in them cannot be taken over and applied to the process of dispersion in streams. Much help may, nevertheless, be obtained from the pioneer work of Fourier.

The analytical and experimental developments to be described relate to dispersion in three dimensions. These are in the downstream, cross-stream, and vertical directions. The rate of diffusion in the downstream direction has been investigated by Taylor (1954) and Elder (1959) by a fluid mechanics approach. A development for vertical dispersion has been described by Vanoni (1953). In a later chapter, it will be shown how these developments can be used to calculate values for the dispersion coefficients introduced into the present development.

For the benefit of those readers who may be unfamiliar with differential equations and their solutions, it may be explained that a differential equation is a mathematical formula which expresses some fundamental requirement. Equation 1 is such a formula. It not only expresses quantitatively how salt is propagated, but it also expresses a requirement that once the salt has entered the stream its amount will remain unchanged. A solution of a differential equation is an expression, such as formula 3, which satisfies the requirements imposed by the differential equation and, in addition, satisfies certain special conditions called boundary conditions. The task of finding a solution of a differential equation which meets an appropriate set of boundary conditions is basically one of great difficulty. The labors of mathematicians over the last two centuries have, however, provided solutions of many basic types. The ones used here come principally from the work of Fourier. His work was presented to the French Academy in the year 1812.

The concepts which form the basis of the analyses may be described in the following way:

1. Salinity can be carried along by the flow of the stream.

2. The variations of velocity through the cross section of a stream disperses any existing salinity concentration in the direction of flow.

3. Turbulence and, to a minor extent, molecular diffusion contribute to the dispersion of a salinity concentration in the vertical, horizontal, and downstream directions.

To put these concepts into a form suitable for analytical use, we may assume that

4. Where a salinity gradient exists, salinity will be propagated in the direction of decreasing salinity.

5. The rate of propagation of salinity across a unit of area is proportional to the salinity gradient along a line normal to the area.

Some of these concepts are certainly true and all are reasonable, but with respect to item 5 we may entertain some reservations concerning the linearity of the relation, as assumed. Here a choice must be made. We may note, first, that if a nonlinear relationship is to be used instead of the linear type assumed, we will be forced to deal with a nonlinear partial differential equation, and this will involve us in mathematical difficulties of the most serious kind. As a matter of tactics, it seems much better to assume a linear relation, to deal with the linear differential equation that results, and to reserve judgment until we can compare the results of this analysis with the results of tests. It may be worthwhile to state here that both test data and the independent analyses to be described later justify this course of action.

The effect of velocity variations as described in item 2 is very important when dealing with a mass of salinity introduced into a stream at a given time and place, such as could occur if a cloud of radioactive material were released into a stream. It has a lesser significance where the salinity is supplied continuously to a stream so that the salinity gradient in the longitudinal direction is low.

Our analyses may be begun by noting that the linear relationship of item 5 permits the rate of transport of salinity across unit areas normal to the coordinate direction x, y, and z to be expressed in the form (see p. for explanation of symbols used):

$$-K_x\frac{\partial S}{\partial x}, -K_y\frac{\partial S}{\partial y}, \text{ and} -K_z\frac{\partial S}{\partial z}.$$

If attention is now fixed upon an elemental volume $dx\,dy\,dz$ through which the fluid flows, the rate of change of its salt content will be determined by the rate salinity is propagated into the volume across three of the faces and the rate it is propagated out of the volume

across the three remaining faces. The expression of this relationship for the x direction is typical:

$$-K_x\frac{\partial S}{\partial x}\,dydz+K_x\left(\frac{\partial S}{\partial x}+\frac{\partial^2 S}{\partial x^2}\,dx\right)dydz=K_x\frac{\partial^2 S}{\partial x^2}\,dxdydz.$$

Since our coordinate system is fixed to the ground, we note that at the point x the combination of a salinity gradient $\dfrac{\partial S}{\partial x}$ and a mean velocity V will produce a rate of salinity change in the element of volume of

$$-V\frac{\partial S}{\partial x}\,dxdydz.$$

The condition that the rate of change of salinity in the elemental volume will be consistent with the rates at which salinity is entering and leaving it is

$$\frac{\partial S}{\partial t}\,dxdydz=\left(K_x\frac{\partial^2 S}{\partial x^2}+K_y\frac{\partial^2 S}{\partial y^2}+K_z\frac{\partial^2 S}{\partial z^2}\right)dxdydz$$
$$-V\frac{\partial S}{\partial x}\,dxdydz.$$

When this expression is simplified by dividing out the common factor $dx\,dy\,dz$, it takes the form

$$\frac{\partial S}{\partial t}=K_x\frac{\partial^2 S}{\partial x^2}+K_y\frac{\partial^2 S}{\partial y^2}+K_z\frac{\partial^2 S}{\partial z^2}-V\frac{\partial S}{\partial x}. \qquad (1)$$

Although in this equation V appears as a constant, the variations of velocity within the stream cross section are not ignored but are accounted for in the magnitudes of the diffusion coefficients K_x, K_y, and K_z.

SOLUTIONS

A solution will first be obtained for the one dimensional case defined by the differential equation:

$$\frac{\partial S}{\partial t}=K_x\frac{\partial^2 S}{\partial x^2}-V\frac{\partial S}{\partial x} \qquad (2)$$

Extension to the two and three dimensional cases will be made later by use of a product law.

Let us assume that a quantity of salt Q_s is dispersed throughout a cross section of a stream having a breadth B and average depth D at $x=0$ when $t=0$. Then a solution satisfying equation 2 is:

$$S=\frac{Q_s}{BD}\frac{e-\left(\frac{x-Vt}{\sqrt{4K_x t}}\right)^2}{\sqrt{\pi}\,\sqrt{4K_x t}} \qquad (3)$$

If a quantity of sale Q_s is dispersed through the volume BDL at the time $t=0$, a solution of equation 2 is:

$$S=\frac{Q_s}{BDL}\frac{1}{2}\left[E\left(\frac{x-Vt}{\sqrt{4K_x t}}\right)-E\left(\frac{x-L-Vt}{\sqrt{4K_x t}}\right)\right] \qquad (4)$$

PRODUCT LAW

The term "Product law" as used here applies to a method of constructing solutions of partial differential equations of diffusion with two or three space variables from solutions of a differential equation of the same type with one space variable. These solutions are formed as products of solutions having a single space variable, and this relation is indicated in the name.

The writer has used this method of constructing solutions to the diffusion equation (Glover, 1934; and Tuthill and others, 1951) over a period of years and has found it to be an effective means for simplifying the treatment of problems which involve diffusion in more than one direction.

The process to be described here should not be confused with the older method which seeks a solution as a product of functions which have only one independent variable. The functions used here can have two or more independent variables. The following developments will clarify the details of this procedure. Suppose we have a solution

$$S_1 = F_1(x, K_x, t, V) \tag{5}$$

which satisfies the differential equation

$$\frac{\partial S_1}{\partial t} = K_x \frac{\partial^2 S_1}{\partial x^2} - V \frac{\partial S_1}{\partial x}, \tag{2}$$

a solution

$$f_2 = F_2(y, K_y, t) \tag{6}$$

which satisfies the differential equation

$$\frac{\partial f_2}{\partial t} = K_y \frac{\partial^2 f_2}{\partial y^2} \tag{7}$$

and a solution

$$f_3 = F_3(z, K_z, t) \tag{8}$$

which satisfies the differential equation

$$\frac{\partial f_3}{\partial t} = K_z \frac{\partial^2 f_3}{\partial z^2}, \tag{9}$$

then the product

$$S = S_1 f_2 f_3 \tag{10}$$

is a solution of the differential equation

$$\frac{\partial S}{\partial t} = K_x \frac{\partial^2 S}{\partial x^2} + K_y \frac{\partial^2 S}{\partial y^2} + K_z \frac{\partial^2 S}{\partial z^2} - V \frac{\partial S}{\partial x}. \tag{1}$$

Similarly, the product

$$S = S_1 f_2 \tag{11}$$

is a solution of the differential equation

$$\frac{\partial S}{\partial t} = K_x \frac{\partial^2 S}{\partial x^2} + K_y \frac{\partial^2 S}{\partial y^2} - V \frac{\partial S}{\partial x}. \tag{12}$$

A proof that the product law holds may be obtained by performing on equations 10 or 11 the differentiations required by equations 1 or 12 and making use of the relations 2, 7, and 9.

USE OF THE PRODUCT LAW

While the product law affirms that the product 10 will be a solution of equation 1 and the product 11 will be a solution of equation 12, it does not assure that the desired initial condition or the boundary conditions will be met. If we take

$$S_1 = \frac{Q_s}{BD} \frac{e^{-\left(\frac{x-Vt}{\sqrt{4K_x t}}\right)^2}}{\sqrt{\pi}\sqrt{4K_x t}}, \tag{3}$$

$$f_2 = \frac{B}{\sqrt{\pi}} \frac{2e^{-\left(\frac{y}{\sqrt{4K_y t}}\right)^2}}{\sqrt{4K_y t}}, \tag{13}$$

and

$$f_3 = \frac{D}{\sqrt{\pi}} \frac{2e^{-\left(\frac{z}{\sqrt{4K_z t}}\right)^2}}{\sqrt{4K_z t}}, \tag{14}$$

the initial condition will be properly met for the products 10 and 11 if the quantity of salt Q_s is introduced at the origin at $x=0$. The boundary conditions will be met so long as the salinity does not reach a barrier which prevents further diffusion in some direction. A chart showing the progress of diffusion, when barriers are present, is included and will be described later. A similar set of solutions can be based on equation 4. If we set

$$S_4 = \frac{Q_s}{BDL} \frac{1}{2}\left[E\left(\frac{x-Vt}{\sqrt{4K_x t}}\right) - E\left(\frac{x-L-Vt}{\sqrt{4K_x t}}\right) \right], \tag{4}$$

then a solution of equation 1 is

$$S = S_4 f_2 f_3 \tag{15}$$

and a solution of equation 12 is

$$S = S_4 f_2. \tag{16}$$

VELOCITY VARIATION AS A DISPERSING MECHANISM

The formulas of the preceding paragraphs are based upon the assumption that dispersion is a diffusion phenomena, and it may be noted here that the experiments

made have confirmed the validity of this assumption. In the early phases of this investigation, however, another type of formula was developed on the basis that variations of velocity [1] in the cross section of the stream were the dominating factor in longitudinal dispersion.

Even though experiments have shown that this basis is inadequate, an account of this development may be warranted because the reasons for its failure throw so much light on the nature of the dispersion mechanism. In developing this formula, it was supposed that the velocity at the bed and banks of the stream was zero and that velocities in the remainder of the section ranged from this minimum value to a maximum near the middle of the cross section near or at the surface. If a quantity of salt were to be distributed uniformly throughout a cross section of the stream at some instant it would be dispersed longitudinally, at later times, by the variations of velocity through the section. All the salt would, however, be found between the point of introduction and the point reached by the thread of maximum velocity. On the basis that the stream has a uniform cross section, it can be inferred that both the average salinity and the maximum salinity in this reach would be inversely proportional to time and that a modicum of salt would always remain at the point of introduction. Because the investigations of Taylor (1954) and Elder (1959) show that these velocity variations are the dominating factor in longitudinal dispersion, as was assumed, it may well be asked why it is that the experiments fail to confirm an inverse first power law as would be implied by the above considerations. The answer seems to be found in the lateral dispersion due, principally, to turbulence. The modicum of salt can not remain at the point of introduction because it is picked up by the turbulence and thrown into the moving portions of the stream where it can be carried along. The salinity remains more closely bunched, and the decay of maximum salinity follows an inverse square-root law rather than an inverse first power law. This process is described in detail by Taylor and Elder. It must be conceded, then, that transverse dispersion due to turbulence is an indispensable factor in the mechanism of longitudinal dispersion. Some other important characteristics of this longitudinal dispersion mechanism will be discussed in later paragraphs.

CONTINUOUS INTRODUCTION OF SALINITY

The difficulties encountered in the laboratory tests, when an attempt was made to study lateral diffusion rates by using salinity samples introduced locally, made

it expedient to introduce the salt solution continuously. By this means, it was possible to minimize the tendency of the salt solutions to cause density currents which produced an excessive rate of lateral spreading. A special solution of the diffusion differential equation is needed in order to reduce these test data. Such a solution also is needed because continuous introductions of salinity must be expected in problems involving actual rivers, and it will be essential to have formulas available for estimating their dispersion. The following development is based upon the idea of dispersion by diffusion.

If a quantity of salinity Q_s is introduced instantaneously at $x=0$, $y=0$, and $t=0$ and allowed to disperse both in the downstream direction and laterally, the distribution of salinity is given by an expression of the type:

$$S = \frac{Q_s}{D} \frac{e^{-\left(\frac{x-Vt}{\sqrt{4K_x t}}\right)^2}}{\sqrt{\pi}\sqrt{4K_x t}} \frac{2e^{-\left(\frac{y}{\sqrt{4K_y t}}\right)^2}}{\sqrt{\pi}\sqrt{4K_y t}}. \qquad (17)$$

This can be put into the form

$$S = \frac{Q_s e^{\frac{xV}{2K_x}}}{2\pi D\sqrt{K_x K_y}} \frac{e^{-\frac{a}{t}-bt}}{t} \qquad (18)$$

where

$$a = \left(\frac{K_y x^2 + K_x y^2}{4K_x K_y}\right) \qquad b = \left(\frac{V^2}{4K_x}\right). \qquad (19)$$

Suppose salinity is introduced continuously, at $x=0$, at the rate q_s, then the effect of a quantity of salt $q_s d\xi$ introduced at the time ξ on the salinity at the point x, y at the time t will be given by an integral of the type:

$$S = \frac{q_s e^{\frac{Vx}{2K_x}}}{2\pi D\sqrt{K_x K_y}} \int_0^t \frac{e^{-\left(\frac{a}{t-\xi}\right)-b(t-\xi)}}{(t-\xi)} d\xi. \qquad (20)$$

Let

$$u = \frac{a}{(t-\xi)} \qquad \beta = ab; \qquad (21)$$

then, by substitution

$$S = \frac{q_s e^{\frac{Vx}{2K_x}}}{2\pi D\sqrt{K_x K_y}} \int_{\frac{a}{t}}^{\infty} \frac{e^{-u-\frac{\beta}{u}}}{u} du. \qquad (22)$$

By use of the relations developed in the chapter on integral relations this expression can be written

$$S = \frac{2q_s e^{\frac{Vx}{2K_x}}}{2\pi D\sqrt{K_x K_y}} K_0(\sqrt{4\beta}). \qquad (23)$$

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[1] The term "velocity" is used here in the sense of a temporal mean at a point. It is a velocity that would be measured with a current meter.

The method used to obtain this expression should insure that it is an exact solution of equation 12, and an independent check will confirm this. An attempt to use it for computation purposes will show, however, that it is not well adapted to these uses.

A simplification can be made if the quantity $\sqrt{4\beta}$ is large enough to justify the use of the first term of the asymptotic series:

$$K_0(z)=\left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1-\frac{1^2}{(8z)\lfloor 1}+\frac{1^2 3^2}{(8z)^2\lfloor 2}-\cdots\right] \quad (24)$$

as an approximation. This will be permissible if $1/8z$ is small compared to unity. Then as an approximation

$$K_0(z)\approx\left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \quad \left(\text{approximate if } \frac{1}{8z}\!<\!<\!1\right). \quad (25)$$

In the present case $z=\sqrt{4\beta}$ and this relation takes the form

$$K_0(\sqrt{4\beta})\approx\left(\frac{\pi}{2\sqrt{4\beta}}\right)^{1/2} e^{-\sqrt{4\beta}} \quad \left(\text{valid if } \frac{1}{8\sqrt{4\beta}}\!<\!<\!1\right). \quad (26)$$

Since

$$4\beta=\frac{V^2 x^2}{4K_x^2}\left(1+\frac{K_x}{K_y}\frac{y^2}{x^2}\right) \quad (27)$$

then

$$\sqrt{4\beta}=\frac{Vx}{2K_x}\sqrt{1+\frac{K_x}{K_y}\frac{y^2}{x^2}} \quad (28)$$

and, if the quantity $\frac{K_x}{K_y}\cdot\frac{y^2}{x^2}$ is also small compared to unity,

$$\sqrt{1+\frac{K_x y^2}{K_y x^2}}\approx 1+\frac{1}{2}\frac{K_x}{K_y}\frac{y^2}{x^2} \quad \left(\text{approximate if } \frac{K_x}{K_y}\frac{y^2}{x^2}\!<\!<\!1\right). \quad (29)$$

If both of these conditions are met, the exact solution of equation 23 can be approximated by the relation

$$S\approx\frac{q_s e^{-\frac{Vy^2}{4K_y x}}}{D\sqrt{\pi K_y Vx}} \quad \left(\text{approximate if } \frac{1}{8\sqrt{4\beta}}\!<\!<\!1\right.$$

$$\left.\text{and } \frac{K_x}{K_y}\frac{y^2}{x^2}\!<\!<\!1\right) \quad (30)$$

It is fortunate that if the limitations given here are exceeded, the salinity will generally be very small, and this makes it possible to use the formula almost without restriction. Doubtful cases can be expected to fall in the zone close to the point of introduction of the salt, and in this region equation 23 could probably be used without much difficulty. This expression applies to a stream of unlimited width, but the method of images may be used to adapt it to use with streams of finite width.

SPECIAL CHART

Solutions of the form of equation 13 are appropriate if the medium is of infinite extent in the direction of y. When the width of the stream is limited, as is always true, these solutions are valid until the parameter

$$\left(\frac{B}{\sqrt{4K_y t_2}}\right)$$

reaches approximately 2. This value of the parameter indicates that the diffusing salinity has reached the far bank of the stream. The distribution of salinity across the stream at this time is indicated by the curve of figure 1 identified by the value

$$\left(\frac{B}{\sqrt{4K_y t}}\right)=2.$$

For later times, it will be necessary to substitute for the original expression 13 an equation 31 of another type which conforms to the boundary conditions imposed by the banks. A suitable form for this use is a Fourier series of cosine terms which can be adjusted to make the salinity gradients zero at the banks. The first three terms of such a series are given in equation 31 and graphed on figure 1. The terms in this expression have been adjusted to give, very nearly, the distribution shown by the curve for

$$\frac{B}{\sqrt{4K_y t_2}}=2$$

on figure 1. Then for times longer than t_2, figure 1 is used instead of the original expressions.

A similar situation exists in vertical diffusion when the bed of the stream halts the downward diffusion of salt. The curves of figure 1 may be used for this problem also. To adapt them for this use replace y by z, K_y by K_z, f_2 by f_3, and B by D.

The treatment described involves a slight approximation because the time when the salinity reaches the far bank is poorly defined and a somewhat arbitrary choice of a changeover time must be made. Such a choice will be acceptable if the total amount of salt, as indicated by the equations 13 and 31, remains essentially unchanged. The amount of salinity which is lost track of by converting from equation 13 to 31 may be judged from the height of the ordinate at $y/B=1.00$ for the curve

$$\frac{B}{\sqrt{4K_y t}}=2$$

on figure 1.

This chart was computed from the Fourier series:

$$\frac{f_2}{\left(\frac{4}{\pi}\right)}=0.4820e^{-\frac{K_y\pi^2}{B^2}(t-t_2)}\cos\frac{\pi y}{B}+0.0722e^{-\frac{4K_y\pi^2}{B^2}(t-t_2)}\cos\frac{2\pi y}{B}$$
$$+0.4432. \quad (31)$$

LABORATORY AND FIELD INVESTIGATIONS

PURPOSES

The equations as described in previous paragraphs contain certain coefficients which must be determined experimentally, and this requirement alone would justify a program of experimentation. There are, however, other ways in which the results of a carefully planned series of experiments can be turned to advantage, and the experiments described here were arranged to gain these additional advantages.

If turbulence is a factor in the dispersion (and there are strong reasons for believing that it is) and the turbulence velocities are proportional to the mean stream velocity, then it should be expected that so long as the cross section of the stream remains unchanged, quantities of the type of K_x, K_y, and K_z, should be proportional to the mean velocity V. Variations of velocity in an unchanging cross section can be arranged in the laboratory. In natural streams, a change of flow will generally change the stage so that these relations will be upset, but where dams maintain water levels these conditions may be approximated. If this expectation is confirmed, it would follow that for a quantity of salt Q_s, introduced at the origin at time zero, the maximum concentration observed at a downstream point would be independent of the velocity, although the time required for the maximum to appear at the downstream point would grow greater as the velocity V is reduced. The maximum salinity observed at any downstream point would vary inversely as the three halves power of the distance x from the origin to the point, as long as the diffusion proceeds in the three directions: downstream, across stream, and vertically. If diffusion takes place only in the downstream and cross-stream directions, the maximum salinity will be inversely proportional to the distance from the origin. After the salinity has become uniformly distributed throughout the stream cross section, the maximum salinities should be inversely proportional to the square root of the distance from the origin.
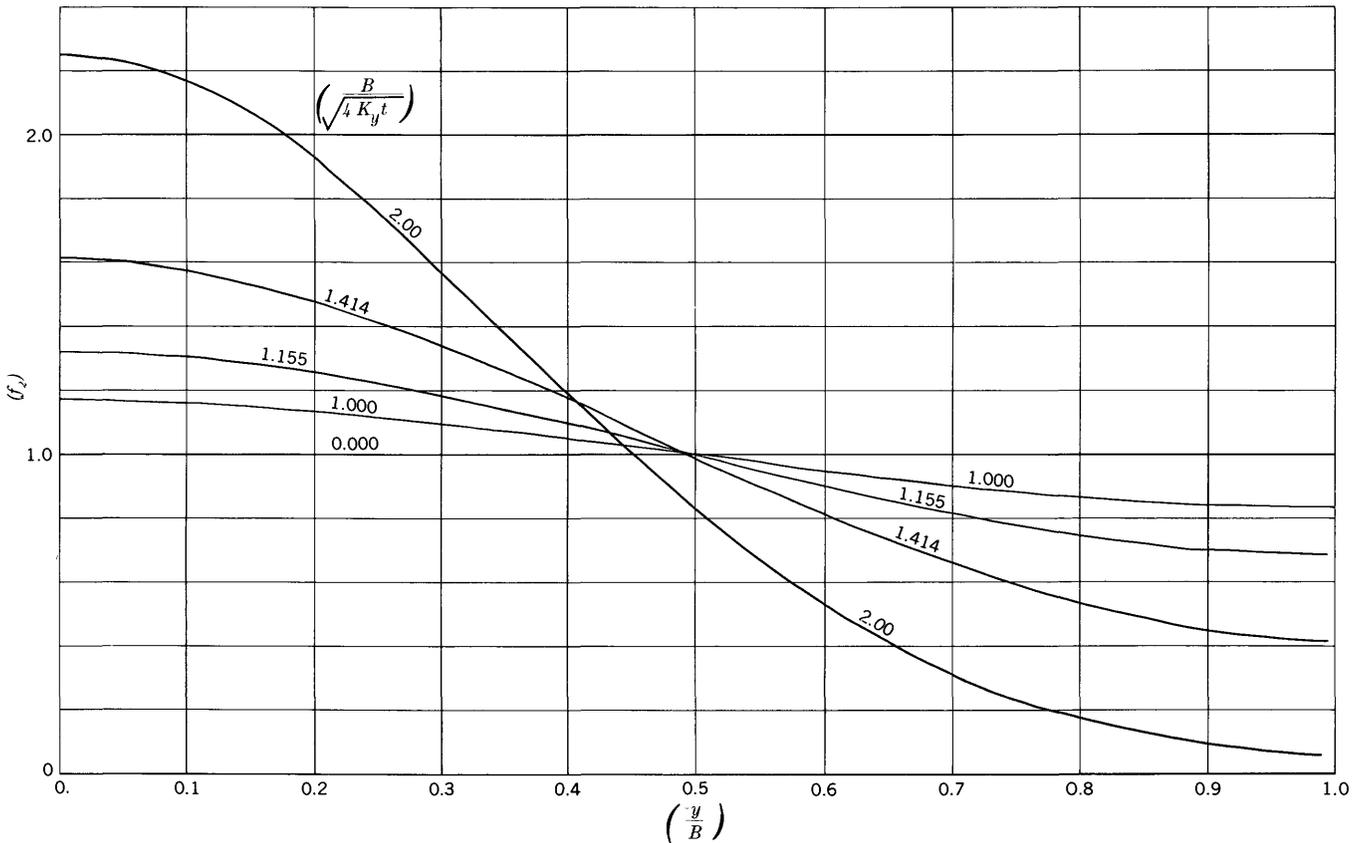

FIGURE 1.—Special chart for estimating salinity changes as modified by boundaries.

With these considerations in mind, the purposes to be served by an experimental program may be outlined in the following way:

1. To evaluate the coefficients of equations 3 and 13 for several stream cross sections having known geometrical and hydraulic properties.
2. To discover the ways in which these coefficients vary with stream geometry and hydraulics.
3. To discover what factors are of primary importance in producing dispersion in streams.

Records.—The original data collected from the Fort Collins, Allenspark and South Platte tests are filed at the Federal Records Center, Denver 25, Colo., where they are available for reference.

PLANNING OF TESTS

In the Fort Collins, Allenspark and South Platte tests, the test program was based upon computations prepared beforehand. Salt concentrations, dispersion patterns, salt quantities, time schedules, and electrode resistances were all computed. The design of circuitry was based upon these computed results. The effectiveness of this method is shown by the test record. Of 77 tests run, only one run was lost. This was due to a poor choice of a stepping relay interval made by the writer. This test was the No. 1 run at Fort Collins. In 1 day at Fort Collins 15 tests were made; the average was 8 per day. The early tests were planned with the aid of the formula based on velocity variations which required no knowledge of a diffusion coefficient. Sampling schedules were prepared for the South Platte test in a similar way.

ELECTRODE RESISTANCE

If two parallel wires of diameter d and length L' are spaced a distance s' apart in a fluid of conductance K', the current I due to an electromotive force E is approximately

$$I = \frac{\pi K' L' E}{\log_e \left(\frac{2s'}{d}\right)}. \tag{32}$$

For dilute solutions, the conductance has almost a linear relation to concentration, and equation 32 may be rewritten,

$$I = K'' C' \tag{33}$$

in which K'' is a constant and C' is the concentration in pounds of salt per cubic foot of water.

DETAILS OF METHODS AND EQUIPMENT

A solution of common salt was used as a tracer material in the flume tests at Fort Collins and at Allenspark.

In these tests, the amount of tracer present was measured electrically through its influence on the conductivity of the flowing water. The electrodes used to measure the salt concentration were round brass bars. Electrode arrangements are shown in figures 7 and 9. An alternating electrical pressure of about 1 volt was applied to these electrodes. The frequency used was 60 cycles per second. In order to avoid danger from electrical shock, the 110-volt supply was stepped down in two stages to about 1 volt by use of an autotransformer which energized bell-ringing transformers. The secondary insulation of the bell-ringing transformers isolated the electrode circuit from the supply. The electrode circuit included the secondary of the transformer, a resistor and the electrode, connected in series. A separate bell-ringing transformer was used on each electrode circuit.

An oscillograph was used to make the actual recordings. The oscillograph galvanometers were connected across the series resistor. The resistance of this series resistor was always kept less than 1 percent of the electrode resistance. Because the electrode resistance varied with the depth of water in the flume, the amount of salt originally present in the water and the amount of salt added, series resistances of 0.27, 0.47, and 1.00 ohms were used at various times to maintain this relationship. The series resistor, a scale setting resistor, and the galvanometer were connected in series in the galvanometer circuit. The galvanometers were electrically damped and had an undamped natural frequency of about 120 cycles per second. Very slow paper speeds were used for these tests, and each galvanometer therefore produced a record in the form of a band whose width indicated the amount of salt present at the electrode. Records of this type are shown on figures 2, 3 and 4. Its initial width indicated the amount of salt naturally present in the flume water. This initial width was used for calibration purposes in a manner to be explained later.

The time scale on all the records was imposed by an electrically operated timer that closed a contact at 1-second intervals. For the high-velocity runs the timer was connected to a spare galvanometer which made a trace near the edge of the record, but for the tests with low velocities the timer was connected to a stepping relay with 40 contacts. When so connected, it advanced the relay one contact every second. By energizing the oscillograph from selected contacts, an intermittent operation of the oscillograph could be arranged. Energized in this way, it would operate for 1 second at prearranged intervals. This device was used to hold the record to a convenient length when the test required a
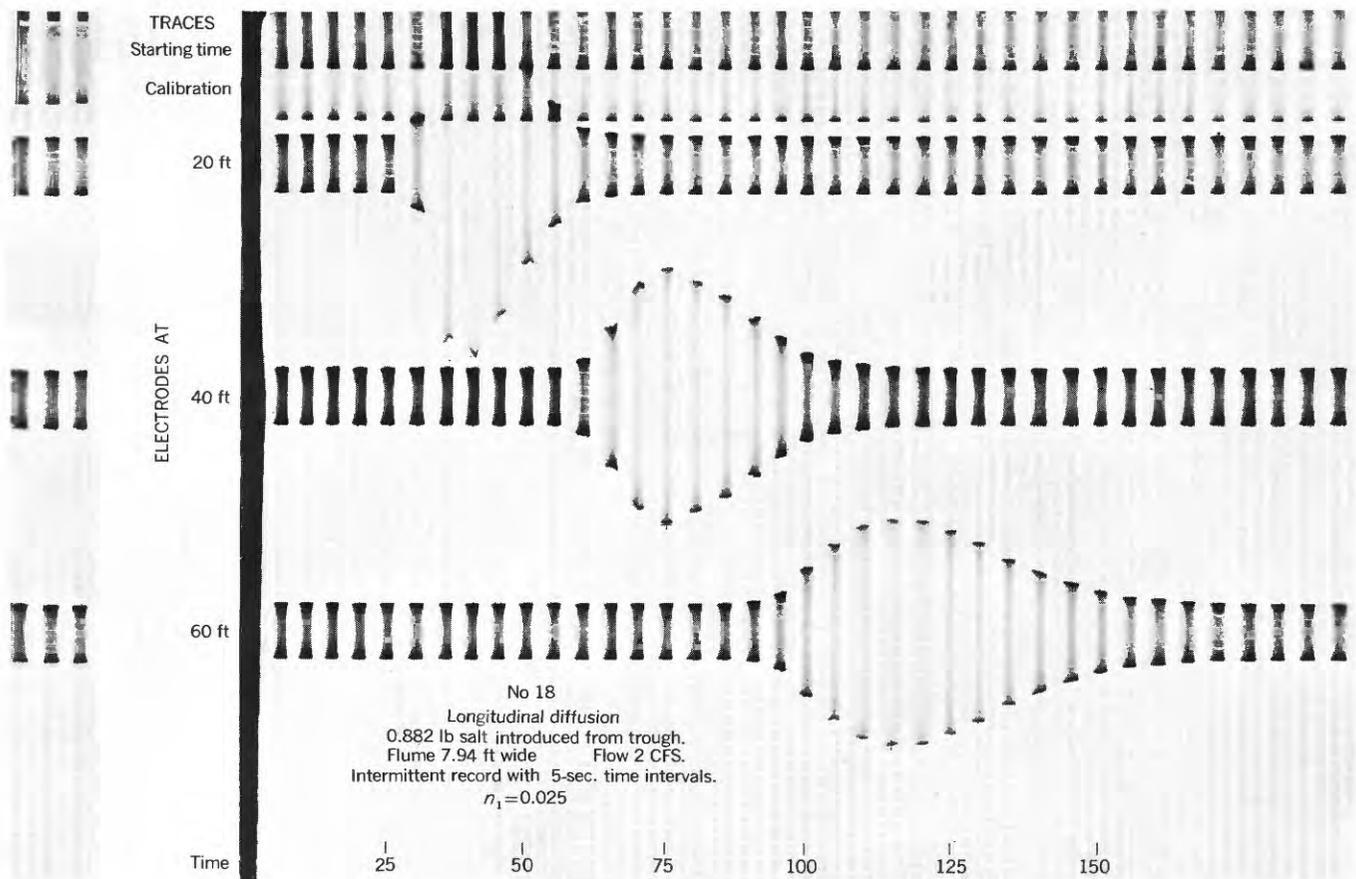
FIGURE 2.—Longitudinal diffusion test record—intermittent operation of recorder.   Oscillograph recorder connected to each fifth contact of a stepping relay actuated by a 1-second timer to shorten the test record.   Fort Collins test.

long time for completion. A record obtained in this way is shown on figure 2.

Some of the tests required the current through each pair of electrodes to be recorded individually while other tests required a response as a group representing a section.   To facilitate the rearrangement of electrical connections, a control panel was arranged so that the desired changes could be made by using switches. This control panel was used for both the Fort Collins and Allenspark tests.

A spare galvanometer was used to indicate the start of a run, and another galvanometer was used for calibration purposes. To make the calibration, a special electrode was transferred from a standard to the flume water during the test. At Fort Collins the city water was used as a standard, but at Allenspark a solution containing 100 ppm of salt in distilled water was used as the standard.

These markings are shown clearly on the record of test 23. This record is reproduced as figure 3. The trace indicating the start of the test is at the top, and the calibration trace is the next one below it. Because

the flume water was pumped from a sump filled from the city mains, the calibration trace does not change width in many of the records. After a number of runs were made, the salt used in the test began to show on the records. The initial width of the trace indicates the amount of salt originally present in the flume water. The 1-second time marks show at the bottom of the record. Run 18, reproduced as figure 2, is an intermittent record made by using eight contacts on the stepping relay. Each bar of the record therefore represents a 5-second time interval. The starting time and calibration records appear at the top of the record, as described above.

A transverse diffusion run is shown on record 25. This is reproduced as figure 4. The starting time and calibration marks show as before. The electrode numbers are shown on this record also. The position of these electrodes is shown on figure 6. The electrodes, galvanometer, and trace correlation is given in table 3.

Salt was introduced from a trough as shown in figure 5. The salt was put in solution, and the solution was placed in the trough. At the start of the test, the
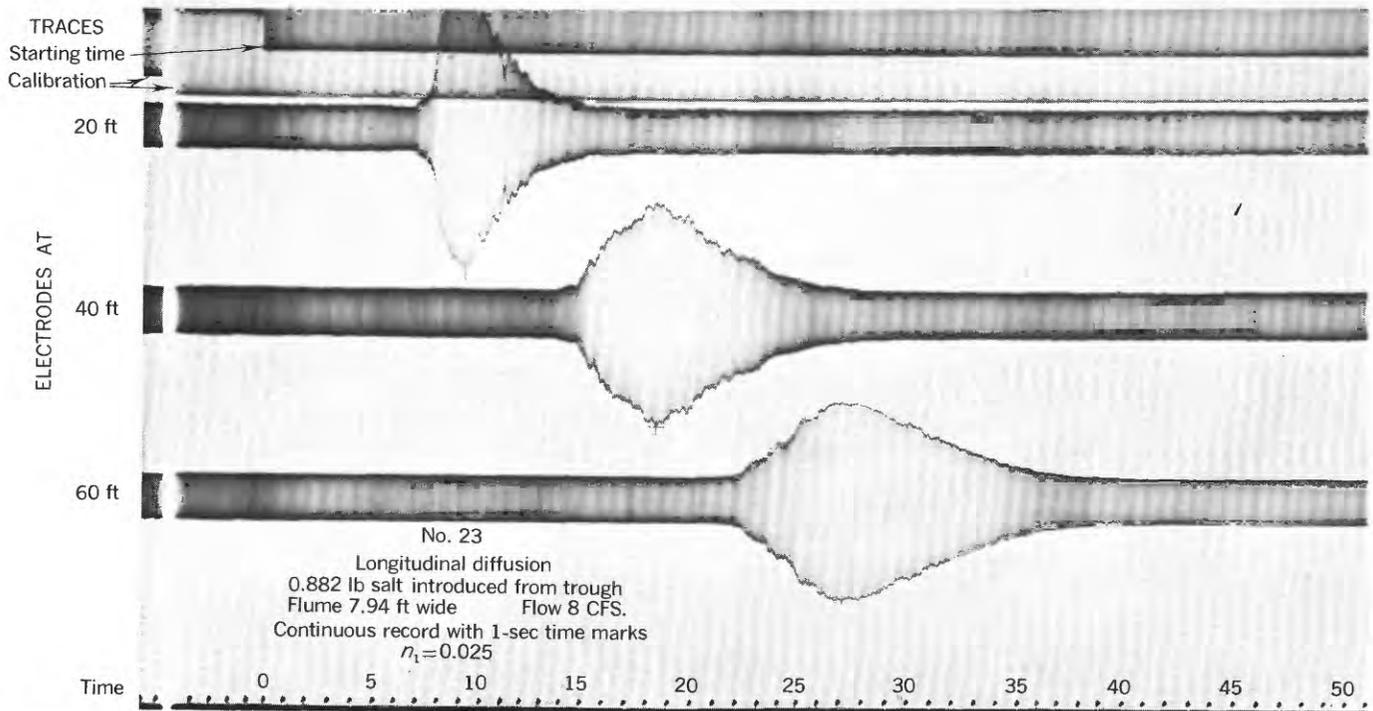
FIGURE 3.—Longitudinal diffusion test record—continuous operation of recorder. Starting time trace at top. Calibration trace next below starting time trace. One-second time marks at bottom. Width of the three electrode traces indicates amount of salt present at the electrode. Fort Collins test.
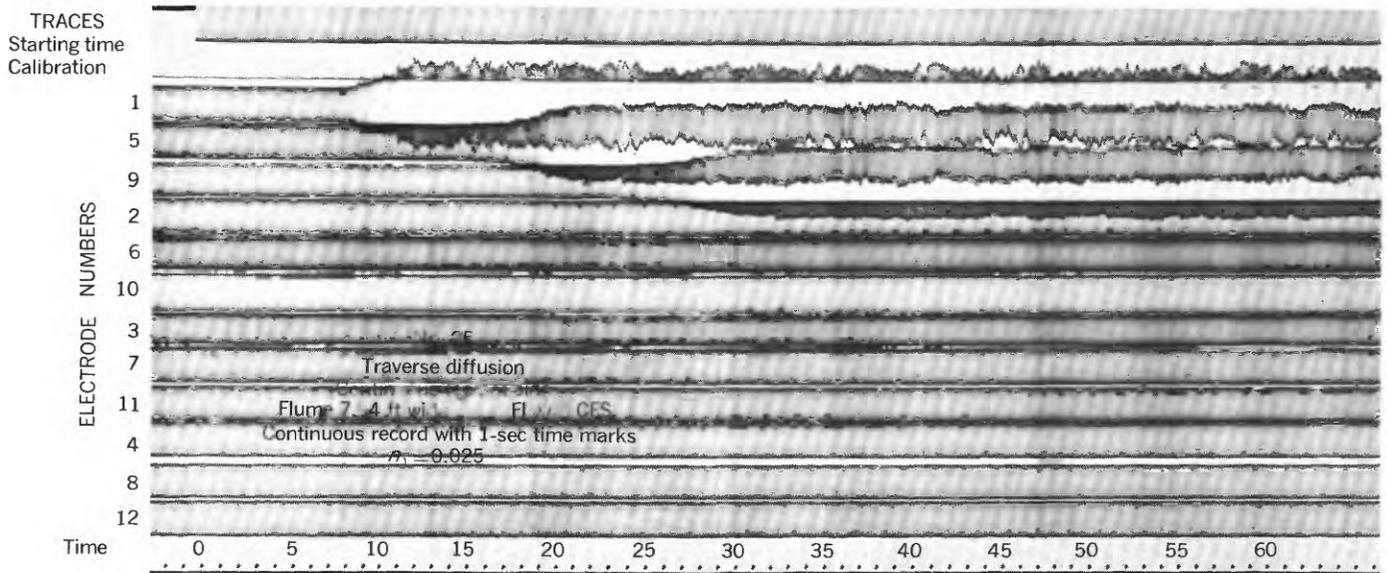
FIGURE 4.—Transverse diffusion test with continuous feed at side. Starting time trace at top. Timer trace at bottom. Trace identification at side. Fort Collins test.

handle was rotated to drop the salt solution into the water. The trough was coated with parafin wax to insure that all of the solution would be released. It was arranged to just clear the water surface at the maximum flow depth so that the salt charge could be released with the minimum amount of fall.

A darkroom was available at Fort Collins for development of the records. At Allenspark a light-tight box with a shelf was used for development. This was fitted with a door and arm holes protected by light-tight sleeves. In use, the record holder from the Oscillograph was placed on the upper shelf and the door closed. The

developing trays were kept in the bottom of the box. The operator worked through the arm holes to unload the record holder and develop the record. The box was of plywood about 1.5 by 2 by 3 feet. Because commerical power was not available at Allenspark, the 60-cycle per second power supply was obtained from a storage-battery-operated vibrator of the type used in car radios.

The South Platte test was made by a different procedure. In this test potassium carbonate was used as a tracer material, and samples were taken in plastic bottles and evaluated by use of a flame photometer. This technique proved to be both expeditious and effective.

Flow measurements were made by a calibrated orifice at Fort Collins, by rectangular and triangular weirs at Allenspark, and by current meter on the South Platte Test.

FORT COLLINS TESTS

In order to provide experimental data which could be used for determination of constants and the effects of stream geometry and hydraulics, a series of experiments were performed in a flume in the Hydraulic Laboratory of the Colorado State University at Fort Collins, Colo. This flume is constructed of plywood sheets and is nominally 8 feet wide and 2 feet deep. A view of the interior of the flume is shown in figure 5.

FIGURE 5.—Test flume 8-feet wide with trough and point gages in place, Fort Collins tests.

Its overall length was 72 feet. The usable length was somewhat less than this because of a stilling-well arrangement at the inlet end which served as a distributor. Of the total length, 60 feet was used in the experiments by introducing salt solution at a point near the inlet end of the flume and measuring the salt concentrations electrically at intervals of 20 feet. A plan view showing the stilling well and electrode stations is shown on figure 6. The flume was supplied with water by a pump which has a capacity in excess of 8 cubic feet per second.
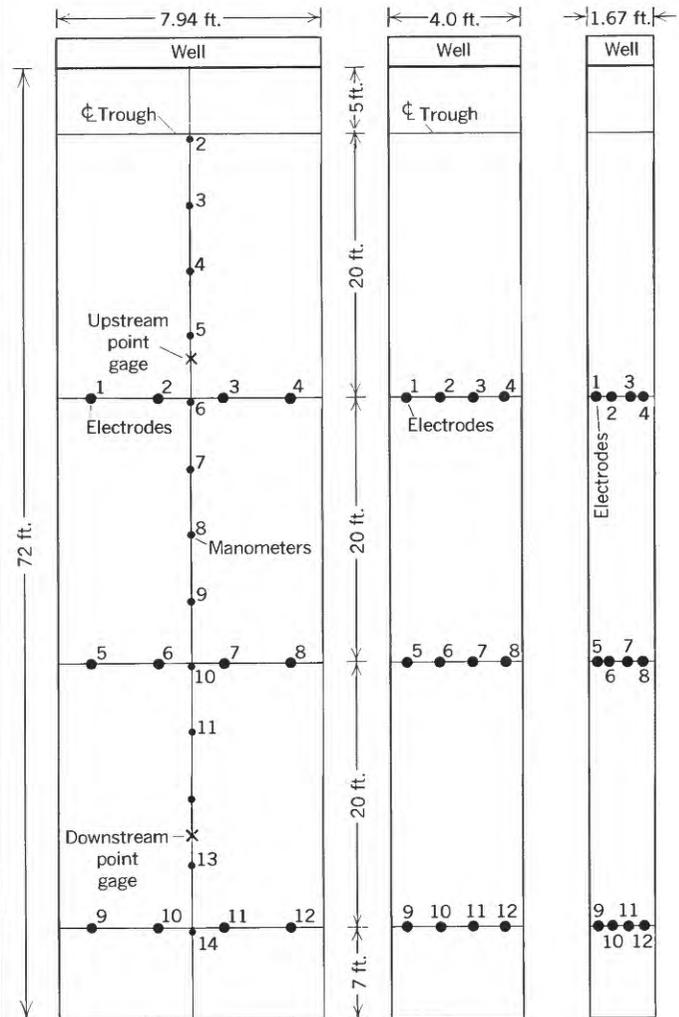
FIGURE 6.—Electrode numbers and positions, Fort Collins tests.

Water-quantity measurements are made by means of a calibrated orifice in the pump discharge line. The Kutter's "n_1" value for the bare flume is about 0.010, but this can be raised to about 0.025 by laying grids of reinforcing bars on the floor. A view of the flume with the bars installed is shown on figure 7.

Measurements of salt concentration were made electrically by imposing an alternating voltage of about 1 volt across electrodes mounted vertically on the floor of the flume and recording the current flow by means of an oscillograph. In dilute concentrations, the current flow is nearly proportional to the amount of salt present. Three sets of four electrodes each were used. These consisted of 3/16-inch-diameter brass rods spaced 1 inch apart and mounted in a plastic base arranged for attachment to the floor of the flume as shown in figure 7. Switches on a control panel permitted a recording of the current flow at each electrode separately,
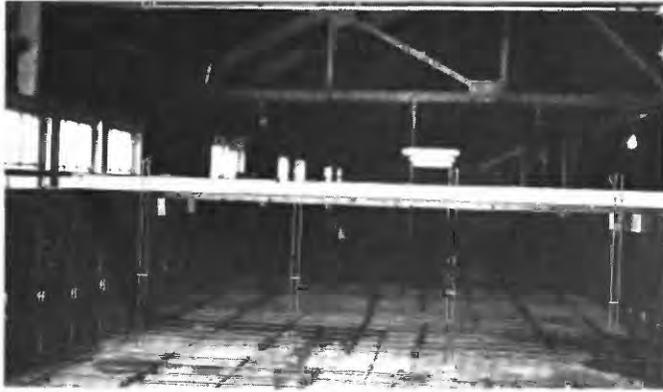
FIGURE 7.—Test flume 8-feet wide showing electrode assembly and bar roughness, Fort Collins tests. The electrode bars are attached to plastic plates mounted on the floor and are supported by a crossbeam at the top. The bar grids on the floor of the flume raise the roughness $n_1=0.025$.

or a simultaneous record of the four arranged in a series-parallel arrangement which had the same overall resistance as a single electrode. For the longitudinal diffusion test, the salt solution was introduced from a trough which spanned the flume 20 feet upstream from the first electrode, as shown in figure 5, but for the lateral diffusion tests the salt solution was introduced continuously, at this same location, by gravity flow through a rubber tube connected to a tank. Water levels were read from a set of piezometer tubes connected by plastic tubing to short sections of metal tube which were flush-mounted in the floor of the flume near the middle. Checks of water level were made at two points by means of point gages which were arranged to take readings on the floor of the flume and at the water surface. At the downstream end of the flume, water levels were controlled by an adjustable slatwork gate.

In order to obtain data for a range of cross sections and hydraulic conditions, experiments were run on sections 8 feet wide and 0.5 foot deep, 4 feet wide and 1 foot deep, and 1.667 feet wide and 1.667 feet deep. The following sequence of flow rates was planned: [1]

Flume width (feet)	Flow rates (cubic feet per second)				
8	8,	4,	2,	1,	0. 5
4	8,	4,	2,	--	---
1.667	5. 6,	2. 8,	1. 4,	--	---

[1] This is the planned sequence. For test value, see table 1.

These planned water depths and flow rates were only approximately met in the tests and the actual conditions obtained were recorded in each test. The test conditions are summarized in table 1. The bar roughness was used to vary the hydraulic conditions in several tests. In all, 65 runs were made.

To insure an accurate recording of salinity by electrical means, the resistance of the current measuring instrument must be small compared to the resistance of the electrode-electrolyte. Computations and tests showed that the minimum resistance of 3/16-inch brass rods with a 6-inch immersion would approximate 300 ohms, and therefore 1 ohm in the current measuring circuit would meet our requirements. Arrangements were made so that for greater depths of immersion, a correspondingly smaller value of resistance could be inserted in the circuit.

The recorder used was a 24-element recording oscillograph. The galvanometers, when properly damped, have a flat characteristic response to 120 cycles per second with a sensitivity in excess of 2,000 millimeters per milliampere at a distance of 1 meter. The galvanometer was connected, with suitable attenuator circuits, across the 1 ohm series resistor.

The paper speed in the oscillograph could be varied from about 1/4 to 30 inches per second. The one-quarter inch per second minimum speed was used throughout the tests. A description of the tests run is given in the table on page B15.

When the velocity of the water was high, it was possible to make a continuous recording of the conductivity without making an excessively long record. In these runs, second-time-marks were put on the record by using one galvanometer connected to a battery through a switch operated by a motor. The motor is a d-c type, but it is controlled by an escapement mechanism for constant speed.

For runs in which the velocity of the water was low, intermittent recordings of 1 second duration were made periodically with the periods varying from 5 seconds to 20 seconds. Intermittent operation was obtained by use of the second-timer with a stepping relay. The relay has 40 contacts. By advancing the relay one step per second, cycles of 5, 10, and 20 seconds each are available by connecting to the proper terminals. The oscillograph, timer motor, and stepping relay were all operated from two 6-volt storage batteries.

The electrodes were aligned in three sets of four pairs each as shown in figure 6. Switching made it possible to operate the individual pairs of electrodes into their corresponding galvanometer in the oscillograph or to connect each set of four pairs in a series-parallel combination for recording on one galvanometer. The individual arrangement was used in the transverse diffusion study, and the series-parallel connections were used for the longitudinal studies.

TABLE 1.—*Flow data, Fort Collins tests*

[L, longitudinal dispersion.   TS, transverse dispersion from a spot source.   TC, transverse dispersion form a continuous source.]

Test Number	Flume width (feet)	Flow (cubic feet per second)	Roughness coefficient n_1	Water depths at indicated manometer (feet)			Indicated water depth at point gage (feet)		Test No. and (or) type
				2	8	14	Upstream	Downstream	
2	7.94	8.00	0.010	0.312	0.447	0.549	0.336	0.508	L.
3	7.94	6.00	.010	.448	.544	.638	.492	.599	L.
4, 5	7.94	4.00	.010	.414	.506	.601	.457	.566	L.
6–12	7.94	2.00	.010	.434	.525	.620	.480	.590	6TS 7L 8–11TS 12TC.
13–15	7.94	1.74	.010	.429	.521	.616	--------	--------	TC.
16	7.94	2.08	.010	.458	.553	.650	--------	--------	TC.
17	7.94	1.40	.025	--------	--------	--------	--------	--------	L.
18–20	7.94	2.00	.025	.409	.493	.581	.446	.551	18L 19–20TC.
21, 22	7.94	4.00	.025	.448	.514	.596	.473	.569	21L 22TC.
23–26	7.94	8.00	.025	.514	.492	.479	.479	.474	23L 24–26TC.
27–30	7.94	8.00	.010	.339	.461	.566	.410	.542	TC.
31–34	7.94	4.00	.010	.384	.482	.579	.429	.545	TC.
35–38	7.94	1.00	.010	.411	.506	.601	.457	.568	35–36L 37–38TC.
39–41	7.94	.50	.010	.368	.461	.555	.413	.521	39L 40–41TC.
42, 43	7.94	1.00	.025	.405	.499	.592	.455	.559	42L 43TC.
44	7.94	1.00	.025	.389	.481	.576	.439	.551	TC.
45	7.94	.50	.025	.359	.452	.546	.402	.507	L.
46–49	7.94	.50	.010	.392	.486	.583	.439	.543	46–47L 48–49TC.
50–53	4.00	7.75	.010	.859	.951	1.044	.910	1.014	50L 51–52TC 53L.
54–56	4.00	4.00	.010	.862	.954	1.046	.905	1.014	54L 55–56TC
57–60	4.00	2.05	.010	.887	.979	1.071	.928	1.040	57–58L 59–60TC.
61, 62	1.67	1.50	.010	--------	--------	--------	1.628	1.733	61L 62TC.
63, 64	1.67	2.80	.010	--------	--------	--------	1.607	1.710	63L 64TC
65	1.67	5.80	.010	--------	--------	--------	1.639	1.708	L.

DESCRIPTION OF TESTS

Since the electrodes were energized from a 60-cycle-per-second alternating voltage source, the resulting current alternations caused the oscillograph galvanometer beam to swing over a width determined by the peak values of the current. The galvanometers had a sufficiently high undamped natural frequency to be able to follow faithfully these 60-cycle-per-second variations. One of the galvanometers was connected to a special electrode which was used for standardization. Fort Collins city water was used as a standard, and a reading was taken in a beaker of this water at the start of each test. During the test, this electrode was transferred to another beaker filled with water from the flume just before the start of the test. Since each electrode recorded the salinity of the flume water, these readings in the Fort Collins water and the flume water provide a means for setting the scale for each of the electrode records. The time scale was set by the 1-second timer in all cases as the paper speed was too slow to permit use of the timer built into the oscillograph. The correlation among electrode numbers, galvanometer numbers, and trace numbers is shown in table 2.

Between tests the water ran continuously to keep the salt content of the water in the sump uniform. The amount of salt used in each longitudinal diffusion test was 400 grams or 0.882 pounds. The dry salt was weighed out on a balance and was then put into solution and poured into the trough. At the beginning of

TABLE 2.—*Correlations among electrode, galvanometer, and trace numbers, Fort Collins tests*

Trace [1]	Electrode	Galvanometer
1 Time	--------	5
2 Standardizations	--------	6
3	1	7
4	5	8
5	9	9
6	2	10
7	6	11
8	10	12
9	3	13
10	7	14
11	11	15
12	4	16
13	8	17
14	12	18

[1] Traces are read from the top of the page down. Galvanometer 5 gives time of injection. Galvanometer 4, trace 15, is used for the 1-second timer record. Galvanometer 6 is used for Fort Collins water and flume water recordings. These trace assignments were used to avoid a confusing overlap of the records.

these tests, the salt solution was dumped uniformly across the width of the flume onto the surface of the running water. The trough consisted of a 120° sector of a cylinder concentric with the axis upon which it was mounted. A paraffin coating on the trough prevented adherence of any appreciable amount of salt solution to the trough surface. The mounting of the trough was such that the salt solution was released very close to the surface of the running water in the flume. In all tests, it is believed that the bottom of the trough was within an inch of the water surface. In the lateral diffusion

tests, the continuous salt solution injection was sometimes made near the bottom in the tests with an 8-foot-wide flume and a 0.5-foot water depth, but in the remainder of the tests the salt solution was introduced at the surface.

A few lateral diffusion tests were made by placing a 6-inch-diameter tube against a rubber mat on the bottom of the flume and filling it with salt solution to approximately the level of the water outside and pulling the tube away quickly at the start of the test. This procedure gave trouble, however, because of the density currents set up by the greater density of the salt solution. Use of permanganate color disclosed that these density currents persisted even though a siphon was used to bring the two levels to equilibrium before the tube was withdrawn. This type of test was therefore abandoned. These tests were Nos. 6, 8, 9, 10, and 11. In test 11, the salt content was reduced to 0.441 pound to reduce the difference in density between the salt solution and the water, but the dye disclosed that the density differential was still sufficient to cause flow.

In another test, an ammonia-salt solution was used to investigate the effects of density on the results of the tests. This test run is No. 47. Test 46 is identical except that it was run with salt. The ammonia-salt solution was made by mixing concentrated ammonia solution, distilled water, and salt in approximately the following proportions: 500 cubic centimeters ammonia solution, 300 cubic centimeters distilled water, and 90 grams of dry salt. This solution can be adjusted to have the same density as water. This solution is not recommended for continuous use because of its acrid odor and the nonlinearity of its conductance-concentration characteristics at high dilutions. However, in this test it served to disclose the presence of density currents in the tests conducted with salt solutions introduced into low velocity flows. Approximately 1 gallon of ammonia-salt solution used in this one test so contaminated the sump that it was necessary to drain the sump and refill it with fresh water. The sump was drained and refilled with fresh water subsequent to tests 41 and 49.

The weight of salt stated in connection with the tests

represents the weight of dry crystalline salt used. The salt was always put into solution before being used in the tests.

CALIBRATIONS

After the tests were completed, a calibration was made by placing a test electrode in beakers containing samples. The series was read four times and the test electrode was rinsed in distilled water after each reading. The following table shows the result of the calibration.

Sample	Average trace width (inches)
Fort Collins water, new sample	1. 024
Distilled water	. 057
Distilled water + 100 ppm of c.p. NaCl	3. 182
Distilled water + 100 ppm of Salt 2	3. 102
Distilled water + 100 ppm of Salt 1	3. 115
Fort Collins old sample + 100 ppm of Salt 1	4. 085

Salt 1 was used in tests up to and including test 42. Salt 2 was used in the remainder of the tests with the exception of test 47, which was run with the ammonia-salt solution.

COMPARISON OF DIFFUSION COEFFICIENTS, FORT COLLINS TESTS

The longitudinal diffusion coefficients obtained from selected Fort Collins tests are shown in table 3. Some transverse diffusion coefficients were also obtained from these tests and will be described later. Of the tests listed in the table, the test conditions approached the ideal more closely in test 23 than in any of the others. Because of the high flow rate and the roughness, the flow was strongly stable and had a surface nearly parallel to the bed of the flume. The gradients were also high enough to permit a good slope determination. The diffusion coefficients, identified as from test, are obtained from the observed salt concentrations by the use of equation 3. The computed values are obtained by using a formula adapted from the work of G. I. Taylor (1954) as described later under the heading "Determination of diffusion coefficients by analytical procedures." The comparative values are shown in the following table:

TABLE 3.—*Comparison of experimentally and analytically determined longitudinal diffusion coefficients, Fort Collins tests*

Test	Flow (cubic feet per second)	Flume width (feet)	Flume depth (feet)	Mean velocity (feet per second)	n_1	r_1 (feet)	C_1	v_* (feet per second)	Diffusion coefficients (square feet per second)	
									From test	Computed[1]
2	8. 0	7. 94	0. 447	2. 254	0. 010	0. 402	128	0. 100	0. 806	0. 810
3	6. 0	7. 94	. 544	1. 389	. 010	. 478	131	. 060	. 703	. 580
18	2. 0	7. 94	. 493	. 511	. 025	. 438	52	. 056	. 461	. 493
23	8. 0	7. 94	. 492	2. 048	. 025	. 438	52	. 225	1. 285	1. 985
58	2. 05	4. 0	. 984	. 533	. 010	. 660	138	. 0217	. 421	. 286
65	5. 80	1. 67	1. 674	2. 072	. 010	. 557	134	. 0884	. 877	. 985

[1] Computed from the formula $K_x = 20.2\, r_1\, v_*$.

ALLENSPARK TESTS

The experimental flume at Allenspark, Colo., is 400 feet long. Because a long conduit permits a more nearly complete development of the pattern of salinity distribution with length, this length is favorable for experimentally observing the variation of maximum salinity with distance. A view of this flume is shown in figure 8. The flume, being triangular in cross section, also

FIGURE 8.—View of Allenspark flume showing the triangular section. Point gages for measuring water-surface elevations were mounted on spacers crossing the flume. Readings were made with a level.

affords an opportunity to make tests to learn whether the shape of the cross section exerts any influence upon the rate of dispersion. A series of 11 tests were made on this flume. They confirm the inverse square root law and indicate that the shape of the cross section does not greatly alter the rate of dispersion. The Allenspark flume is supplied from North St. Vrain Creek. The water has only a small content of dissolved solids which favors the use of salt as a tracer material. The amount of tracer present was measured through its

effect on the conductivity of the water, as at Fort Collins.

The flume used for these tests is constructed of plywood sheets which are 4-feet wide and 8-feet long. These are assembled to form a V-shaped trough with an angle of 90° in the bottom. The cross-section of a stream flowing in this flume is, therefore, triangular in shape. The slope of this flume can be varied by use of jacks which slide it up or down on ways on one side. At the time of these tests, the flume was set to a slope of 0.00048. The interior of the flume is coated with an aluminum base paint, and great care was used by the builders to insure a smooth and uniform interior and a good alinement. Elevations of the bottom of the flume were read by means of a graduated bar. A similar bar was lowered until a needle point at the end made contact with the water surface, and the graduations were then read with a level. For the dispersion tests, a measured quantity of chemically pure sodium chloride, hereafter called salt, was dissolved in water and placed in a trough. This trough was the shortest of the three used at Fort Collins. At the start of each test, this salt solution was dropped into the water in the flume by rotating the trough around the axis by means of a handle. Passage of the salt cloud was recorded electrically by connecting electrodes in series with a source of alternating voltage and an oscillograph galvanometer element, as at Fort Collins. The electrodes were of the form shown in figure 9. They were made of $5/16$-inch-diameter rods,
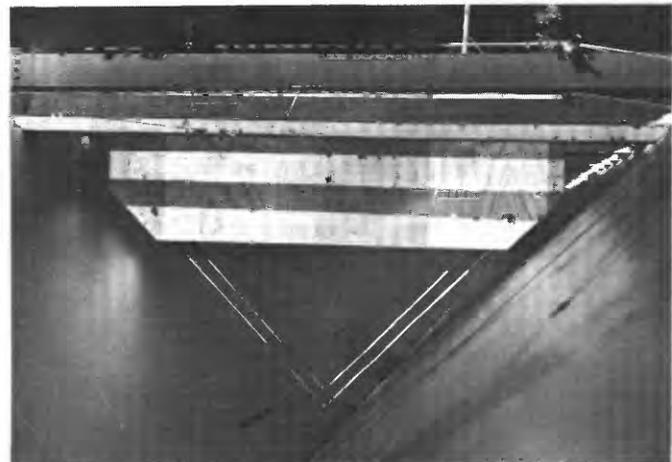
FIGURE 9.—Allenspark electrode arrangement. Holes in electrode support permit adjustment to depth of flow. Plastic sleeves prevent spurious readings due to wave motion.

spaced at 1.5-inch centers and clamped to a plywood frame. This assembly was supported from a brace which spanned the flume at the top. Holes in the plywood frame permitted an adjustment of the electrode position so that at the center of the flume the electrode

was at 0.6 depth of the stream cross section. Small plastic shields were placed around the electrodes near the waterline to prevent spurious variations of conductivity, due to wave motion, as shown in figure 9. The five electrodes used in these tests were spaced 75 feet apart. The salt trough was 75 feet upstream from the first electrode and 11 feet downstream from the upper end of the flume. The downstream electrode was 14 feet upstream from the lower end of the flume. Since the conductivity of a solution of salt is closely proportional to the concentration, in weak solutions, the amplitude of the galvanometer swing is a measure of the salt present. A shunt was used to adjust the galvanometer sensitivity. Use of an alternating voltage, obtained from a battery-powered vibrator, causes the galvanometer spot to sweep over a band width which is proportional to the amount of salt present. Galvanometer sensitivities were adjusted to give a band width of about 0.5 inch with the salinity naturally present in the creek water, and the amount of salt introduced was adjusted to increase this double amplitude to about 6 inches at the first electrode. Calibration was obtained by means of a supplementary test electrode which was placed alternately in a beaker of creek water and a beaker containing a 100 ppm salt solution. A separate galvanometer was used to indicate the time when the salt solution was dropped into the flume. This galvanometer was actuated by a switch which was thrown at the instant the salt solution struck the water. Another galvanometer was used with a 1-second interval timer to put time marks on the record. On tests 9, 10, and 11 the start indicating circuit was used to provide an independent time check where periods of 2.5 and 2.0 minutes were indicated on the record as timed by a stop watch. Communication facilities were provided between the oscillograph and salt injection stations by voice-powered phones.

Water flows were measured with rectangular and triangular weirs. The recording equipment was housed in a station wagon. Wires from the electrodes were brought in overhead and distributed to the same control panel that was used in the Fort Collins tests. Power supply for the galvanometers came from the 6-volt d-c to 110-volt a-c converter equipment, and storage batteries supplied the power for operating the oscillograph and the converter. To insure that a proper record was obtained before the flow in the flume was changed, a portable darkroom was used to develop the records as soon as they were taken.

LONGITUDINAL DIFFUSION COEFFICIENTS OBTAINED FROM ALLENSPARK TESTS

The diffusion coefficients obtained from the Allenspark tests are shown in table 4. The test value was

obtained by analyzing the recorded salinity patterns with the aid of equation 3. The analytically determined values are computed from the flume properties by use of the formula adapted from the work of G. I. Taylor (1954), as described later under the heading "Determination of diffusion coefficients by analytical procedures." The mean velocities shown were obtained by applying salt-velocity procedures. The flow rates were obtained by applying the observed velocities to the area of the cross section. The table on page B19 shows the coefficients as determined from the Allenspark test data.

When the peak salinities observed at the five electrode stations are plotted against distance on logarithmic cross-section paper, it is found that the relation of salinity versus length is of the form

$$S = A_1 x^{-n}$$

The values of n obtained from these plots are shown on the following tabulation:

Values of n as obtained from logarithmic plots of the test data

Test	Nominal water depth (feet)	n
8	1.0	0.331
9	1.0	.354
10	1.0	.318
11	1.0	.316
1	1.5	.552
2	1.5	.523
3	1.5	.544
4	2.0	.433
5	2.0	.485
6	2.0	.338
7	2.0	.463
Average		.423

The developments described indicate that the value of n should be one-half. The tabular values vary from 0.316 to 0.552 and average 0.423. These are considered to be in substantial agreement with the ideal value. Tests on very long conduits yield a value close to the ideal (Taylor, 1924).

THE SOUTH PLATTE DISPERSION TEST

A dispersion test was made on the South Platte River in October 1958. The reach chosen for this test extends from the Kersey bridge to the Kuner bridge. In this reach the river runs southeasterly, the Kuner bridge being 4 miles east and about 2 miles south of the Kersey bridge. The length of river between the two bridges is about 5 miles. The South Platte runs over a sandy bed in this area and generally follows more than one channel. At the Kersey bridge, for example, there were five channels. This stream is, therefore, of the braided type.

On October 22, the flow at the Kersey bridge was 600 cubic feet per second. Based upon this flow, detailed

TABLE 4.—*Longitudinal diffusion coefficients obtained from the Allenspark tests*

Test	Flow (cubic feet per second)	Maximum depth (feet)	Mean velocity (feet per second)	Hydraulic radius (feet)	Slope	n_1	v_* (feet per second)	K_x (square feet per second)	
								From test [1]	Computed $20.2\, r_1 v_*$
8, 9, 10, 11	1. 50	1. 005	1. 49	0. 355	0. 000480	0. 0092	0. 0622	0. 55	0. 45
1, 2, 3	4. 99	1. 473	2. 30	. 522	. 000480	. 0092	. 0901	. 88	. 95
4, 5, 6, 7	10. 22	1. 990	2. 58	. 704	. 000480	. 0092	. 0961	1. 32	1. 37

[1] Based on trace amplitudes corrected to conform to the known amounts of salt used in the tests.

plans were worked out for the test. A diffusion coefficient was selected, based upon the best data available, and the dispersion pattern to be expected at the points where the north-south section lines crossed the stream was computed. The Kersey bridge was designated point 0. Points 1, 2 and 3 are between the Kersey and Kuner bridges. The Kuner bridge was at point 4. A detailed program of sampling and stream gaging was then prepared and entered in the five field notebooks to be used by the observers at the five observation stations. Because the observers would be out of touch with each other during the test period, the program was based upon a time schedule. In addition to the samples and stream gagings, both air and water temperatures were read.

The tracer material chosen for this test was potassium carbonate. This material was chosen because it is non-poisonous and highly soluble in water and because potassium yields a brilliant flame color. This latter consideration was important since a flame photometer was to be used to analyze the samples. A preliminary test of the sample collected on October 22 showed 9 ppm of potassium ion as a natural constituent of the stream. To produce sufficient contrast to this background it was estimated that 1,000 pounds of potassium carbonate would be required. In preparation for the test, six 50-gallon drums were fitted with a valve and a nipple.

Six men participated in the test. The drums were placed on the downstream walkway of the bridge with the valve and nipple directed downstream, and about 40 gallons of solution was made up in each drum.

At 11:00 o'clock the men simultaneously opened a valve on each of the six drums to release the tracer solution into the stream. When the drums were nearly empty, the valve was closed and the remaining tracer material was dumped over the bridge rail. This operation took 80 seconds. The observers then proceeded to the downstream observation points. Samples were collected in 4-ounce size plastic bottles. The readings and samplings were complete at 3:35 p.m. The data obtained are shown in tables 5 and 6.

TABLE 5.—*Analysis of samples for potassium ion, South Platte test*

Location	Laboratory No.	Time of collection	(ppm) K+
Point 0	2919 [1]	12:50 p.m.	9. 2
	2927A	10:00 a.m.	8. 4
		10:30	8. 0
		11:30	8. 0
		12:00 m.	8. 4
		12:30 p.m.	8. 0
		1:00	8. 0
		1:30	8. 0
		2:00	8. 0
		2:30	8. 0
Point 1	2927B	11:00 a.m.	7. 8
		11:10	7. 8
		11:20	7. 8
		11:30	7. 8
		11:40	9. 4
		11:45	27. 0
		11:47	22. 8
		11:49	18. 6
		11:51	16. 8
		11:54	17. 8
		12:00 m.	12. 2
		12:10 p.m.	8. 8
		12:20	8. 2
		12:30	8. 0
Point 2	2927C	11:30 a.m.	8. 0
		11:40	8. 0
		11:50	7. 8
		12:00 m.	7. 8
		12:10 p.m.	7. 8
		12:15 (34[1])	7. 8
		12:15 (62[1])	12. 0
		12:15 (90[1])	14. 4
		12:15 (118[1])	14. 6
		12:20	16. 4
		12:30	16. 6
		12:40	13. 4
		12:50	11. 2
		1:00	9. 2
		1:10	8. 4
		1:20	8. 0
		1:30	9. 0
Point 3	2927D	12:00 m.	8. 2
		12:15 p.m.	8. 2
		12:30	8. 2
		12:45	8. 4
		1:00	9. 6
		1:10	13. 6
		1:12½	14. 8
		1:14	14. 8
		1:18	14. 6
		1:22½	13. 2
		1:30	12. 8
		1:45	10. 0
		2:00	9. 2
		2:15	8. 2
		2:30	8. 6
Point 4	2927E	12:30 p.m.	8. 0
		12:45	8. 0
		1:00	8. 0
		1:15	8. 0
		1:30	8. 0
		1:45	8. 0
		1:52	8. 0
		1:55	8. 0
		1:59	9. 6
		2:04	12. 4
		2:12	12. 6
		2:16	11. 8
		2:30	10. 2
		2:45	8. 6
		3:00	8. 0
		3:15	7. 8
		3:30	7. 8

[1] Large sample collected in a glass bottle. This sample contained no tracer.

A large sample collected at the Kersey bridge during the test was found, on analysis, to contain 1,420 ppm of dissolved solids. The day of the test was overcast with a slight wind blowing. Water temperatures remained at about 50° F while air temperatures remained at about 49° F. A traverse of the river at point 2 showed a maximum water temperature differential of only 0.27° F.

The potassium carbonate used was of technical grade having a purity of 98 to 100 percent. It is of the dry anhydrous form and visually is a white, free-running powder. The 1,000 pounds of tracer material are estimated to have contained 566 pounds of potassium ion.

Levels run subsequent to the test gave the following figures:

Station	Water-surface elevation
Kersey Bridge	4579. 17
Kuner Bridge	4550. 76
Fall in reach	28. 41

A diffusion coefficient of 174 ft²/per sec was obtained for the conditions of this test. Some loss of tracer was accounted for in making the determination. Results of stream gagings made during the test are shown in the table 6.

MOHAWK TEST

A test of stream dispersal characteristics was made in August 1954 by introducing 4.53 curies of radioactive phosphorus into the Mohawk River at the Knolls Atomic Power Laboratory site (Simpson, Beetem, Ruggles, 1958). This laboratory is located on the right bank of the stream downstream from Schenectady and about 1.7 miles upstream from the Vischers Ferry dam. The progress of the tracer was watched for about 2 weeks. The observations covered the approximately 13 miles of stream from the Knolls Atomic Power Laboratory site to the junction of the Mohawk and Hudson Rivers at Cohoes, N.Y. In this reach, the Vischers Ferry dam and the Crescent dam below it create pools which are a part of the Erie Canal system. Traffic moves from the Hudson River at Waterford, N.Y.,

through locks 1 to 6 into the pool above the Crescent dam. At the Vischers Ferry dam, lock 7 lifts the traffic to the pool above this dam. The reach between the Crescent dam and the Hudson River has two more dams with the Cohoes Falls between them. The river makes several sweeping bends in its passage through the test reach.

The test period beginning August 9, 1954, was one of low flow, the average flow being around 1,100 cubic feet per second (Thomas, 1956). Hydroelectric power plants at the Vischers Ferry and Crescent dams utilized this flow intermittently. The tracer was released into a sewer outlet. The water being discharged from the sewer was colder than the water in the river, and the density difference caused the formation of a density current which carried the tracer along the bottom to the deepest part of the channel before it became dispersed through the cross section of the stream.

This test has been described by Thomas (1956), Simpson, Beetem, and Ruggles (1958), and Parker (1958). Diffusion coefficients were obtained by Thomas (1956) and Parker (1958). The data of table 10 for this test are adapted from the analysis of Thomas who obtained 67 ft² per sec for the first 7.5 miles of the test reach. Parker obtained a coefficient of 100 ft² per sec for the river to mile 13 at the Cohoes waterplant.

Plankton samples collected during the test showed a high activity (New York State Department of Health, 1954), indicating that the phosphorus was being collected by these organisms. There was no evidence however that dispersion rates were altered by their activity since they are very small and float freely in the water. Agencies cooperating in this test were the U.S. Geological Survey, New York State Department of Health, The Knolls Atomic Power Laboratory, and Harvard University.

LATERAL DISPERSION

Fort Collins test flume.—When the Fort Collins tests were made, no data were available to indicate how rapid the lateral dispersion might be. The test equipment was therefore arranged to permit the salt concentration

TABLE 6.—*Discharge measurements and stream channel data, South Platte test*

Location	Time of flow measurement (p.m.)	Discharge (cubic feet per second)	Channels	Total top width (feet)	Area (square feet)	Velocity (feet per second)	Hydraulic radius (feet)	River distance from Kersey Bridge (feet)
Point 0 (Kersey Bridge)	1:50 to 3:00	588	5	213	242	2. 43	1. 14	0
Point 1	12:40 to 1:10	553	1	160	247	2. 24	1. 54	6, 100
Point 2	1:40 to 2:43	551	1	141	232	2. 38	1. 64	12, 400
Point 3	1:45 to 2:30	559	2	180	271	2. 06	1. 51	19, 900
Point 4 (Kuner Bridge)	12:05 to 12:50	553	2	189	272	2. 03	1. 44	27, 000

at each electrode to be recorded separately so that tests could be made from which the lateral dispersion rates could be determined. In the first lateral diffusion tests, the salt solution was placed in a tube pressed against a sponge rubber pad on the bottom of the flume, and the levels outside and inside brought to equilibrium by means of a syphon. The salt solution was then released by pulling the cylinder upward. Some potassium permanganate color was also included with the salt, and observation showed that the lateral spreading being obtained was due to a difference in density because the salt solution went to the bottom of the flume where it spread out. To overcome this difficulty, the salt was put in solution in a container and supplied continously to a selected point through a rubber tube. Analyses of the records of these tests indicate that the effects of density were still present to some degree since the tests made at low velocities show relatively high dispersion rates. However, by selecting tests in which the velocity was high, some lateral diffusion coefficients can be obtained. These are shown in the table 7.

COLUMBIA RIVER TEST

Data for this test were supplied through the courtesy of the General Electric Co. The reactor cooling water discharge used in the test is released at essentially a single point in the bottom of the river about 300 feet from the plant shore. Approximately 38 percent of the river flowed between the point of discharge and the plant shore. A mixture of short-lived radioisotopes in the reactor effluent was used as a tracer. The effluent is warmer than the water in the river.

The ten traverses sampled were selected to be as nearly normal to the direction of flow of the main current as could be judged visually. They were selected at distances of 950, 1,750, 2,850, 3,590, 6,230, 8,870, 14,150, 14,430, 24,710 and 29,990 feet from the point of discharge. The river makes a very gradual S-curve in the reach encompassed by these traverses. Velocities were measured with a cup-type velocity meter, and enough measurements were obtained to permit velocity contours to be plotted for each section.

The data relate to concentrations based on a value of 100 units per second arbitrarily assigned to the reactor effluent at the time of discharge to the river. The data were all corrected for decay. The samples were collected from a uniform depth of 3 feet below the surface of the river, as it had been previously determined that vertical concentration variations are negligible on the scales used for plotting. The test was performed near the middle of the annual low-water period of the Columbia, and no major changes of flow rate had occurred during the preceding three months. The flow rate of the river as reported by an adjacent USGS gaging station during the day of the test was 62,000 cubic feet per second.

The lateral diffusion coefficient, as obtained by the writer, is based upon the data from the first four stations. In this 3,590-foot reach, the river has a nearly uniform width of 1,000 feet. The maximum depth varies from about 16 to 24 feet. The concentration curve for station 4 is quite regular.

The measured flows in the reach including traverses 1 to 4, inclusive, range from 40,400 to 45,900 cubic feet per second and average 43,625. The flow of the river in this reach is affected by the operation of powerplants. The flow reported by the U.S. Geological Survey represents a mean for the day while the measured flows represent the conditions at the time the test was being made. The measured flows were used in making the analyses described here.

Several difficulties were cited in the original report. Because the river is large and swift, it is difficult to locate sampling points with exactitude. Because the time of transit was somewhat uncertain, it was difficult to allow precisely for the decay of the short-lived isotopes. There was some variation at the source. A standard deviation of 10 percent was noted in hourly samples. There was a marked temperature difference between the effluent and the river water. A small rapids is present just downstream from traverse 5. This is 6,230 feet downstream from the discharge point.

In spite of these difficulties the data are correlated well by a diffusion coefficient K_y=2.1 ft² per sec. If the

TABLE 7.—*Lateral diffusion coefficients obtained from selected Fort Collins tests*

Test	n_1	Flow (cubic feet per second)	Flume		Mean velocity (feet per second)	Diffusion coefficient (square feet per second)		Ratio of coefficients
			Width (feet)	Depth (feet)		From test [1]	0.067 $r_1 v_*$	
25	0. 025	8. 00	7. 94	0. 496	2. 04	0. 035	0. 015	2. 3
27	. 010	8. 00	7. 94	. 461	2. 19	. 0066	. 0065	1. 0
31	. 010	4. 00	7. 94	. 482	1. 05	. 044	. 0031	14. 2
51	. 010	7. 75	4. 00	. 951	2. 04	. 0050	. 0053	1. 0

[1] These analyses were made by means of equation 30.

diffusion coefficients for transverse and vertical diffusion are comparable, an estimated diffusion coefficient would be, as will be explained later:

$$K_y = 0.067\ Dv_*$$

With $D = 10.03$ feet and $n_1 = 0.025$, Manning's formula gives $C_1 = 87.29$ and $v_* = 0.287$ ft per sec. The mean velocity is $V = 4.418$ ft per sec. These values yield the product 2.88 and a diffusion coefficient of 0.193 ft² per sec. In this case, therefore, the value observed in an actual stream is approximately 10 times the value obtained from investigations of vertical diffusion rates.

INTEGRAL RELATIONSHIPS

Oscillograph charts resembling those obtained from the experiments described here have often been obtained during salt velocity determinations of the flow passing through penstocks. This method has been widely used, and considerable literature is available on developments in this art. If it can be shown that the relationships developed in the practice of this art are in accord with those derived from the concept of stream dispersion as a diffusion process, then the belief that stream dispersion is of this nature would be strongly confirmed. It seems worthwhile to do the small amount of work needed to make this test. It will be seen later that this concept is confirmed and that some important relations are also obtained as a by-product.

The relations of interest for the present purpose are:
1. For two stations where a salinity-time pattern is recorded, the time interval between the centers of gravity of the two patterns represents the time required for the mean velocity to traverse the distance between the two stations.
2. The maximum salinity observed varies inversely as the square root of the distance travelled by the salt cloud.

To investigate the possibility of correlating the relationship of item 1 we need two integrals. These are:

$$J_1 = BDV \int_0^\infty S\,dt \tag{34}$$

and

$$J_2 = BDV \int_0^\infty St\,dt \tag{35}$$

or

$$J_1 = \frac{Q_s V}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\left(\frac{x-Vt}{\sqrt{4K_x t}}\right)^2}}{\sqrt{4K_x t}}\,dt \tag{36}$$

and

$$J_2 = \frac{Q_s}{\sqrt{\pi}} \int_0^\infty \frac{Vt\,e^{-\left(\frac{x-Vt}{\sqrt{4K_x t}}\right)^2}}{\sqrt{4K_x t}}\,dt. \tag{37}$$

The integrals of equations 34 and 35 represent, respectively, the area under the salinity time curve and the moment of this area about the origin of the time scale. The ratio $J_2 : J_1$ represents the time to the center of gravity of the salinity-time area. This interpretation is appropriate when, as is usually the case, the salinity time curve is represented by a graph in rectangular coordinates with salinity as ordinate and with time represented by a distance measured along the abscissa.

The substitution of variables

$$u = \frac{x}{\sqrt{4K_x t}} \qquad \alpha = \frac{xV}{4K_x} \tag{38}$$

reduces these integrals to the form

$$J_1 = \frac{Q_s x V e^{2\alpha}}{2K_x \sqrt{\pi}} \int_0^\infty \frac{e^{-u^2 - \frac{\alpha^2}{u^2}}}{u^2}\,du \tag{39}$$

and

$$J_2 = \frac{Q_s x^3 V e^{2\alpha}}{8K_x{}^2 \sqrt{\pi}} \int_0^\infty \frac{e^{-u^2 - \frac{\alpha^2}{u^2}}}{u^4}\,du. \tag{40}$$

The two integrals which appear here are evaluated in the following chapter. By using the relations developed there, it can be shown that

$$J_1 = Q_s \tag{41}$$

and

$$J_2 = \frac{Q_s x}{V}\left(1 + \frac{2C}{x}\right) \tag{42}$$

where $C = (K_x/V)$ is a constant. The relation is confirmed if the velocity is determined by the use of two electrode stations and the time of transit from one to the other is determined by subtraction. The time of transit represented by the center of gravity is

$$t_m = \frac{J_2}{J_1} \text{ or } t_m = \frac{x + 2C}{V}.$$

There term $\frac{2C}{V}$ is a constant which will disappear if the difference between two values of t_m is taken. This is the customary procedure in salt-velocity work. It is clear, then, that the concept of dispersion in streams as a diffusion process is in accord with salt velocity experience.

We may now consider item 2. The condition that the mean velocity will have reached the point x is expressed by the relation $x - Vt = 0$. This relation will reduce the exponential term of equation (3) to unity. At this time also the salinity will be very near its maximum. Then when $x - Vt = 0$, the salinity varies inversely as the square root of the time. This relation is closely con-

firmed by tests in long pipes (Taylor, 1924). It is clear, therefore, that the concept of stream dispersion as a diffusion process is in complete accord with salt-velocity experience.

Having established that the concept of stream dispersion is in accord with salt velocity practice and experience, our attention may now be turned to some relationships which involve these integrals. At the outset we may note that the salinity introduced is certainly represented by the expression

$$W_1 = BD \int_{-\infty}^{+\infty} S dx \qquad (43)$$

or

$$W_1 = \frac{Q_s}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-\left(\frac{x-Vt}{\sqrt{4K_x t}}\right)^2}}{\sqrt{4K_x t}} dx. \qquad (44)$$

The substitution of variable

$$u = \frac{x-Vt}{\sqrt{4K_x t}} \qquad (45)$$

reduces this integral to the form

$$W_1 = \frac{Q_s}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} du. \qquad (46)$$

The integral involved here has the value $\sqrt{\pi}$ and the result of this computation is then

$$W_1 = Q_s. \qquad (47)$$

This is the expected result.

If equations 34, 41, 43, and 47 are now compared, it will be seen that we have the integral of the salinity time curve for a given point x and the integral of the salinity-space curve at a given time both representing the original amount of salt Q_s introduced into the stream. In this connection, it is of interest to note that a similar development, based upon the concept of dispersion by velocity differentials, yields Q_s from the salinity-space integral, as it should, but that the salinity-time integral does not yield Q_s. If the integration is carried over a finite time, the result is often close to Q_s; but if the integration is carried from zero to infinity, an infinite result is obtained. The data from an actual test would, of course, involve only a finite time interval. This difference suggests that, while the integrals described above show that the salinity-time integral should yield Q_s, if the dispersion mechanism is of the diffusion type, a determination of this quantity from test data may be sensitive to experimental errors due to the manner of making the salinity measurements. A comparison of the results of the Fort Collins and Allenspark tests indicates that this is true. The Fort Collins records yield salinity-time integrals which indicate a lesser amount of salt than was known to have been used, while the salinity-time curves from the Allenspark tests yield total salinities in excess of the amount of salt introduced. The discrepancy reaches 56 percent in the Fort Collins tests and 30 percent in the Allenspark tests. In both tests the sampling was done by electrodes distributed through the stream cross section. The salinity-time curves should have been well defined in these tests, but, as indicated above, the area of the salinity-time curve did not determine closely the amount of salt present. This possibility should be taken into account when making interpretations of test data.

In return for the assistance received from salt-velocity experience, it may be pointed out that the combination of analytical and experimental identification of stream dispersion as a diffusion process and the integral relations described above can be used to provide a new theoretical basis for the salt-velocity method of velocity determination.

EVALUATION OF INTEGRALS

To evaluate integrals of the type

$$W_2 = \int_0^\infty \frac{e^{-u^2-\frac{\alpha^2}{u^2}}}{u^2} du \qquad (48)$$

and

$$W_4 = \int_0^\infty \frac{e^{-u^2-\frac{\alpha^2}{u^2}}}{u^4} du, \qquad (49)$$

consider the integral:

$$W_0 = \int_0^\infty e^{-u^2-\frac{\alpha^2}{u^2}} du. \qquad (50)$$

By differentiation with respect to a (Osgood, 1933, p. 486),

$$\frac{dW_0}{d\alpha} = -2\alpha \int_0^\infty \frac{e^{-u^2-\frac{\alpha^2}{u^2}}}{u^2} du. \qquad (51)$$

Let

$$u = \frac{\alpha}{\eta}, \quad \eta = \frac{\alpha}{u}, \quad \text{and} \quad du = \frac{-\alpha}{\eta^2} d\eta.$$

By substitution

$$\frac{dW_0}{d\alpha} = -2 \int_0^\infty e^{-\eta^2-\frac{\alpha^2}{\eta^2}} d\eta. \qquad (52)$$

Then

$$\frac{dW_0}{d\alpha} = -2W_0. \qquad (53)$$

This is a differential equation whose solution is:

$$W_0 = \frac{\sqrt{\pi}}{2} e^{-2\alpha}. \tag{54}$$

The arbitrary constant of the solution was determined from the known value of the probability integral to which the integral 50 reduces when $\alpha = 0$.

By the use of equations 51 and 54 the value of integral 48 can be determined as

$$W_2 = \frac{\sqrt{\pi}}{2\alpha} e^{-2\alpha}. \tag{55}$$

By equating the second derivatives of W_o with respect to u, as obtained from equations 50 and 54, equation 49 can be evaluated as

$$W_4 = \frac{\sqrt{\pi} e^{-2\alpha}}{2\alpha^2} \left(1 + \frac{1}{2\alpha}\right). \tag{56}$$

To evaluate an integral of the type

$$W_1 = \int_0^\infty \frac{e^{-u - \frac{\beta}{u}}}{u} \, du \tag{57}$$

consider the integral

$$W_{-1} = \int_0^\infty e^{-u - \frac{\beta}{u}} du. \tag{58}$$

By differentiation with respect to β (Osgood, 1933)

$$\frac{dW_{-1}}{d\beta} = -\int_0^\infty \frac{e^{-u - \frac{\beta}{u}}}{u} \, du \tag{59}$$

and

$$\frac{d^2 W_{-1}}{d\beta^2} = \int_0^\infty \frac{e^{-u - \frac{\beta}{u}}}{u^2} \, du. \tag{60}$$

If

$$v = \frac{\beta}{u} \tag{61}$$

then

$$\int_0^\infty \frac{e^{-u - \frac{\beta}{u}}}{u^2} \, du = -\frac{1}{\beta} \int_\infty^0 e^{-v - \frac{\beta}{v}} dv \tag{62}$$

and

$$\frac{d^2 W_{-1}}{d\beta^2} = \frac{1}{\beta} W_{-1}. \tag{63}$$

A solution of this differential equation which satisfies the requirement that, when $\beta = 0$ (McLachlan, 1955, p. 133),

$$W_1 = \int_0^\infty e^{-u} du = -e^{-u} \Big|_0^\infty = 1 \tag{64}$$

is

$$W_{-1} = \sqrt{4\beta} K_1(\sqrt{4\beta}) \tag{65}$$

where K_1 represents the modified Bessel function of the second kind of order 1. (British Association for the Advancement of Science, 1937.) By differentiation with respect to β, use of the relation

$$K(_1'z) = -\frac{K_1(z)}{z} - K_0(z) \tag{66}$$

together with equation 59, it can be shown that

$$\int_0^\infty \frac{e^{-u - \frac{\beta}{u}}}{u} \, du = 2K_0(\sqrt{4\beta}). \tag{67}$$

The substitution of variable

$$v^2 = u \tag{68}$$

in the integral of equation 67 will yield the result

$$\int_0^\infty \frac{e^{-v^2 - \frac{\beta}{v^2}}}{v} \, dv = K_0(\sqrt{4\beta}). \tag{69}$$

This transformation shows that the integrals evaluated by the two developments described above are closely related. The first group has the even powers of the variable in the denominator, while those of the second group have the odd powers, when they are put into a comparable form.

DETERMINATION OF DIFFUSION COEFFICIENTS BY ANALYTICAL PROCEDURES

Some noteworthy progress has been made in the analytical determination of diffusion constants for flowing fluids. Diffusion in pipes has been treated in two papers by Taylor. (Taylor, 1953, p. 186–203; 1954, p. 446–468.) The latter of the two relates to the disperson of matter in turbulent flow through a pipe. For this flow he determines a virtual coefficient of diffusion:

$$K = 10.1av_*, \tag{70}$$

where a represents the radius of the pipe and (Prandtl, 1952, p. 126)

$$v_* = \sqrt{\frac{\tau_0}{\rho}} \tag{71}$$

is the shear velocity. The symbol τ_0 represents the friction stress exerted by the fluid, of density ρ, on the wall. This virtual coefficient represents the combined action of variation of velocity over the cross section of the tube

and turbulent diffusion in the radial direction. It is therefore of a type comparable to the longitudinal diffusion constant K_x as used here.

A later development (Elder, 1959) applies to flow in a wide channel. His analysis is based upon a logarithmic form of the velocity profile between the bed of the stream and the free surface. The result he obtains, if expressed in the present notation, is

$$K_x = 5.93 r_1 v_*. \tag{72}$$

In writing this expression, Elder's stream depth h has been replaced by the hydraulic radius r_1. Of the factor 5.93, the part contributed by the combination of velocity variations and lateral dispersion due to turbulence is 5.86, while the part contributed by longitudinal dispersion due to turbulence is 0.07. The above expression has the same form as that found by Taylor (1954), but the factor is only about one-third as large. Elder has made experiments on a small flume in which a dispersing dye concentration was recorded photographically. He found a part of the dye being carried by the turbulent portion of the stream and a part by the laminar sublayer. For the turbulent portion he obtained a factor 6.3 which is about 8 percent above the theoretical value 5.9.

Elder (1959) has also discussed the dispersion of particles of finite size and has noted that, for nonbuoyant particles previous experiments indicate that very small discrete particles are dispersed at the same rate as the fluid particles.

The quantities involved in the diffusion coefficient are clearly identified by these investigations. They include a dimension descriptive of the cross section of the stream, here represented by the hydraulic radius r_1 and the shear velocity. It therefore seems appropriate to express the results of experiments by the ratio

$$(K_x : r_1 v_*).$$

The virtual coefficient of longitudinal diffusion is also given by Taylor (1954) in the form in the present notation:

$$K = 14.28 r_1 V \sqrt{\gamma} \tag{73}$$

where γ is a friction factor of the Darcy-Wiesbach type. The results obtained by Taylor (1954) may be adapted to our present purposes by expressing the pipe radius a in terms of the hydraulic radius r_1. On this basis we obtain

$$K_x = 20.2 r_1 v_* \tag{74}$$

Some further relations will facilitate the use of this expression. If we write equation 71 in the form:

$$v_* = \sqrt{\frac{\tau_0 g}{w}} \tag{75}$$

where w represents the weight of a unit volume of water and g the acceleration of gravity and note that in a stream of hydraulic radius r_1 and slope s_1

$$\tau_0 = w r_1 s_1, \tag{76}$$

we can obtain by substitution (see also Boyer, 1954)

$$v_* = \sqrt{g r_1 s_1}. \tag{77}$$

If we compare this with the well-known Chezy formula [2] (King, 1954, p. 7–15)

$$V = C_1 \sqrt{r_1 s_1}, \tag{78} \dagger$$

we may write, by introducing the quantity g:

$$V = \frac{C_1}{\sqrt{g}} \sqrt{g r_1 s_1}. \tag{79} \dagger$$

Then the shear velocity is given by relations of the type:

$$V = \frac{C_1}{\sqrt{g}} v_* \text{ or } v_* = \frac{\sqrt{g}}{C_1} V. \tag{80} \dagger$$

In the cases under consideration here the ratio $v_* : V$ would range, approximately, between the limits 0.03 to 0.2. For some purposes it will be advantageous to express equation 74 in the form

$$K_x = 20.2 r_1 V \left(\frac{\sqrt{g}}{C_1} \right). \tag{81} \dagger$$

This equation shows that, for a given cross section, the diffusion coefficient should be proportional to the mean velocity. It is interesting to note that Taylor's researches confirm the inverse square root law for longitudinal diffusion phenomena.

We may now turn our attention to the transverse dispersion rates. These are represented by constants of the type K_y and K_z. Because they depend upon turbulence only, instead of a combination of velocity variations and turbulence, they may be expected to be much smaller than the longitudinal dispersion constant K_x The case of vertical dispersion has been considered by

[2] Tabulations of C_1 values are generally dimensioned quantities. The quotient $\frac{C_1}{\sqrt{g}}$ is dimensionless. Care should be taken to have C_1 and g in the same units when computing the quotient.

  † See note at end of notation list.

Vanoni (1953, pp. 141–144) in connection with the phenomenon of sediment suspension. He finds that the dispersion rate is given by an expression of the type

$$\epsilon_m = k_1 v_* \left(1 - \frac{\zeta}{D}\right) \zeta \tag{82}$$

where $k_1 = 0.4$ is von Karman's constant and ϵ_m of Vanoni's notation corresponds to our K_z. This expression has been used as a basis for derivation of formulas for the distribution of sediment in streams, and it is stated that these have been confirmed by numerous laboratory and field measurements.

For our purposes we may use an average value

$$K_z = \frac{k_1}{6} D v_* \tag{83}$$

or

$$K_z = 0.067 D v_*. \tag{84}$$

A comparison of this form and that for the longitudinal dispersion constant K_x will disclose that their magnitudes stand in the ratio of about 300 to 1 with K_x being the larger. Because both K_y and K_z are due to turbulence alone, they may be of the same order of magnitude.

SUMMARY OF LONGITUDINAL AND LATERAL DIFFUSION COEFFICIENTS

A digest of the analytical and experimental data on diffusion coefficients developed in this study are contained in table 8. Some results obtained by other investigators are also listed in this table for purposes of comparison, and the sources of these data are indicated. Analytical investigations based upon fluid mechanics considerations show that the diffusion coefficient K_x and K_y should be proportional to the product of the shear velocity v_* and the hydraulic radius r_1. The data are presented in terms of a ratio to this product.

EXAMPLES

The following examples have been prepared as a guide for applying the equations described here. By making the examples apply to streams for which test data are available, comparisons of computed and test data can be made, and the factor of loss of tracer can be examined. The computations will be made as for a new problem, and the values so obtained will then be compared with those obtained from the tests.

LONGITUDINAL DISPERSION

The first of these computations will be made for the conditions prevailing on the 27,000-foot reach of the South Platte River between Kersey and Kuner, Colo., on the date of October 28, 1958. The South Platte is

TABLE 8.—*Comparison of computed and observed values of longitudinal and transverse diffusion coefficients*

$\dfrac{K_z}{r_1 v_*}$	Source	Remarks
		Longitudinal Diffusion
5.86	Analysis	Value obtained from the analytical and experimental investigations of Elder (1949) on dispersion in open channels.
20.2	Analysis	Value adapted from the analytical and experimental work of Taylor (1954) on dispersion in pipes.
18	Triangular flume experiments.	Value obtained from tests 1, 2, and 3 of the Allenspark series. Flume 400 ft long with plywood sides at 90°; depth, 1.473 ft; flow, 4.99 ft³ per sec; average velocity of flow, 2.30 ft per sec; hydraulic radius, 0.521 ft; slope, 0.000480; Kutters n_1 value, 0.0092 (Powell and Posey, 1959); diffusion coefficient, 0.825 ft² per sec.
24	Rectangular flume experiments.	Value obtained from test 3 of the Fort Collins series. Plywood flume, 7.94 ft wide and 75 ft long; average depth, 0.544 ft; flow, 6.0 ft³ per sec; mean velocity of flow, 1.39 ft per sec; hydraulic radius, 0.478 ft; Kutters n_1 value 0.010; diffusion coefficient 0.702 ft² per sec.
13	Experiment on a rectangular flume with bar roughness on the bottom.	Value obtained from test 23 of the Fort Collins series. Plywood flume, 7.94 ft wide and 75 ft long with a mesh of welded reinforcing bars laid on the floor; average depth, 0.492 ft; hydraulic radius, 0.438 ft; flow, 8.0 ft³ per sec; average velocity of flow, 2.05 ft per sec; Kutters n_1 value, 0.025; diffusion coefficient, K_z=1.26 ft² per sec.
500	Test on a natural stream (South Platte test).	Value obtained from a test on the South Platte River near Kersey, Colo. Braided stream running over a sandy bed; flow, 554 ft³ per sec; average velocity of flow, 2.16 ft per sec; hydraulic radius, 1.52 ft; effective slope. 0.00105; n_1=0.028; diffusion coefficient K_z=174 ft² per sec. (Some loss of tracer accounted for); test reach 27,000 ft long.
800	Test on a developed stream (Mohawk River test).	Value obtained from a test on the Mohawk River near Cohoes, N.Y. Conditions of great complexity with dams, intermittent flow through powerplants, expanding sections, crookedness, barge traffic, lockage, and density currents. Average flow, 1,100 ft³ per sec; diffusion coefficient, 67 ft² per sec; average depth 20 ft (Thomas, 1956.) with r_1=20 ft, n_1=0.025, V=0.07525 ft per sec, v_*=0.00435 ft per sec, $\dfrac{K_x}{r_1 v_*}$=770. This value accounts for loss of tracer.
		Transverse Diffusion
0.36	Experiment on a rectangular flume with bar roughness on the bottom.	Value obtained from test 25 of the Fort Collins series. Bar roughness laid in bottom of flume to raise the roughness to n_1=0.025; flume, 7.94 ft wide, 0.484 ft deep; flow, 8.0 ft³ per sec, K_y=0.036 ft per sec.
.22	Rectangular flume experiment.	Value obtained from test 27 of the Fort Collins series. Bare flume with n^1=0.010. Flume, 7.94 ft wide and 0.453 ft deep; flow, 8.0 ft³ per sec; K_y=0.0089 ft² per sec.
.14	Rectangular flume experiment.	Value obtained from test 51 of the Fort Collins series. Bare flume with n^1=0.010. Flume, 4 ft wide and 0.943 ft deep; flow, 7.75 ft³ per sec; K_y=0.0077 ft² per sec.
.72	Test on a natural stream (Columbia River test).	Value obtained from a test on the Columbia River near Richland, Wash. Top width about 1,000 ft; maximum depth 16 to 24 ft; diffusion coefficient obtained from observations on a 3,590-ft reach; flow, 43,600 ft³ per sec; mean velocity, 4.42 ft per sec; hydraulic radius, 10.03 ft; K_y=2.0 ft² per sec.

a braided stream that runs over a sandy bed. On the day of the test the flows averaged 554 ft³ per sec, the areas, A, of the cross sections 256 sequare feet, the top widths 168 feet, the hydraulic radii, r_1, 1.532 feet, and the mean velocity, V, 2.164 ft per sec. These are averages obtained from four discharge measurements made in the reach during the test. A reference to table 6 will indicate that there was not much variation in the flow in the test reach.

Potassium carbonate was used as a tracer. Samples were taken and measurements were made by means of

a flame photometer. This device measured the potassium ion of which the 1,000 pounds of tracer contained 566 pounds. The sensitivity of the flame photometer is about 0.1 ppm. The tracer was put in solution in six drums and was released into the river at the Kersey bridge at 11:00 a.m. The time required for the release was 1 minute and 20 seconds.

We will compute the salinity pattern at points 3 and 4. A longitudinal diffusion coefficient must first be selected. By reference to table 8, it is seen that for a stream of this type a ratio of $K_x : r_1 v_* = 500$ is appropriate.

A suitable value for the shear velocity v_* can be obtained from equation 55:

$$V = \frac{C_1}{\sqrt{g}} v_*. \qquad \dagger$$

For a stream of this kind, a reasonable value for the coefficient of roughness would be $n_1 = 0.025$.

By use of Manning's formula with $n_1 = 0.025$,

$$C_1 = \frac{1.486}{n_1}, \text{ and } r_1^{1/6} = 63.9, \qquad \dagger$$

then

$$\frac{V}{v_*} = \frac{C_1}{\sqrt{g}} = \frac{63.9}{5.674} = 11.26 \qquad \dagger$$

and

$$v_* = \frac{V}{11.26} = \frac{2.164}{11.26} = 0.193. \qquad \dagger$$

The estimated diffusion coefficient is

$$K_x = (500) r_1 v_* = (500) \ (1.536) \ (0.193) = 147 \ ft^2 \text{ per sec.}$$

This value is somewhat smaller than the one obtained from the analysis of the test data. This was based upon a measured gradient. The difference is due to the choice of n_1. A value of about $n_1 = 0.028$ would be required to reproduce the test value. It will be found, however, that a reasonably good correlation can still be obtained because the presence of K_x under the square root sign in equation 3 makes it somewhat insensitive to variations of the diffusion coefficient.

The distance from the point of introduction of the tracer to point 3 was 19,900 feet. The time required for the mean velocity to reach this point may be estimated as $19900 \div 2.178 = 9140$ seconds, or 2 hours 32 minutes and 20 seconds. This would put the estimated time of arrival of the peak of the salinity pattern at about 1:32 p.m. The observed time of arrival was at about 1:15 p.m., or 17 minutes earlier. Differences of this

magnitude should be expected when dealing with natural streams.

The computations will cover the period from 12:30 p.m. to 2:30 p.m. at 5-minute intervals. If

$$Q_s = 566 \text{ pounds of potassium ion,}$$

$$BD = 256 \ ft^2 = A,$$

and

$$K_x = 147 \ \frac{ft^2}{\text{sec}},$$

then

$$\frac{Q_s}{BD\sqrt{4\pi K_x}} = \frac{566}{(256) \ (42.93)} = 0.0515.$$

The computation is shown in table 9. It is based on the use of equation 3.

TABLE 9.—*Sample computation for point 3, South Platte test*

Time (hr, min)	t (sec)	$x-Vt$ (ft)	$\frac{(x-Vt)^2}{4K_x t}$	$e^{-\frac{(x-Vt)^2}{4K_x t}}$	S (lb per cu ft · 10^3)	*K^+ (ppm)
12:50	6600	5618	8.1472	0.00029	0.00018	0.003
12:55	6900	4968	6.094	.00226	.00140	.022
1:00	7200	4319	4.414	.01215	.00737	.119
1:05	7500	3670	3.060	.04689	.02788	.447
1:10	7800	3021	1.993	.13629	.07947	1.274
1:15	8100	2372	1.183	.30636	.17531	2.809
1:20	8400	1722	.601	.54827	.30807	4.937
1:25	8700	1073	.225	.79852	.44080	7.064
1:30	9000	424	.034	.96658	.52471	8.409
1:35	9300	−225	.009	.99075	.52909	8.479
1:40	9600	−874	.136	.87320	.45897	7.355
1:45	9900	−1524	.400	.67052	.34705	5.562
1:50	10200	−2173	.789	.45448	.23175	3.714
1:55	10500	−2822	1.292	.27472	.13807	2.213
2:00	10800	−3471	1.901	.14942	.07405	1.187
2:05	11100	−4120	2.605	.07390	.03612	.579
2:10	11400	−4770	3.400	.03337	.01610	.258
2:15	11700	−5419	4.276	.01390	.00662	.106
2:20	12000	−6068	5.228	.00537	.00252	.040
2:25	12300	−6717	6.249	.00193	.00090	.014
2:30	12600	−7366	7.336	.00065	.00030	.005

* Equivalent potassium ion concentration.

A comparison of these estimated salinities with the observed salinities is shown on figure 10.

The observed salinities are listed in table 5. It will be noted that while the observed pattern is about as wide as the computed pattern, the observed salinities reach only 6.6 ppm as compared to the estimated 8.7 ppm. Some loss of tracer is indicated.

A similar condition is found at point 4 where the estimated travel time is

$$\frac{27,000}{2.178} = 12,400 \text{ seconds}$$

and the estimated maximum salinity is

$$S_m = \frac{Q_s}{BD\sqrt{4\pi K_x t}} = \frac{0.05253}{111.35} = 0.000472 \text{ lb per ft}^3$$

or 7.6 parts per million of potassium, by weight. This
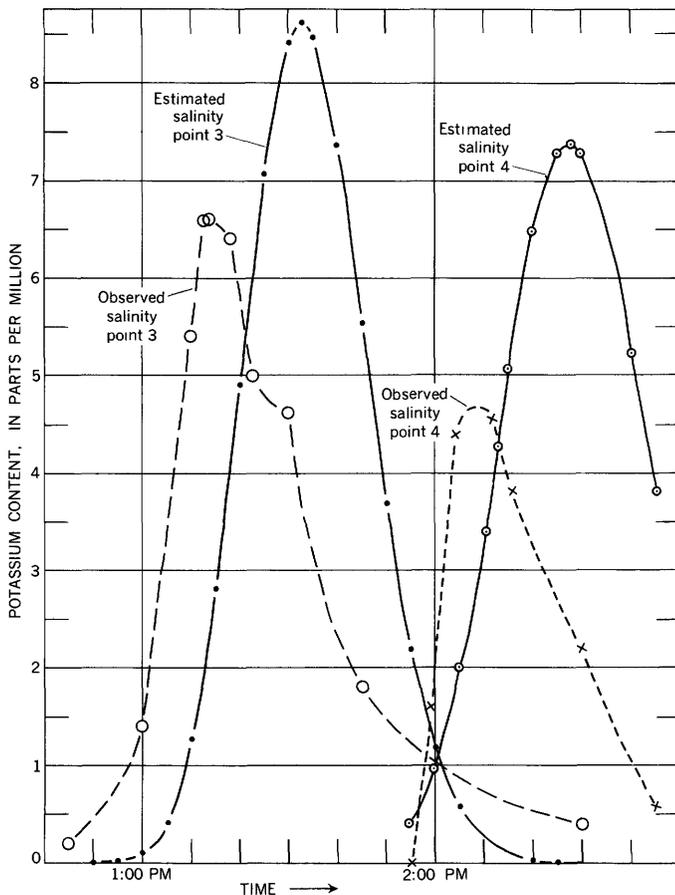
FIGURE 10.—Comparison of observed and computed salinities at points 3 and 4, South Platte test.

conversion is based upon a density of water of 62.4 pounds per cubic foot. The maximum observed salinity at this point was $12.6 - 8.0 = 4.6$ ppm.

A good correlation of peak amplitudes can be obtained by using a diffusion coefficient of about 320 square feet per second, but the computed patterns are then much too wide. The areas of the observed salinity-time curves in terms of potassium ion at the four observation points are given in the following table.

Point	Indicated amount of potassium ion (lb)	Remarks
Kersey bridge	566	Amount introduced.
1	501	
2	571	
3	510	
4	304	

If the difficulty of determining the amount of salinity from the salinity-time curve, as revealed by the Fort Collins and Allenspark tests, is kept in mind, the first three stations show a reasonably consistent behavior. The amount of potassium ion indicated at point 4 is low by comparison. This may be due to the way the

samples were collected at point 4. The sampling schedule had been worked out for a flow of 600 cubic feet per second as determined from a recording gage reading obtained by telephone. However, on the morning of the test the flow was down to about 554 cubic feet per second. This had an effect on the timing and, at point 4, caused the salt cloud to pass while a series of readings across the stream were being made. Because these samples were taken from a bridge, this series took more time than those made at the other stations where they were obtained by wading. A similar evidence of tracer loss was also present in the Mohawk test.

A bump is present in the descending limb of the curves of observed salinity at points 1, 3, and 4. More closely spaced readings at point 2 would probably have shown its presence there also. The reason for this may be found in the way the tracer was introduced. The tracer was put in solution in drums placed on a walkway of the Kersey bridge. Initial release was made by opening a valve. As soon as the contents of the drum was reduced to the point where a man could lift the drum, the valve was quickly closed, and the remaining contents were dumped over the bridge rail. The bump is probably due to the dumped part.

This comparison should indicate the type of behavior to be expected in a natural stream. The propriety of the analytical formulation used is supported by the work of Taylor (1954) and Elder (1959).

TRANSVERSE DISPERSION

A computation of lateral diffusion will be made for the conditions on a 3,590-foot reach of the Columbia River near Richland. On the day of the test, a tracer material was being continuously released at a point about 300 feet from the bank of the stream. A rate of 100 units per second was arbitrarily assigned to this release. The observations were reported in terms of this arbitrary designation. This is an adequate specification for our purposes. The top width of the stream at this point is about 1,000 feet, and approximately 38 percent of the river flow was passing between the point of release and the bank. Over the test reach the flows averaged 43,600 cubic feet per second, the areas of the cross section 9,870 square feet, the hydraulic radii 10.03 feet, and the mean velocities 4.418 feet per second.

If a roughness coefficient of 0.025 is appropriate, then Manning's formula gives

$$C_1 = \frac{1.486}{n_1} r_1^{1/6} = 87.3 \qquad \dagger$$

and

$$\frac{V}{v_*} = \frac{C_1}{\sqrt{g}} = \frac{87.3}{5.675} = 15.38, \qquad \dagger$$

from which

$$v_* = \frac{v}{15.38} = \frac{4.418}{15.38} = 0.287 \text{ ft per sec.} \qquad \dagger$$

From table 8 an appropriate value for K_y can be obtained:

$$\frac{K_y}{r_1 v_*} = 0.72$$

Then $K_y = (0.72)(10.03)(0.287) = 2.07 \text{ ft}^2$ per sec. With

$$q_s = \frac{100}{2} = 50 \text{ (units per sec) each way,}$$

$$D = 10.03 \text{ feet,}$$
$$V = 4.418 \text{ ft per sec,}$$
$$x = 3590 \text{ feet,}$$

$$\frac{q_s}{D\sqrt{\pi K_y V x}} = \frac{50}{10.03\sqrt{\pi(2.07)(4.418)(3590)}} = 0.015725$$

and

$$\frac{V}{4K_y x} = \frac{4.418}{(4)(2.07)(3590)} = 0.00014863.$$

The computation of salinity is made as shown in the following table. Because the tracer can diffuse both ways, the q_s value is taken as one-half of the total supply. The computation is made by use of equation 30 which was obtained by use of the Product Law.

TABLE 10.—*Computation of lateral diffusion from a continuous point source, Columbia River test*

y (feet)	$\frac{Vy^2}{4K_y x}$	$e^{-\frac{Vy^2}{4K_y x}}$	S (units per cu ft)
0	0	1.0000	0.015725
25	0.093	.9112	.01433
50	.372	.6894	.01084
75	.836	.4335	.00682
100	1.486	.2263	.00356
125	2.32	.0982	.00154
150	3.34	.0354	.00056
175	4.55	.0106	.00017
200	5.95	.0026	.0000

A comparison of the computed and observed values is shown on figure 11.

It will be of interest to estimate the dispersion at a point much farther downstream. This will also illustrate the use of an image. The last observations were made at a point 29,900 feet downstream from the source.

For this point

$$\frac{q_s}{D\sqrt{\pi K_y V x}} = \frac{50}{10.03\sqrt{\pi(2.07)(4.42)(29,900)}} = 0.00538$$

and

$$\frac{V}{4K_y x} = \frac{4.42}{(4)(2.07)(29,900)} = 0.0000178.$$

Table 11 is computed with the aid of equation 30:

† See note at end of notations list.
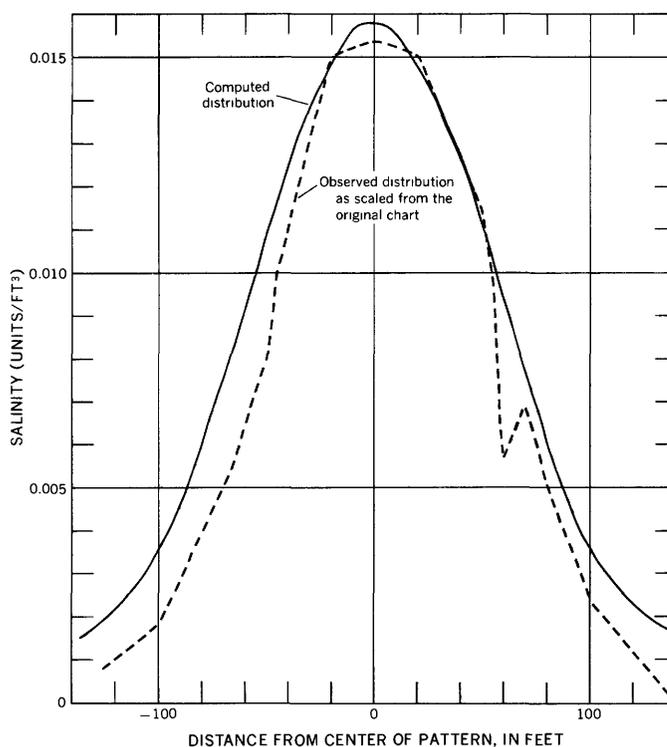
FIGURE 11.—Comparisons of computed and observed distributions at 3,590 feet from the source—Lateral dispersion, Columbia River test.

TABLE 11.—*Estimated salinities 29,900 feet from the source, Columbia River test*

y (feet)	$\frac{Vy^2}{4K_y x}$	$e^{-\frac{Vy^2}{4K_y X}}$	S (units per cu ft)	Adjusted width (feet)
0	0	1.00000	0.005378	0
50	.0404	.961	.005169	72.5
100	.1616	.850	.004571	145
150	.3635	.695	.003738	217.5
200	.6463	.524	.002818	290
300	1.454	.234	.001258	435
400	2.585	.104	.000559	580
500	4.039	.0176	.000095	725
600	5.8165	.0030	.000016	870
700	7.9169	.00036	.000002	1015
800	10.340	.000032	----------	1160
900	13.087	----------	----------	1305
1,000	16.157	----------	----------	1450

The results of this computation and the observed salinities are shown on figure 12. The curve a–a' represents the salinity as computed from equation 30 on the basis that the width of the stream is unchanged and that the bank of the stream does not halt the lateral spread of salt. Curve b–b' represents the estimated salinity distribution, as described for a–a', if the near bank were absent. A reference to the figure will show, however, that the near bank does halt the spread of salinity in this direction. To account for this, it may be assumed that the stream extends beyond the bank and that a second plume of tracer material is released 300
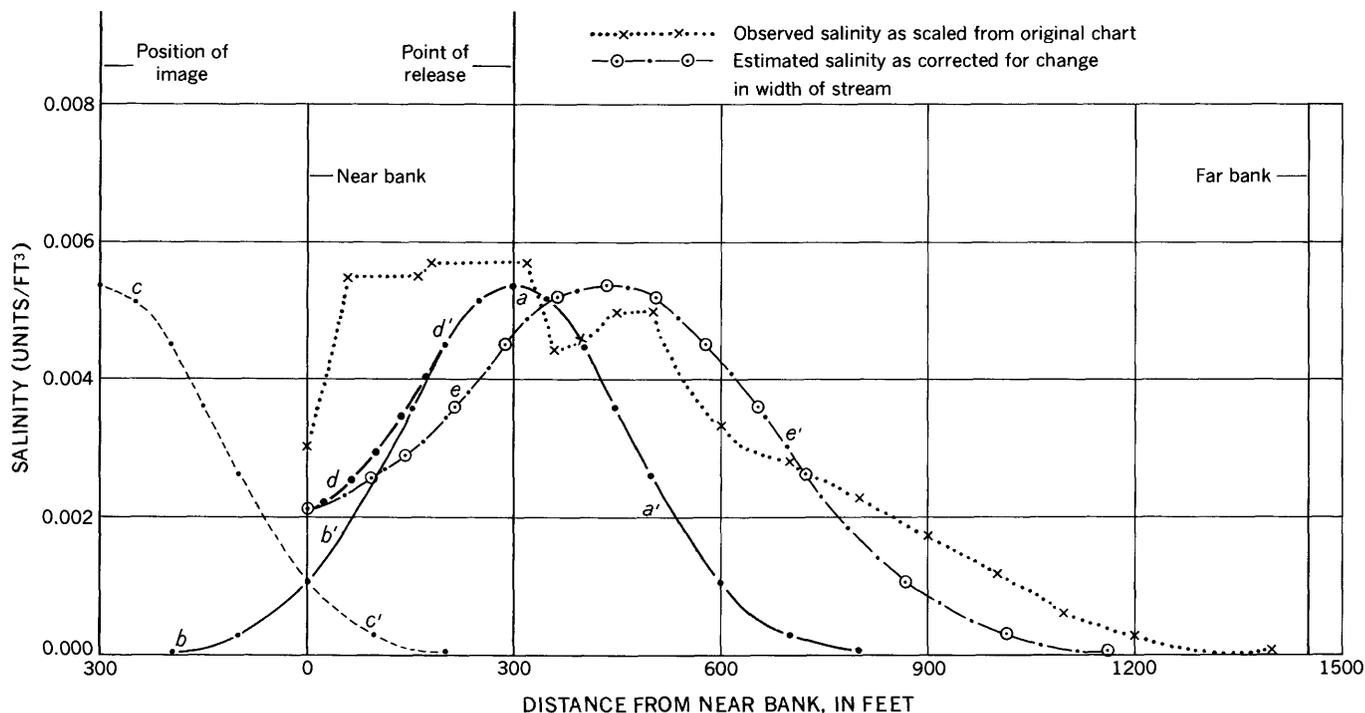
FIGURE 12.—Comparison of observed and computed salinity distributions at 29,900 feet from the source—lateral dispersion, Columbia River test. Curves a–a', b–b', c–c' and d–d' illustrate steps in constructing curve of estimated salinity (see section on "Transverse dispersion").

feet from it to create a condition that is everywhere symmetrical about the line of the near bank. Such a hypothetical symmetrical source is called an image. The real source and its image produce effects which nullify each other at the bank, thereby satisfying the condition of no flow of salt through the bank. The effect of the image is represented by curve c–c'. The sum of curves b–b' and c–c' yields curve d–d'. This curve has a zero gradient at the bank and accounts for the presence of the bank. Then the curves a–a' and d–d' represent the estimated salinity distribution if the stream had its original top width of 1,000 feet. At this station, however, the stream has a top width of 1,450 feet. To account for the change of width, the plotted points of curves a–a' and d–d' were displaced toward the far bank, the new distance from the near bank being in the ratio of 1,450 to 1,000 to the old distance. This correction is based upon the reasonable assumption that the lateral spreading of the stream carried the salinity along with it.

Although the observed and estimated salinity distributions are somewhat different, the maximum values of salinity and the amount pf lateral dispersion are not greatly different.

These comparisons represent the type of correlations obtained in actual tests. They indicate what might be expected from an application to a new test.

Use of the chart of figure 1 may be illustrated if it is assumed, as an approximation, that the tracer used in this test was released near the bank. This chart was prepared for use with the product law where equation 3 is to be used with equation 13 or 14, or both, but it may be adapted to the present use by interpreting x as the distance travelled by the mean velocity V in the time t and by interpreting the factor f_2 on the ordinate of the chart as the quantity

$$\frac{S}{\left(\dfrac{q_s}{BDV}\right)}.$$

Based upon a mean velocity of 4.418 feet per second, the time required to traverse the distance $x=29,900$ feet would be 6,768 seconds. If the original width $B=1,000$ feet is used, the parameter $B/\sqrt{4K_yt}$ takes the value 4.23. A glance at figure 1 will show that the salinity has not reached the far bank and will not do so short of an x distance of about 135,000 feet. At this distance, the travel time is $t=(135,000/4.418)=30,562$ seconds, and the chart parameter has the value

$$\frac{B}{\sqrt{4K_yt}}=\frac{1,000}{\sqrt{(4)(2.07)(30,562)}}=2.0.$$

The salinity has then reached the far bank and the salinity profile across the stream is as shown on the curve having this value of the parameter. At the near bank, where $y/B=0.0$, this curve has an ordinate of 2.25, the salinity at this point will be

$$S=\frac{q_s}{BDV}(2.2568)=\frac{(100)(2.2567)}{(9,874)(4.418)}$$

$$=0.00517 \text{ unit per cubic foot}.$$

At the middle of the stream where $y/B=0.5$ the ordinate is 0.83 and

$$S=(0.00517)\frac{0.83}{(2.2568)}=0.00186 \text{ unit per cubic foot.}$$

At a distance of 528,000 feet or 100 miles downstream from the source where $t=119,500$ seconds, or 1.37 days, the parameter would have the value 1.01 and the cross-stream profile would be as shown on the chart for this parameter. The maximum salinity at this point would be 0.00264 unit per cubic foot. Ultimately the parameter approaches zero, the ordinate approaches 1.000, and the salinity becomes uniformly distributed through the cross section with a value of

$$S=\frac{q_s}{BDV}(1.000)=\frac{q_s}{BDV}.$$

The ultimate value is

$$\frac{q_s}{BDV}=\frac{100}{(9,874)(4.418)}=0.002293 \text{ unit per cubic foot.}$$

The tracer used in this test was composed of short-lived isotopes. The above ultimate value does not, therefore, represent the observed activity, which might be very low, but rather the activity which would be observed if the tracer had retained its original strength. An increase of stream width could be accounted for in the manner described previously. A continuous supply of salinity released adjacent to a bank of a stream will not disperse as rapidly as it will if released at a point away from the bank because, when released near the bank, it can diffuse in only one direction. A comparison of values obtained in this example will show that the estimated maximum salinity attained at $x=135,000$ feet for material released adjacent to the bank is comparable to the maximum salinity found at 29,900 feet when the salinity is released at a point out in the stream.

OUTLINE OF COMPUTATION PROCEDURE

For the benefit of those who wish to apply these methods to a new problem it will be helpful to outline the procedure followed in these examples. The steps to be taken are as follows:
1. Obtain the flow of the stream and its cross section.
2. From available hydraulic data select an n_1 value appropriate for the stream conditions and compute the Chezy coefficient C_1 from Manning's formula, using the hydraulic radius obtained from the data of step 1 preceding.
3. Use equation 80 to estimate the shear velocity v_* using the mean stream velocity obtained from the data of step 1 and the C_1 value of step 2 preceding.
4. From table 8 select appropriate factors and use them with the values r_1 and v_* previously obtained to estimate values for the K_x, K_y, and K_z diffusion coefficients.
5. For the case of a finite quantity of salinity suddenly released into a stream under conditions where longitudinal dispersion is important use equation 3. If lateral as well as longitudinal dispersion must be accounted for, use equations 3 and 13 combined according to the product law.
6. For a sustained release use equation 30.

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