A Stochastic Model for Predicting the Probability Distribution of the Dissolved-Oxygen Deficit in Streams

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SYMBOLS

Symbol	Definition	Symbol	Definition
a	Parameter equal to $\overline{K}_2 - \overline{K}_1 - K_3$, in reciprocal days.	BOD	Biochemical-oxygen-demand, in milligrams per litre.
B	Assumed deterministic part of the dissolved-oxygen deficit, defined by equation 40, in milligrams per litre.	C	Dissolved-oxygen concentration, in milligrams per litre.

Symbol	Definition

Symbol	Definition

H_6 ------------------Term giving the effect on the variance of the dissolved-oxygen deficit of random variations in the deoxygenation coefficient along the reach, defined by equation 84, in (milligrams per litre) squared.

H_7 ------------------Term giving the effect on the variance of the dissolved-oxygen deficit of random variations in the reaeration coefficient, defined by equation 85, in (milligrams per litre) squared.

H_8 ------------------Term giving the effect on the variance of the dissolved-oxygen deficit of correlation between the deoxygenation coefficient along the reach and the reaeration coefficient, defined by equation 86, in (milligrams per litre) squared.

H_9 ------------------Term giving the effect on the variance of the dissolved-oxygen deficit of random variations in the BOD at the upstream end of the reach, defined by equation 87, in (milligrams per litre) squared.

H_{10} ------------------Term giving the effect on the variance of the dissolved-oxygen deficit of random variations in the deoxygenation coefficient at the upstream end of the reach, defined by equation 88, in (milligrams per litre) squared.

H_{11} ------------------Term giving the effect on the variance of the dissolved-oxygen deficit of correlation between the BOD and the deoxygenation coefficient at the upstream end of the reach, defined by equation 89, in (milligrams per litre) squared.

i ------------------Denotes a summation index.

I ------------------Term giving the effect of the addition of BOD along the reach, defined by equation 27, in milligrams per litre.

j ------------------Denotes a summation index.

k ------------------Deoxygenation coefficient along the reach or the rate constant for the biochemical oxidation of carbonaceous material along the reach; assumed to be a function of the distance downstream (x) or equivalently of time of travel (T), in reciprocal days.

k_i, k_{i+1}, k_j, k_n --------Deoxygenation coefficient along the reach after i, $i+1$, j, and n steps, respectively, of the random walk process, in reciprocal days.

$\bar{k}$ ------------------Mean value of the deoxygenation coefficient along the reach, in reciprocal days.

K_1 ------------------Deoxygenation coefficient or the rate constant for the biochemical oxidation of carbonaceous material; assumed to be a function of the distance downstream (x), or equivalently of the time of travel (T), in reciprocal days.

K_{1_1} ------------------Initial value of the deoxygenation coefficient or the deoxygenation coefficient at the upstream end of the reach, in reciprocal days.

K_{1_i}, $K_{1_{i+1}}$, K_{1_j}, K_{1_n} ---Deoxygenation coefficient after i, $i+1$, j, and n steps, respectively of the random walk process; equivalently, K_{1_j} is the deoxygenation coefficient at the end of time of travel T_j, in reciprocal days.

$\bar{K_1}$ ------------------Mean value of the deoxygenation coefficient, in reciprocal days.

$\bar{K_1}(T)$ ------------------Mean value of the deoxygenation coefficient at any time of travel T, in reciprocal days.

K_2 ------------------Reaeration coefficient or the rate constant for oxygen absorption from the atmosphere, in reciprocal days.

K_{2_i}, $K_{2_{i+1}}$, K_{2_j}, K_{2_n}----Reaeration coefficient after i, $i+1$, j, and n steps, respectively, of the random walk process; equivalently, K_{2_j} is the reaeration coefficient at the end of time of travel T_j, in reciprocal days.

$\bar{K_2}$ ------------------Mean value of the reaeration coefficient, in reciprocal days.

$\bar{K_2}(T)$ ------------------Mean value of the reaeration coefficient at any time of travel T, in reciprocal days.

K_{2_0} ------------------Reaeration coefficient at 20° Celsius, in reciprocal days.

K_{2_p} ------------------Reaeration coefficient predicted by equation 15, in reciprocal days.

K_3 ------------------Rate constant for the removal of BOD by sedimentation and adsorption, in reciprocal days.

L ------------------BOD or biochemical-oxygen-demand of carbonaceous material, in milligrams per litre.

L_j, L_{j-1} ------------------BOD at the end of times of travel of T_j and T_{j-1}, respectively, in milligrams per litre.

$\bar{L}$ ------------------Mean value of the BOD, in milligrams per litre, defined by equation 33.

L_a ------------------Rate of addition of BOD along the reach, in milligrams per litre per day.

L_0 ------------------BOD at the upstream end of the reach, in milligrams per litre.

L_{0_1}, L_{0_2}, ... L_{0_n} ------BOD at the upstream end of the reach at the end of 1, 2, ... n steps of the random walk process, in milligrams per litre.

Symbol	Definition
L_0	Mean value of the BOD at the upstream end of the reach, in milligrams per litre.
m	Number of times that the Monte Carlo simulation procedure is repeated.
n	Number of steps in the random walk process.
p	Rate of production of dissolved oxygen by photosynthesis, in milligrams per litre per day.
p_1 or $p_1(\Delta K_1)$	Probability that K_1 takes a positive step in the random walk process.
p_2 or $p_2(\Delta K_2)$	Probability that K_2 takes a positive step in the random walk process.
$P[Z=y]$	Probability that the outcome of the event Z is y.
q_1 or $q_1(\Delta K_1)$	Probability that K_1 takes a negative step in the random walk process.
q_2 or $q_2(\Delta K_2)$	Probability that K_2 takes a negative step in the random walk process.
r_p	Rate of consumption of dissolved oxygen by plant respiration, in milligrams per litre per day.
$r(Z_1, Z_2)$	Correlation coefficient between the random variables Z_1 and Z_2.
$r(BOD, C)$	Correlation coefficient between the BOD and the dissolved-oxygen concentration.
$r(BOD, D)$	Correlation coefficient between the BOD and the dissolved-oxygen deficit.
R_{1j}	Set of n uniformly distributed random numbers used in the Monte Carlo simulation of random variations in the deoxygenation coefficient.
R_{2j}	Set of n uniformly distributed random numbers used in the Monte Carlo simulation of random variations in the reaeration coefficient.
t	Time, in days.
T	Time of travel or $\int_0^x \frac{dx}{V}$; subscripts j and $j-1$ indicate specific times of travel, in days.
T_c	Critical time of travel or the time at which the dissolved-oxygen deficit is maximum, in days.
T_{c_D}	Critical time of travel for the deterministic model (eq 4), in days.
T_{c_s}	Critical time of travel estimated from the Taylor series approximation of the stochastic model (eq 74), in days.
Var (Z)	Variance of the random variable Z.
V	Mean velocity of flow, in feet per second or equivalent.
x	Longitudinal position or distance downstream, in feet.
X	Amount of BOD removed or the amount of dissolved oxygen consumed, in milligrams per litre.
X_j, X_{j-1}	Amounts of BOD removed at the end of times of travel T_j and T_{j-1}, respectively; in milligrams per litre.
Z	Random variable.
Z_m	Mean value of a random sample of size m.
α_1	Drift coefficient for the deoxygenation coefficient, defined by equation 30.
α_2	Drift coefficient for the reaeration coefficient, defined by equation 37.
β_1	Variance of the total deoxygenation coefficient (K_1+k), in (reciprocal days) squared.
β'_1	Variance of the deoxygenation coefficient at the upstream end of the reach, in (reciprocal days) squared.
β''_1	Variance of the deoxygenation coefficient along the reach, in (reciprocal days) squared.
β_2	Variance of the reaeration coefficient, in (reciprocal days) squared.
δ	Confidence limit.
Δ	Parameter of the stochastic model of Thayer and Krutchkoff (1966) for estimating the distribution of the biochemical-oxygen-demand and the dissolved-oxygen deficit, in milligrams per litre.
ΔK_1	Step length of the random walk for the deoxygenation coefficient, computed from equation 31.
ΔK_2	Step length of the random walk for the reaeration coefficient, computed from equation 38.
ΔT	Incremental value of the time of travel, equal to T/n; in days.
ε	Expected deviation from the mean value of the random variable Z.
μ	Mean value.
$\mu_4(j)$	Fourth moment of a binomial distribution.
$\mu_4(K_1T, N)$	Fourth central moment of the limiting normal distribution of K_1T, in reciprocal days to the fourth power.
$\mu_4(K_1T, \Delta T)$	Fourth central moment of the scaled binomial distribution describing K_1T for finite ΔT, in reciprocal days to the fourth power.
ζ	$t-\int_0^x \frac{dx}{V}$ or $t-T$, in days.
π	The constant 3.14.
σ	Standard deviation.
σ^2	Variance.
τ, τ'	Dummy variables of integration.
Ψ	Function of the deoxygenation and reaeration coefficients and the time of travel for estimating the variance of the dissolved-oxygen deficit, defined by equation 69.

METRIC-ENGLISH EQUIVALENTS

Metric unit		English equivalent
Length		
millimetre (mm)	= 0.03937	inch (in)
metre (m)	= 3.28	feet (ft)
kilometre (km)	= .62	mile (mi)
Area		
square metre (m²)	= 10.76	square feet (ft²)
square kilometre (km²)	= .386	square mile (mi²)
hectare (ha)	= 2.47	acres
Volume		
cubic centimetre (cm³)	= 0.061	cubic inch (in³)
litre (l)	= 61.03	cubic inches
cubic metre (m³)	= 35.31	cubic feet (ft³)
cubic metre	= .00081	acre-foot (acre-ft)
cubic hectometre (hm³)	=810.7	acre-feet
litre	= 2.113	pints (pt)
litre	= 1.06	quarts (qt)
litre	= .26	gallon (gal)
cubic metre	= .00026	million gallons (Mgal or 10⁶ gal)
cubic metre	= 6.290	barrels (bbl) (1 bbl=42 gal)
Weight		
gram (g)	= 0.035	ounce, avoirdupois (oz avdp)
gram	= .0022	pound, avoirdupois (lb avdp)
tonne (t)	= 1.1	tons, short (2,000 lb)
tonne	= .98	ton, long (2,240 lb)
Specific combinations		
kilogram per square centimetre (kg/cm²)	= 0.96	atmosphere (atm)
kilogram per square centimetre	= .98	bar (0.9869 atm)
cubic metre per second (m³/s)	= 35.3	cubic feet per second (ft³/s)

Metric unit		English equivalent
Specific combinations—Continued		
litre per second (l/s)	= .0353	cubic foot per second
cubic metre per second per square kilometre [(m³/s)/km²]	= 91.47	cubic feet per second per square mile [(ft³/s)/mi²]
metre per day (m/d)	= 3.28	feet per day (hydraulic conductivity) (ft/d)
metre per kilometre (m/km)	= 5.28	feet per mile (ft/mi)
kilometre per hour (km/h)	= .9113	foot per second (ft/s)
metre per second (m/s)	= 3.28	feet per second
metre squared per day (m²/d)	= 10.764	feet squared per day (ft²/d) (transmissivity)
cubic metre per second (m³/s)	= 22.826	million gallons per day (Mgal/d)
cubic metre per minute (m³/min)	=264.2	gallons per minute (gal/min)
litre per second (l/s)	= 15.85	gallons per minute
litre per second per metre [(l/s)/m]	= 4.83	gallons per minute per foot [(gal/min)/ft]
kilometre per hour (km/h)	= .62	mile per hour (mi/h)
metre per second (m/s)	= 2.237	miles per hour
gram per cubic centimetre (g/cm³)	= 62.43	pounds per cubic foot (lb/ft³)
gram per square centimetre (g/cm²)	= 2.048	pounds per square foot (lb/ft²)
gram per square centimetre	= .0142	pound per square inch (lb/in²)
Temperature		
degree Celsius (°C)	= 1.8	degrees Fahrenheit (°F)
degrees Celsius (temperature)	=[(1.8×°C)+32]	degrees Fahrenheit

A STOCHASTIC MODEL FOR PREDICTING THE PROBABILITY DISTRIBUTION OF THE DISSOLVED-OXYGEN DEFICIT IN STREAMS

By I. I. ESEN and R. E. RATHBUN

ABSTRACT

A random walk model was developed for predicting the distribution of the biochemical-oxygen-demand for points downstream from a waste source for a stream system in which the deoxygenation coefficient is a normally distributed random variable. A Monte Carlo technique for simulating a random walk process was used for estimating the distribution of the dissolved-oxygen deficit at downstream points in a stream in which both the deoxygenation and reaeration coefficients are normally distributed random variables. Equations for approximating the mean oxygen deficit and the variance of the oxygen deficit were developed by expanding the basic equation of the stochastic model in a Taylor series.

The random walk model gave a lognormal distribution function for the biochemical-oxygen-demand. The frequency distributions of the oxygen deficit predicted by the stochastic model became flatter and skewed to the right as time of travel increased. The critical time of travel estimated from the stochastic model was always larger than the critical time of travel computed from the deterministic model; however, the percentage difference decreased as the ratio of the reaeration and deoxygenation coefficients decreased.

The variance of the oxygen deficit at the critical time of travel was largest for small ratios of the reaeration and de-oxygenation coefficients and smallest for the large ratios. The variance showed the greatest dependence on the ratio at large values of the ratio and the smallest dependence at small values of the ratio.

The variances of the oxygen deficit computed from the Taylor series approximation of the stochastic model were comparable to the variances obtained from the stochastic model for small times of travel; as the time of travel increased, the Taylor series approximation underestimated the variance. For computations at the critical time of travel, the variance estimated from the Taylor series approximation was less than that of the variance of the stochastic model over the entire range of conditions considered.

The ability to predict the variances of the biochemical-oxygen-demand and the dissolved-oxygen deficit at points downstream from a waste source is extremely important, in view of the ever-increasing concern with the maintaining of water-quality standards. The stochastic model of this report is a valuable tool for predicting variances; however, further development and refinements of this and other models is still needed.

INTRODUCTION

The pollution of our streams and rivers caused by the discharge of excessive amounts of municipal and industrial wastes into them has been of increased concern in recent years. As a result of this concern, the Water Quality Act of 1965 was enacted. This law required all States to classify rivers and streams according to intended use and to adopt water-quality standards for each of the intended uses. Specific limits were required for 10 water-quality parameters for each of 9 designated water uses; these standards, as of 1969, for the different States were tabulated by the American Public Health Association (1969). Of the various water-quality parameters, the single parameter used most frequently to indicate the relative state of pollution or health of a stream is the dissolved-oxygen concentration. The dissolved-oxygen concentration is the amount of free oxygen, that is, not chemically combined with other elements, available in the water for the respiration processes of the flora and fauna of the stream and for the oxidation of organic waste materials. Hence, knowledge of the response of a stream's dissolved-oxygen concentration to the addition of organic wastes is essential to maintaining dissolved-oxygen concentrations that are adequate to support a desirable flora and fauna.

The longitudinal profiles of the concentration of organic wastes, expressed in terms of BOD (biochemical-oxygen-demand), and of the dissolved-oxygen concentration, may be predicted from deterministic models based on the principle of the conservation of mass. The differential equations are

$$\frac{\partial L}{\partial t} + V\frac{\partial L}{\partial x} = D_x\frac{\partial^2 L}{\partial x^2} - (K_1 + K_3)L + L_a \qquad (1)$$

$$\frac{\partial C}{\partial t} + V\frac{\partial C}{\partial x} = D_x\frac{\partial^2 C}{\partial x^2} + K_2(C_s - C)$$

$$- K_1 L + C_a - D_B + p - r_p \qquad (2)$$

1

where L = biochemical-oxygen-demand of carbonaceous material;

t = time;

x = distance in the longitudinal or direction of flow;

V = mean flow velocity in the longitudinal direction;

K_1 = deoxygenation coefficient or the rate constant for biochemical oxidation of carbonaceous material;

K_2 = reaeration coefficient or the rate constant for oxygen absorption from the atmosphere;

K_3 = rate constant for the removal of BOD by sedimentation and adsorption;

D_x = longitudinal dispersion coefficient;

C = dissolved-oxygen concentration;

C_s = dissolved-oxygen concentration at saturation;

C_a = rate of addition of dissolved oxygen along the reach by all processes other than reaeration and photosynthesis;

L_a = rate of addition of BOD along the reach;

D_B = rate of removal of dissolved oxygen by the benthal layer on the stream bottom;

p = rate of production of dissolved oxygen by photosynthesis; and

r_p = rate of consumption of dissolved oxygen by plant respiration.

The development of equations 1 and 2 assumes that:

1. The dissolved-oxygen concentration and BOD are uniformly distributed over each cross section so that the equations can be written in the one-dimensional form.

2. The processes described by the rate constants K_1, K_2, and K_3 are first-order processes; that is, the rate of removal of BOD is proportional to the amount of BOD remaining, and the rate of reaeration is proportional to D, the dissolved-oxygen deficit, which is the difference between the dissolved-oxygen concentration at saturation and the actual dissolved-oxygen concentration.

3. Only the carbonaceous demand of the waste is significant. If the nitrogenous demand is important, an additional term must be added to equations 1 and 2.

If D_x, K_3, L_a, C_a, and D_B have negligible effects on the BOD and dissolved-oxygen profiles, and if steady-state and uniform-flow conditions exist, then equations 1 and 2 reduce to the classical equations of Streeter and Phelps (1925). The solutions under these conditions are

$$L = L_0 \exp \ (-K_1 T) \tag{3}$$

and

$$D = C_s - C = \frac{K_1 L_0}{K_2 - K_1} \ [\exp \ (-K_1 T)$$
$$- \exp \ (-K_2 T) \ ] + D_0 \exp \ (-K_2 T) \tag{4}$$

where L_0 = BOD of the carbonaceous material at the upstream end of the reach;

D_0 = dissolved-oxygen deficit at the upstream end of the reach; and

T = time of travel from the upstream end of the reach to the point of interest at longitudinal position x, or

$$T(x) = \int_0^x \frac{dx}{V} = \frac{x}{V} \tag{5}$$

The various other modifications of equations 1 and 2 and the various types of analytical and numerical solutions of these equations that have appeared in the literature were reviewed by Bennett and Rathbun (1972).

The deterministic equations for the longitudinal distributions of the BOD and dissolved-oxygen concentration, that is, equations 1 and 2 or similar equations with constant coefficients, give one value for the dissolved-oxygen concentration at each downstream point for a specific set of conditions. Similarly, most water-quality standards state that the dissolved-oxygen concentration must not drop below one specific concentration. However, because of the presence of random components in natural processes, there is a nonzero probability that the dissolved-oxygen concentration will fall below the concentration predicted by the deterministic equations, and a dissolved-oxygen concentration that is on the average sufficient to assure healthy fish does not prevent fish kills. Hence, an interest has developed in determining the probability distributions of the BOD and dissolved-oxygen concentration at downstream points in the reach.

Previous studies have described three methods for the estimation of the probability distributions of the BOD and dissolved-oxygen concentration.

The first of these methods, developed by Loucks and Lynn (1966), predicts the probability that a dissolved-oxygen concentration less than some specific concentration will exist for some specified time period at a point downstream from a waste discharge. A Markov model was used and a first-order transition probability matrix for the low-flow sea-

son was formed by making use of the available streamflow data. The term in the jth row and kth column of the transition matrix of a first-order, discrete-time, discrete-outcome Markov process gives the probability that a process which is in state j at time t will be in state k at time $t+1$. To each discrete average daily streamflow, one set of parameters such as K_1, K_2, K_3, temperature, L_0, and D_0 were assigned. Then, the critical or maximum dissolved-oxygen value for each set of parameters was assigned a probability in the transition matrix. Loucks and Lynn (1966) also give methods for determining the probabilities of violating a particular stream standard for two or more consecutive days and of violating a standard when sewage flow depends on streamflow.

The second of these methods considers the problem essentially as a stochastic birth and death process. Thayer and Krutchkoff (1966) assumed that the BOD and dissolved-oxygen concentration changed by a small amount Δ in a short time interval owing to additions of BOD and dissolved oxygen along the river, benthal demand, sedimentation of solids to the river bottom, deoxygenation, and reaeration. In their model, a change of size Δ in the concentration was considered to constitute a change of one state, and it was assumed that a change of more than one state in time ΔT had a probability of $o(\Delta T)$. The probability of a change of one state was assumed to be proportional to ΔT.

Thayer and Krutchkoff (1966) determined the mean, variance, and probability distribution of BOD and the dissolved-oxygen deficit for a number of different stream and waste conditions and showed that the predicted mean value of the BOD and the oxygen deficit is the same as those computed by deterministic equations for the same stream and waste conditions. The variances of the BOD and the deficit, at any time of travel T, were found to be linear functions of Δ, and a numerical value for Δ could be determined by experimentally determining the variances at some point downstream from the point of addition of the waste to the stream.

The third of these methods is the Monte Carlo method proposed by Kothandaraman (1968). He considered the deoxygenation coefficient, K_1, and the reaeration coefficient, K_2, to be random variables and showed them to be normally distributed. Then, he randomly selected values of K_1 and K_2 from normal distributions with known means and variances and computed the oxygen deficit from the Streeter-Phelps equations for a preselected number of flow times. In this manner, he was able to estimate the probability distribution of the oxygen deficit at downstream points.

Thus, the problem of predicting the response of the dissolved-oxygen concentration of a stream to an organic waste load can be approached in either of two ways: deterministic or probabilistic. In the deterministic approach, the dissolved-oxygen concentration is predicted by solving two coupled differential equations (equations 1 and 2 or modifications of these) with appropriate assumptions and boundary conditions. On the other hand, the random nature of those factors such as turbulence, mean velocity, depth of flow, and type and concentration of waste that determine the coefficients of these equations and random variations in the type and concentration of the input waste loads suggest that a probabilistic approach should be more appropriate to the problem. However, the probabilistic models developed thus far are either too restrictive in the assumptions necessary or require extensive field data at some point upstream to predict the probability distributions of the BOD and oxygen deficit at downstream points.

The purpose of this report is to:

1. Present a discussion of the random variations to be expected in the deoxygenation coefficient, K_1, and the reaeration coefficient, K_2, and possible correlation between these parameters.
2. Describe the development of a random walk model for estimating the probability distribution of the BOD when K_1 is a random variable.
3. Describe the use of a Monte Carlo technique for estimating the probability distribution of the dissolved-oxygen deficit when K_1 and K_2 are random variables.
4. Describe the application of these techniques to the estimation of the variance and the percentile limits of the dissolved-oxygen deficit at downstream points.
5. Describe the extension of the model so that the input BOD could be considered as a random variable, with particular emphasis on the critical time of travel when the maximum oxygen deficit occurs.

This study was a part of the U.S. Geological Survey research program on reaeration in open-channel flow.

VARIATIONS IN THE DEOXYGENATION AND REAERATION COEFFICIENTS

The processes of deoxygenation and reaeration are generally considered to be first order; that is, the rate of deoxygenation is directly proportional to the amount of BOD remaining to be oxidized, and the rate of reaeration is directly proportional to the dissolved-oxygen deficit remaining to be satisfied. Therefore, deoxygenation may be described by the equation

$$\frac{dL}{dt} = -K_1 L \qquad (6)$$

and reaeration by the equation

$$\frac{dD}{dt} = -K_2 D \qquad (7)$$

The deoxygenation coefficient depends on the type of waste and the reaeration coefficient on the hydraulic conditions in the channel. Because variations in the type of waste and in the hydraulic conditions are in general random, there is ample basis for considering K_1 and K_2 as random variables.

DEOXYGENATION COEFFICIENT, K_1

Possible explanations for variations in the deoxygenation coefficient, K_1 were discussed in detail by Kothandaraman (1968). He pointed out that because the characteristics of municipal wastes vary considerably with time, the rate parameter, K_1, also varies considerably. This parameter characterizes biological processes which depend on the response of living organisms to their environment and hence, these processes do not have the uniformity of a chemical reaction. He concluded that because most of the contributing factors were random in nature, the variations in K_1 could also be considered random and hence treated in a probabilistic manner.

Kothandaraman (1968) determined K_1 values and ultimate or total first-stage BOD values of carbonaceous material for the Ohio River data collected and published by the U.S. Public Health Service (1960). He applied several statistical tests to 83 average values of K_1 which were determined by the least-square procedure of Reed and Theriault (1931), and he accepted at the 5 percent significance level the hypothesis that the K_1 values were normally distributed with a mean of 0.173 days^{-1} and a variance of 0.0044 days^{-2}.

We analyzed the Ohio River data to determine the variation of K_1 with distance downstream or travel time through the reach. The BOD versus time data given by Kothandaraman (1968) were used in this analysis, and these data are presented in table 10 in "Supplemental Data." Also given in table 10 are the mean values of the deoxygenation coefficient (K_1) and the total first-stage or carbonaceous BOD (L_0) determined by Kothandaraman (1968).

If equation 6 is integrated and the result rearranged, then it can be shown that

$$X = L_0 [1 - \exp(-K_1 T)] \qquad (8)$$

where X $= L_0 - L$, the amount of BOD removed in travel time T (also the amount of dissolved oxygen consumed up to time T);

L_0 = total first stage or carbonaceous BOD; and

L = BOD at time T.

It follows that

$$K_1 = -\frac{1}{T} ln[1 - X/L_0] \qquad (9)$$

The value of K_1 for each increment of travel time was computed from

$$K_{1j} = -\frac{1}{T_j - T_{j-1}} ln \left[1 - \frac{x_j - x_{j-1}}{L_{j-1}} \right] \qquad (10)$$

where

$$L_j = L_{j-1} \exp[-K_{1j}(T_j - T_{j-1})] \qquad (11)$$

and $j = 1, 2, 3, 4,$ and 5. The index j indexes the value of the variable at the end of the travel time T_j, and L_0 was determined by the Reed and Theriault (1931) procedure. Values of K_1 obtained from equation 10 for the Ohio River data are given in table 10 in "Supplemental Data." Travel times larger than 5 days were not considered because biological processes become less predictable at large times. The variances of the K_1 values along the reach were computed and these variances are presented also in table 10.

Inspection of the deoxygenation coefficient values given in table 10 shows that K_1 in general decreases with increase in T, or equivalently, distance downstream. A possible explanation for this is that the more easily degraded material is oxidized first in the stream. No attempt was made in the present study to develop a relation between K_1 and T; however, the theoretical developments to be presented were generalized to take this variation into consideration.

It was also observed that the variance of the initial values of K_1 at $T = 0$ was considerably larger than the variance of K_1 along the reach for a specific initial value of K_{1_1}. A Chi-square test of the initial values of K_1 showed that, at the 5 percent significance level, the data were compatible with the assumption that they were normally distributed with a mean of 0.205 days^{-1}, and a variance of 0.00538 days^{-2}. Two extreme values of K_1, computed from observations numbered 37 and 80, were neglected in the analysis. The coefficient of variation of the initial K_1 values was 0.357. The average variance of the K_1 values along the reach was 0.0015 days^{-2}. It was assumed that the K_1 values along the reach were normally distributed with the above-mentioned variance, but this hypothesis could not be tested, because for $T < 5$ days, each series of observations contained only four data points.

The coefficient of variation, C_v, of the mean values of the deoxygenation coefficient, K_1, for the Ohio River data was 0.066/0.173 or 0.38. In the present study, no attempt was made to substantiate the presence of the same K_1 value in all streams; however, as a first approximation, the coefficient of variation of K_1 is assumed to be

$$C_v(K_1) = 0.35 \qquad (12)$$

Thus, the assumption is that the dispersion or spread of the K_1 values about the mean is essentially the same as in the Ohio River data. Although the data in table 10 indicate that there is a difference between the variance of the initial values of K_1 and the variance of the K_1 values along the river reach, a coefficient of variation of 0.35 will in general be used to estimate both of these variances. The differences obtained in the probability distribution of the dissolved-oxygen deficit by considering different values of $C_v(K_1)$ for the initial values of K_1 and for the values of K_1 along the reach, and by assuming the same $C_v(K_1)$ for both variances, are discussed in the examples presented later.

The probabilistic model to be described in the next section is capable of considering both variations in the initial value of K_1 and variations in K_1 along the river reach. Another possibility is that the initial K_1 values are deterministic, but the variance of K_1 is very large during the first few hours. No data were available to test this supposition.

REAERATION COEFFICIENT, K_2

In general, the reaeration coefficient, K_2, of a stream may be considered as a property of the flow in the channel. As such, the variation with time of K_2 may be defined as the sum of a constant term, a periodic component, and a random component (Matalas, 1971). The constant term may under some circumstances itself be a function of time or distance downstream. The periodic component results from seasonal changes or perhaps diurnal changes, for example, in the quantity of flow. The random component has no simple physical explanation but is the resultant of a very large number of physical causes; for practical purposes, this random component may be considered as the inherent characteristic of turbulent flow (Matalas, 1971). Because the intensity of turbulence at a point in a stream has an approximately normal distribution (Batchelor, 1959), there is in turn a basis for considering K_2 as a normally-distributed random variable.

The K_2 data of Churchill, Elmore, and Buckingham (1962) for streams of the Tennessee Valley are generally considered to be the best available data for natural streams. The reaeration coefficient was computed from

$$K_{2_j} = -\frac{1}{T_j - T_{j-1}} ln\left[\frac{C_s - C_j}{C_s - C_{j-1}}\right] \qquad (13)$$

where C_s = saturation concentration of dissolved oxygen;

C = dissolved-oxygen concentration;

$j, j-1$ = value of variable at the end of the time of travel T_j, T_{j-1}; and

$C_s - C_o$ = dissolved-oxygen deficit, where C_o is the dissolved-oxygen concentration at the upstream end of the reach, assumed to be known.

Several equations were developed by multiple regression analysis, and each of these equations was tested by comparing computed K_2 values with the geometric means of the experimental K_2 values for each reach studied. The equation recommended for use by Churchill, Elmore, and Buckingham (1962) was

$$K_{2_{20}} = 5.026 V^{0.969} H^{-1.673} \qquad (14)$$

where V = mean velocity of flow in feet per second;

H = mean depth of flow in feet; and

$K_{2_{20}}$ = reaeration coefficient in days^{-1} at 20°C.

For V in metres per second and H in metres, the constant term in equation 14 is 2.178. Equation 14 had a correlation coefficient of 0.822.

Kothandaraman (1968) conducted a regression analysis of the Churchill, Elmore, and Buckingham

(1962) data using arithmetic mean values for each reach and obtained

$$K_{2_{20}} = 5.827 V^{0.924} H^{-1.705} \qquad (15)$$

with a correlation coefficient of 0.917 (for V in metres per second and H in metres, the constant term in equation 15 is 2.304). He then analyzed the distribution of the percentage errors, with the percentage error defined as $(K_2 - K_{2p})(100)/K_{2p}$ where K_2 is the arithmetic mean value, and K_{2p} is the value predicted by equation 15. He found that the percentage errors were approximately normally distributed with a mean of zero and a standard deviation of 0.368. His analysis assumes that the value of $K_{2_{20}}$ given by equation 15 is correct and that deviations from it are probabilistic in nature; it also assumes that once a process starts with a certain value of K_2, the rate of reaeration is constant for the reach.

In the present study, we assumed that the deviations of measured values of K_2 for a specific reach of stream from the mean of all the measured K_2 values for that reach were normally distributed. This assumption is in contrast with that of Kothandaraman (1968) in that he assumed that deviations of the means of the experimental K_2 values from K_2 values computed from equation 15 were normally distributed. The arithmetic means, variances, and the coefficients of variation estimated from the K_2 data of Churchill, Elmore, and Buckingham (1962) are presented in table 11 in "Supplemental Data." The mean value of the coefficients of variation for all streams studied was 0.307. Therefore, for our study, we have assumed that the coefficient of variation of the reaeration coefficient is 0.3, or

$$C_v(K_2) = 0.3 \qquad (16)$$

CORRELATION BETWEEN DEOXYGENATION AND REAERATION COEFFICIENTS

Increased turbulence and mixing in a stream result in a more rapid rate of reaeration and hence a larger K_2 and also a larger K_1 as a result of increased bacterial degradation of wastes; conversely, lower levels of turbulence result in smaller K_1 and K_2 values. Similarly, an increase in temperature increases both K_1 and K_2. Therefore, a positive correlation between K_1 and K_2 is expected.

Although the interdependence between BOD and dissolved-oxygen deficit, D, is known, little effort has been directed toward determining the numerical value of the correlation coefficient for BOD and D. There has, however, been one theoretical approach to the problem by Moushegian and Krutchkoff

(1969) who used the stochastic model of Thayer and Krutchkoff (1966) to estimate the correlation coefficient. The correlation coefficient was defined as

Correlation coefficient
$$= \frac{\text{covariance}}{[(\text{BOD variance})(D \text{ variance})]^{1/2}} \qquad (17)$$

The correlation study was run with an initial BOD input (L_0) of 12.4 mg/l (milligrams per litre), a saturation concentration (C_s) of 10.4 mg/l, an initial dissolved-oxygen concentration (C_0) of 5 mg/l, and a value of 0.1 mg/l for the Δ-parameter of Thayer and Krutchkoff (1966). Different values of K_1, K_2, and T were used, and the resultant correlation coefficients between BOD and dissolved-oxygen concentration obtained by Moushegian and Krutchkoff (1969) are presented in table 12 in "Supplemental Data." The correlation coefficient is positive at all times which implies that the correlation coefficient between BOD and D is negative.

Moushegian and Krutchkoff (1969) found that interpretation of the results of the correlation coefficient study was difficult, but in general it was observed that the correlation coefficient between BOD and dissolved-oxygen concentration decreased as time of travel increased and the difference between K_2 and K_1 increased. On the other hand, as the time of travel becomes small, correlation between BOD and D approaches -1. This may be seen as follows: for an initial dissolved-oxygen deficit of zero and small travel times, the exponentials in equations 3 and 4 may be approximated by their Taylor series expansions. Hence

$$L \approx L_0(1 - K_1 \Delta T) \qquad (18)$$

and

$$D \approx L_0 K_1 \Delta T \qquad (19)$$

where $\Delta T =$ an incremental value of the time of travel.

The Moushegian and Krutchkoff (1969) study yielded no information on the possible correlation between K_1 and K_2. Therefore, in the present study, the correlation coefficient between K_1 and K_2 was assumed to have the value

$$r(K_1, K_2) = 0.5 \qquad (20)$$

This correlation coefficient was applied to the K_1 and K_2 values at the same instant of time, although the presence of a small lag period is possible. The effect of different correlation coefficient values on the oxygen-deficit distribution will be discussed in the examples to be presented.

DEVELOPMENT OF THE PROBABILISTIC MODEL

SOLUTION OF EQUATIONS FOR LONGITUDINAL PROFILES OF BOD AND DISSOLVED-OXYGEN CONCENTRATION

The longitudinal profiles of the BOD and dissolved-oxygen concentration downstream from a source of organic biodegradable waste are described by equations 1 and 2, respectively. The assumptions inherent in these equations were discussed previously. With the additional assumptions of (1) the effect of longitudinal dispersion on the BOD and dissolved-oxygen concentration profiles is negligible relative to other factors; and (2) the effect of photosynthesis and respiration is negligible; equations 1 and 2 reduce, respectively, to

$$\frac{\partial L}{\partial t} + V\frac{\partial L}{\partial x} = -(K_1 + K_3)L + L_a \qquad (21)$$

and

$$\frac{\partial C}{\partial t} + V\frac{\partial C}{\partial x} = K_2 C_s - K_1 L + C_a - D_B - K_2 C. \qquad (22)$$

These equations, as well as equations 1 and 2, assume that the water temperature, and hence the saturation concentration of dissolved oxygen, is constant.

The effect of longitudinal dispersion on the BOD and dissolved-oxygen concentration profiles for steady-state and uniform flow conditions was discussed by Dobbins (1964) who concluded that the effect was negligible for the largest value of D_x known at that time. A much larger value was found later by Yotsukura, Fischer, and Sayre (1970), but the mean flow velocity was also large; and according to Dobbins' analysis, the effect of dispersion would still be negligible. In estuaries where dispersion becomes large and velocities small, the effect of longitudinal dispersion cannot be neglected. The effect of photosynthesis may be extremely important in certain situations; the literature and procedures for treating photosynthesis were discussed by Bennett and Rathbun (1972). The water temperature may increase in the downstream direction as a result of both natural and manmade causes; this problem was considered by Liebman and Lynn (1966). Thus, longitudinal dispersion, photosynthesis, and changes in water temperature may be important in certain situations; for the present study, however, they were neglected.

Following Li (1962), equations 21 and 22 were solved for a stream system with the following characteristics:

1. Hydraulic conditions may vary with distance downstream but are steady at each cross section.
2. The BOD and dissolved-oxygen concentration at the cross section at which the waste is added to the stream may be functions of time; inherent in the assumption of one-dimensionality is the requirement that the distance necessary for complete lateral and vertical mixing of the BOD be small relative to the distance downstream to the cross section(s) of interest.
3. The rate coefficients K_1, K_2, and K_3 are functions of the time of travel or distance downstream.
4. The distributed source and sink terms, that is, C_a, L_a, and D_B, vary with distance downstream but are steady at each cross section.

Time of travel, T, and distance downstream, x, may be interchanged through the relation

$$T = \int_0^x \frac{dx}{V} \qquad (23)$$

Details of the solution have been presented previously (Esen, 1971), and only the results will be presented here.

For the longitudinal profile of the BOD, the equation obtained was

$$L = e^{-\int_0^T (K_1+K_3)d\tau}\left[f(\zeta) + \int_0^T L_a e^{+\int_0^\tau (K_1+K_3)d\tau'} d\tau \right] \qquad (24)$$

where ζ is given by

$$\zeta = t - \int_0^x \frac{dx}{V}; \qquad (25)$$

$f(\zeta)$ = function describing the variation with time ζ of the BOD at the upstream end of the reach; and

τ, τ' = dummy variables of integration.

Comparison of equation 24 with equation 3, which describes the BOD profile for the approach of Streeter and Phelps (1925), shows the differences in the results. If the BOD at the upstream end of the reach is independent of time, K_3 and L_a are zero, and K_1 is independent of time of travel in the reach, then equation 24 reduces to equation 3.

For the longitudinal profile of the dissolved-oxygen deficit, the equation obtained was

$$D = C_s - C = e^{-\int_0^T K_2 d\tau} \left[C_s - F(\zeta) + \int_0^T (D_B - C_a) e^{\int_0^T K_2 d\tau'} d\tau \right.$$
$$\left. + f(\zeta) \int_0^T K_1 e^{\int_0^T (K_2 - K_1 - K_3) d\tau'} d\tau + \int_0^T K_1 I e^{\int_0^T (K_2 - K_1 - K_3) d\tau'} d\tau \right]$$

$$(26)$$

where

$$I = \int_0^T L_a e^{\int_0^T (K_1 + K_3) d\tau'} d\tau; \text{ and} \qquad (27)$$

$F(\zeta)$ = function describing the variation with time ζ of the dissolved-oxygen concentration at the upstream end of the reach.

In equation 26, the term $C_s - F(\zeta)$ is the dissolved-oxygen deficit at the upstream end of the reach; the first integral term is the net effect of the addition of dissolved oxygen along the reach by all processes other than reaeration and photosynthesis and the removal of dissolved oxygen by the benthal layer on the stream bottom; the second integral term is the BOD added at the upstream end of the reach; and the third integral term is the BOD added along the reach.

Comparison of equation 26 with equation 4, which describes the longitudinal profile of the dissolved-oxygen deficit for the approach of Streeter and Phelps (1925), shows the differences in the results. If the dissolved-oxygen concentration at the upstream end of the reach is independent of time, L_a, D_B, C_a, and K_3 are zero, and K_1 and K_2 are independent of time of travel in the reach, then equation 26 reduces to equation 4.

Equations 24 and 26 describe the longitudinal profiles of the BOD and dissolved-oxygen deficit, respectively, for a stream system in which the rate coefficients K_1, K_2, and K_3 are unknown functions of the time of travel. Application of these equations to the estimation of the probability distributions of the BOD and the dissolved-oxygen deficit for a stream system in which K_1 and K_2 are normally distributed random variables is described in the following sections.

RANDOM WALK MODELS

There is considerable justification, as discussed previously, for considering the deoxygenation coefficient, K_1, and the reaeration coefficient, K_2, as ran-

dom variables. Therefore, it is possible to consider that the values of the integrals $\int_0^T K_1 d\tau$ and $\int_0^T K_2 d\tau$ are attained as a result of a simple random walk.

DEOXYGENATION COEFFICIENT, K_1

Assume that in a time interval of ΔT the quantity $K_1 \Delta T$ can take only two possible values, $\overline{K_1(T)} \Delta T + \Delta K_1$ and $\overline{K_1(T)} \Delta T - \Delta K_1$. To these two values the following probabilities are assigned:

$$P[K_1(T)\Delta T = \overline{K_1(T)}\Delta T + \Delta K_1] = p_1(\Delta K_1) \qquad (28)$$
$$P[K_1(T)\Delta T = \overline{K_1(T)}\Delta T - \Delta K_1] = q_1(\Delta K_1) \qquad (29)$$

where $K_1(T)$ = mean value of K_1 at time of travel T and $p_1(\Delta K_1) + q_1(\Delta K_1) = 1$. For the present analysis, it will be considered that ΔK_1 is constant.

Let us further assume that

$$\lim_{\Delta K_1 \to 0} \frac{[q_1(\Delta K_1) - p_1(\Delta K_1)]}{\Delta K_1} = \frac{\alpha_1}{T} \qquad (30)$$

$$\lim_{\substack{\Delta T \to 0 \\ \Delta K_1 \to 0}} \frac{(\Delta K_1)^2}{\Delta T} = \beta_1 T \qquad (31)$$

where α_1 and β_1 are finite. Then the probability distribution of K_1 is found to be normal (Bailey, 1964; Feller, 1968) with a mean of $\overline{K_1} - \alpha_1 \beta_1$ and a variance of β_1. For the limiting variance to remain finite, $(\Delta K_1)^2/\Delta T$ must be of the order of unity, and for the mean to remain finite, $p_1 - q_1$ must be of the order of ΔK_1. The equations of this section satisfy these conditions.

These concepts were applied to equation 24 to estimate the probability distribution of the BOD for a stream system in which K_1 is a random variable. In this analysis, the following assumptions were made:

1. The BOD added along the reach is much smaller than the BOD added at the upstream end of the reach, that is

$$\int_0^T L_a d\tau \ll f(\zeta);$$

and hence the variance of the term $Ie^{-\int_0^\tau (K_1 + K_3) d\tau}$ is negligible, where I is defined by equation 27; and

2. The rate constant for the removal of BOD by sedimentation and adsorption, K_3, is deterministic throughout the river reach under consideration, that is, independent of flow time.

These assumptions are valid for many situations observed in natural streams; in fact, the effects of L_a and K_3 were completely neglected by several in-

vestigators (Streeter and Phelps, 1925; Kothandaraman, 1968). Considering K_3 as a random variable does not introduce any complications into the analysis if K_3 is considered as normally distributed with known mean and variance. For appreciably large values of L_a, the probability distribution of the BOD cannot be found analytically, but Monte Carlo methods can be efficiently used.

Details of the development of the probability distribution of the BOD have been presented previously (Esen, 1971). It was found that the BOD was distributed according to the lognormal distribution, or

$$f_L(\cdot) = \left[\frac{1}{\sqrt{2\pi T^2 \beta_1}}\right]\left[\frac{1}{[L-I \exp{(-\int_0^T (\overline{K}_1+K_3)\,d\tau)}]}\right]$$

$$\exp\left[-\frac{\{ln[L-I \exp{(-\int_0^T (\overline{K}_1+K_3)\,d\tau)}]\ -\int_0^T (\overline{K}_1+K_3)\,d\tau+T\alpha_1\beta_1-ln\,f(\zeta)\}^2}{2T^2\beta_1}\right]$$

for

$$L>I \exp{\left(-\int_0^T (\overline{K}_1+K_3)\,d\tau\right)}$$

and

$$f_L(\cdot) = 0 \text{ for } L \leq I \exp{\left(-\int_0^T (\overline{K}_1+K_3)\,d\tau\right)} \quad (32)$$

The mean value of L for any time of travel T was found by integrating the product of L and the density function between the limits $-\infty$ and $+\infty$. The result was

$$L=\exp\left[ln\,f(\zeta) - \int_0^T (\overline{K}_1+K_3)\,d\tau+T\alpha_1\beta_1+\frac{T^2\beta_1}{2}\right]$$
$$+I \exp{\left(-\int_0^T (\overline{K}_1+K_3)\,d\tau\right)} \quad (33)$$

Similarly the variance of L at any time of travel T was found by integrating the product of $(L-L)^2$ and the density function between the limits $-\infty$ and $+\infty$, and the result was

$$\text{var}\,(L) = \exp\left[2[ln\,f(\zeta) - \int_0^T (\overline{K}_1+K_3)\,d\tau+T\alpha_1\beta_1]\right.$$
$$\left.+2T^2\beta_1\right] - \exp\left[2[ln\,f(\zeta) - \int_0^T (\overline{K}_1+K_3)\,d\tau\right.$$
$$\left.+T\alpha_1\beta_1] + T^2\beta_1\right] \quad (34)$$

The probability density function of L given by equation 32 reduces to that of Kothandaraman (1970), if K_3 and L_a are 0 and K_1 is considered to be independent of the time of travel. However, one of the advantages of the random walk model from which equation 32 was developed is that it can con-

sider K_1 as well as the mean and variance of K_1 as a function of time of travel. Thus in the most general case, β_1 and α_1 are functions of the time of travel. If the variance of K_1 is considered as constant throughout the river reach of interest, β_1 is constant, and if it is further considered that the mean value of K_1 is also constant, then α_1 can be taken as zero. The mean and variance of K_1 are measurable quantities, and numerical values can be assigned to K_1, α_1, and β_1.

REAERATION COEFFICIENT, K_2

Proceeding exactly analogously as for the deoxygenation coefficient, we assume that in a time interval of ΔT the quantity $K_2\Delta T$ can take only two possible values, $\overline{K_2(T)}\Delta T+\Delta K_2$ and $\overline{K_2(T)}\Delta T-\Delta K_2$. To these two values, the following probabilities are assigned:

$$P[K_2(T)\Delta T = \overline{K_2(T)}\Delta T+\Delta K_2] = p_2(\Delta K_2) \quad (35)$$
$$P[K_2(T)\Delta T = \overline{K_2(T)}\Delta T-\Delta K_2] = q_2(\Delta K_2) \quad (36)$$

where $K_2(T)$ is the mean value of K_2 at time of travel T and $p_2(\Delta K_2) + q_2(\Delta K_2) = 1$. Let us further assume that

$$\lim_{\Delta K_2 \to 0} \frac{[q_2(\Delta K_2) - p_2(\Delta K_2)]}{\Delta K_2} = \frac{\alpha_2}{T} \quad (37)$$

$$\lim_{\substack{\Delta T \to 0 \\ \Delta K_2 \to 0}} \frac{(\Delta K_2)^2}{\Delta T} = \beta_2 T \quad (38)$$

where α_2 and β_2 are finite. Then the probability distribution of K_2 is found to be normal (Bailey, 1964; Feller, 1968) with a mean of $K_2-\alpha_2\beta_2$ and a variance of β_2. These equations also satisfy the conditions previously given for the limiting variance and the mean to remain finite.

Because of mathematical complexities, it was not possible to apply this random walk model for the reaeration coefficient to equation 26 and obtain analytically the distribution function of the dissolved-oxygen deficit. Therefore, the distribution of the oxygen deficit was estimated by using a Monte Carlo simulation technique.

MONTE CARLO SIMULATION TECHNIQUE FOR DISSOLVED-OXYGEN DEFICIT

The Monte Carlo simulation technique is a method by which a complex system with random components is numerically operated by random numbers chosen in such a manner that they simulate the physical behavior of these components. In the general sense, each random component of the system is represented by a numerical value randomly chosen from some probability distribution.

Monte Carlo simulation essentially involves three steps:

1. Select representative probability distributions which will simulate the physical behavior of the random components of the system.
2. Generate numerical values of the random variables from the probability distributions.
3. Use variance-reducing techniques to accelerate the computations.

A random walk model is well suited to simulation by Monte Carlo methods because each random component (the increments added to K_1 or K_2 in the present study) is allowed to take only two possible discrete values. This requires the simplest of random variable generation techniques, the generation of uniformly distributed random variables. The methods for the generation of uniformly distributed random variables are discussed in various books (such as Hammersley and Handscomb, 1964), and almost all digital computers have a built-in subroutine available for this purpose.

The determination of the probability distribution of the dissolved-oxygen deficit is based on equation 26 which may be written

$$D = C_s - C = B + f(\zeta) e^{-\int_0^T K_2 d\tau} \int_0^T K_1 e^{\int_0^T (K_2 - K_1 - K_3) d\tau'} d\tau \tag{39}$$

where

$$B = e^{-\int_0^T K_2 d\tau} \left[C_{s} - F(\zeta) + \int_0^T (D_B - C_a) e^{\int_0^T K_2 d\tau'} d\tau \right. $$
$$\left. + \int_0^T K_1 I\, e^{\int_0^T (K_2 - K_1 - K_3) d\tau'} d\tau \right] \tag{40}$$

Additional assumptions are necessary besides those made in the development of the model for the probability distribution of the BOD. These are:

1. the initial dissolved-oxygen deficit, $C_s - F(\zeta)$, is small and the variance of the term $[C_s - F(\zeta)]$ $e^{-\int_0^T K_2 d\tau}$ is small; and
2. the difference between the benthal demand term, D_B, and the term for the addition of dissolved oxygen along the reach, C_a, is small, and the variance of the term $e^{-\int_0^T K_2 d\tau}$
$$\left[\int_0^T (D_B - C_a) e^{\int_0^T K_2 d\tau'} d\tau \right]$$ is small.

With these assumptions and approximating the integrals in equation 39 by finite sums, it follows that

$$C_s - C - B = f(\zeta)\, \exp\left(-\sum_{i=1}^{n} K_{2_i} \Delta T\right) \left[\frac{-K_{1_1}}{K_{2_1} - K_{1_1} - K_3} \right.$$
$$+ \sum_{i=1}^{n-1} \frac{K_{1_i}(K_{2_{i+1}} - K_3) - (K_{2_i} - K_3) K_{1_{i+1}}}{(K_{2_i} - K_{1_i} - K_3)(K_{2_{i+1}} - K_{1_{i+1}} - K_3)}$$
$$\exp\left\{ \sum_{j=1}^{i} (K_{2_j} - K_{1_j} - K_3) \Delta T \right\}$$
$$\left. + \frac{K_{1_n}}{K_{2_n} - K_{1_n} - K_3} \exp\left\{ \sum_{j=1}^{n} (K_{2_j} - K_{1_j} - K_3) \Delta T \right\} \right] \tag{41}$$

Note that for $n\Delta T = T$, $K_{1_1} = K_{1_2} = \ldots = K_{1_n}$, $K_3 = 0$, and $K_{2_1} = K_{2_2} = \ldots = K_{2_n}$, equation 41 reduces to the classical oxygen sag equation of Streeter and Phelps (1925) (recall eq 4).

If K_3 should be a normally distributed random variable like K_1 and K_2, then Monte Carlo simulation requires three sets of uniformly distributed random numbers to be generated (one each for K_{1_j}, K_{2_j}, and K_{3_j}). There is no evidence at present, however, to suggest that K_3 is a random variable. Hence, K_3 was considered to be deterministic in the present study.

The determination of the probability distribution of the oxygen deficit at any time of travel, T, by Monte Carlo simulation involves the following steps:

1. Experimentally determine the mean and variance of K_1 and K_2 and the correlation coefficient between K_1 and K_2; or by experimentally determining the means, estimate the variance of K_1 and K_2 from equations 12 and 16, respectively, and the correlation coefficient between K_1 and K_2 from equation 20.
2. Choose an integer n and determine ΔT from $\Delta T = T/n$.
3. Determine ΔK_1 and ΔK_2 from equations 31 and 38, respectively, as

$$\Delta K_1 = \sqrt{\beta_1 T \Delta T}$$
$$\Delta K_2 = \sqrt{\beta_2 T \Delta T}$$

where $\beta_1 =$ variance of K_1; and

$\beta_2 =$ variance of K_2.

4. If the mean values of K_1 and K_2 are assumed to be time-independent, then $\alpha_1 = \alpha_2 = 0$ and $p_1 = p_2 = 0.5$. For time-dependent mean values of K_1

and K_2, determine α_1 and α_2 from mean (K_1) $=K_1-\alpha_1\beta_1$ and mean $(K_2)=K_2-\alpha_2\beta_2$ (K_1 and K_2 will be estimated as the values of K_1 and K_2 at $T=0$), and compute p_1 and p_2 from equations 30 and 37 as

$$p_1 = (1-\alpha_1\Delta K_1/T)/2$$

$$p_2 = (1-\alpha_2\Delta K_2/T)/2.$$

5. Compute B from equation 40 with known values of C_s, $F(\zeta)$, D_B, C_a, L_a, K_3, and mean values of K_1 and K_2, and consider B as deterministic at any time of travel, T. If B is not deterministic, see below for the procedure to use.

6. Generate two sets of n uniformly distributed random numbers, R_{1j} and R_{2j}, between 0 and 1, and compute $K_{1j}\Delta T$ and $K_{2j}\Delta T$ as

$K_{1j}\Delta T = K_1\Delta T + \Delta K_1$ $R_{1j} < p_1$

$K_{1j}\Delta T = K_1\Delta T - \Delta K_1$ $R_{1j} \geqq p_1$

$K_{2j}\Delta T = K_2\Delta T + \Delta K_2$ $R_{1j} < p_1$, $R_{2j} \geqq [1-r(K_1, K_2)]/2$

     or

     $R_{1j} \geqq p_1$, $R_{2j} < [1-r(K_1, K_2)]/2$

$K_{2j}\Delta T = K_2\Delta T - \Delta K_2$ $R_{1j} < p_1$, $R_{2j} < [1-r(K_1, K_2)]/2$

     or

     $R_{1j} \geqq p_1$, $R_{2j} \geqq [1-r(K_1, K_2)]/2$

7. With the value of B determined in step (5) and $K_{1j}\Delta T$ and $K_{2j}\Delta T$ values determined in step (6), compute the oxygen deficit or (C_s-C) from equation 41.

8. Repeat steps 6 and 7 m times to obtain an estimate of the probability distribution of the oxygen deficit.

The assumptions made in the development of equations 32 and 41 are reasonable for many streams. However, if the values of L_a, C_a, $C_s-F(\zeta)$, and D_B-C_a are large and K_3 is normally distributed rather than deterministic, Monte Carlo methods can still be used efficiently. However, writing the integrals involving I as finite sums requires very complicated expressions, and it is more convenient to use a series of difference equations which can be solved easily on a digital computer. The difference equations for the equations describing the BOD and oxygen deficit distributions (eqs 24 and 26, respectively) are

$$L(j) = e^{-(K_{1j}+K_{3j})\Delta T}[L(j-1) + \frac{L_a(j)}{K_{1j}+K_{3j}}$$
$$(e^{(K_{1j}+K_{3j})\Delta T} -1)]$$

$$j=1, 2, \ldots \qquad (42)$$

$$L(0) = f(\zeta) \qquad (43)$$

$$C_s - C(j) = e^{-K_{2j}\Delta T}\left[C_s - C(j-1) + \frac{D_B(j)}{K_{2j}}(e^{K_{2j}\Delta T} -1)\right.$$
$$+ L(j-1)\frac{K_{1j}}{K_{2j}-K_{1j}-K_{3j}}(e^{(K_{2j}-K_{1j}-K_{3j})\Delta T} -1)$$
$$+ \frac{K_{1j}L_a(j)}{K_{2j}(K_{1j}+K_{3j})}(e^{K_{2j}\Delta T} -1)$$
$$\left. - \frac{K_{1j}L_a(j)}{(K_{1j}+K_{3j})(K_{2j}-K_{1j}-K_{3j})}(e^{(K_{2j}-K_{1j}-K_{3j})\Delta T} -1)\right]$$

$$j=1, 2, \ldots \qquad (44)$$

$$C(0) = F(\zeta) \qquad (45)$$

where $T=j\Delta T$ for a specific value of the argument j.

Therefore, if the variance of B cannot be neglected, equations 42, 43, 44, and 45 are used recurrently in step (7) of the procedure rather than equation 41. If the variance of the initial K_1 values is different from the variance of the K_1 values along the river reach, then the initial K_1 values can be selected from a normal distribution with known mean and variance. The total variance of K_1 equals the sum of the variance of the initial values of K_1 and the variance of the K_1 values along the reach.

The sampling method used in this study is referred to as "Straightforward Sampling." This method is based on the premise that the uncertainty in the mean value obtained by a Monte Carlo technique is always reduced when the sample size is increased. A number of methods are available for reducing the variance of the results obtained with the same m value used in the straightforward sampling (Kahn, 1957; Hammersley and Handscomb, 1964). However, because of the complexity of the equations used in the determination of the probability distributions of the oxygen deficit, these methods could not be used in the present study. A flow chart of the straightforward Monte Carlo simulation procedure used in the present study is presented in figures 22, 23, and 24 in "Supplemental Data."

APPLICATION OF THE TECHNIQUE AND DISCUSSION OF RESULTS

To demonstrate the application and versatility of the stochastic model, a hypothetical example consisting of three parts was used. In addition, the model was applied to data from the Sacramento River. Details of these examples are presented in the following paragraphs.

HYPOTHETICAL EXAMPLE

The basic data used in the hypothetical example consisted of

$\overline{K}_1 = 0.15$ days^{-1};

$\overline{K}_2 = 0.50$ days^{-1};

$f(\zeta) = 10$ mg/l; and

$C_s - F(\zeta) = 0$ mg/l.

In addition, the effects of D_x, K_3, L_a, and C_a on the profiles of the BOD and oxygen deficit were assumed to be negligible. The quantities $f(\zeta)$ and $C_s - F(\zeta)$ are, respectively, the BOD and oxygen deficit at the upstream end of the reach.

The probability distribution of the oxygen deficit was estimated using three sets of conditions. In the first case, the variances of the initial values of K_1 and of the values of K_1 along the reach were assumed to be equal, K_1 and K_2 were assumed to be uncorrelated, and the variance of K_2 was assumed to be constant. In the second case, the variances of the initial values of K_1 and of the values of K_1 along the reach were different, K_1 and K_2 were assumed to be uncorrelated, and the variance of K_2 was assumed to be constant. The third case assumed that the variances of both K_1 and K_2 were constant and also a constant correlation coefficient between K_1 and K_2 was assumed. In all three cases, the mean values of K_1 and K_2 were assumed to be independent of time of travel, and therefore $\alpha_1 = \alpha_2 = 0$.

Details of the three cases are as follows:

Case 1: The coefficients of variation, $C_v(K_1)$ and $C_v(K_2)$, were assumed to be 0.35 and 0.3, respectively, as discussed previously (recall eqs. 12 and 16). From these values, the variances of K_1 and K_2 were computed as

var $(K_1) = \beta_1 = [C_v(K_1)\overline{K}_1]^2$
$= [(0.35)(0.15)]^2 = 0.00276$ days^{-2}

var $(K_2) = \beta_2 = [C_v(K_2)\overline{K}_2]^2$
$= [(0.3)(0.50)]^2 = 0.0225$ days^{-2}

Case 2: The analysis of the Ohio River data discussed previously showed that the variance of the K_1 values along the reach was 0.0015 days^{-2} for an initial mean K_1 value of 0.205 days^{-1}. Hence, the variance of the K_1 values along the reach for the hypothetical example was estimated as

$\beta''_1 = [(\sqrt{0.0015}/0.205)0.15]^2 = 0.00080$ days^{-2}

The variance of the initial values, K_{1_1}, therefore is

$\beta'_1 = \beta_1 - \beta''_1$
$= 0.00276 - 0.00080 = 0.00196$ days^{-2}

The initial values, K_{1_1} were selected randomly from a normal distribution with a mean of 0.15 and a variance of 0.00196. Twelve uniformly distributed random numbers, R_{1_i}, were generated and K_{1_1} was computed from

$$K_{1_1} = \sqrt{0.00196} \left[\sum_{i=1}^{12} R_{1_i} - 6 \right] + 0.15$$

Then the Monte Carlo techniques were applied with $\beta''_1 = 0.00080$ days^{-2} and $\beta_2 = 0.0225$ days^{-2}.

Case 3: The third case was essentially the same as the first case except that K_1 and K_2 were assumed to be linearly correlated with a correlation coefficient of 0.5.

Equation 41 with $B = 0$ was used in all three cases for the estimation of the distribution of the oxygen deficit. The mean values of K_1 and K_2 were assumed to be independent of time, hence $\alpha_1 = \alpha_2 = 0$. It follows that $p_1 = p_2 = 0.5$.

The mean, variance, and the 10 percentile and 20 percentile limits of the oxygen deficit obtained for the three cases along with the oxygen-deficit values computed from the Streeter and Phelps (1925) equation (recall eq 4) using mean values of K_1 and K_2 are presented in table 1. The 10 percentile and 20 percentile limits of the oxygen-deficit distribution indicate the oxygen-deficit values for which 10 percent and 20 percent, respectively, of the deficit values are larger. The frequency distributions of the oxygen deficits are plotted in figures 1, 2, and 3. The mean values of the oxygen deficits and the 10 percentile and 20 percentile limits are plotted in figure 4 as a function of time of travel.

TABLE 1.—Dissolved-oxygen deficit from equation 4 and significant parameters of the stochastic, oxygen-deficit distribution

Time, T (days)	Deficit (eq 4) (mg/l)	Mean deficit (mg/l)	Variance of deficit (mg/l)^2	10 percentile limit (mg/l)	20 percentile limit (mg/l)
Case 1					
1 -------	1.09	1.04	0.121	1.48	1.31
2 -------	1.60	1.56	.315	2.29	2.04
3 -------	1.78	1.73	.528	2.64	2.33
4 -------	1.77	1.80	.575	2.84	2.34
5 -------	1.67	1.82	.931	3.07	2.45
Case 2					
1 -------	1.09	1.07	0.129	1.54	1.36
2 -------	1.60	1.67	.347	2.46	2.19
3 -------	1.78	1.72	.400	2.51	2.23
4 -------	1.77	1.81	.668	2.91	2.43
5 -------	1.67	1.64	.580	2.62	2.20
Case 3					
1 -------	1.09	1.03	0.101	1.41	1.29
2 -------	1.60	1.49	.191	2.06	1.86
3 -------	1.78	1.67	.266	2.36	2.09
4 -------	1.77	1.83	.372	2.68	2.28
5 -------	1.67	1.61	.367	2.44	2.09
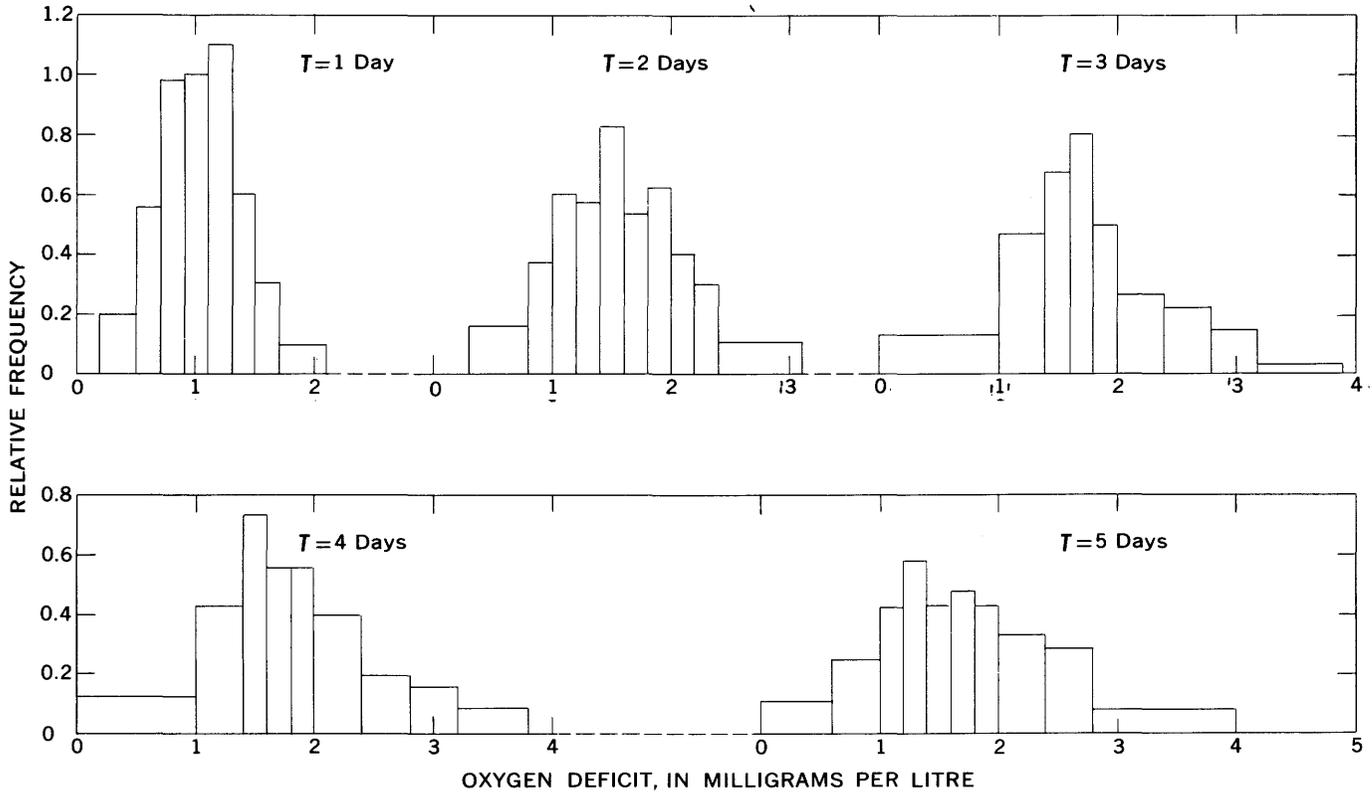
FIGURE 1.—Distribution of the oxygen deficit estimated for the conditions of case 1.
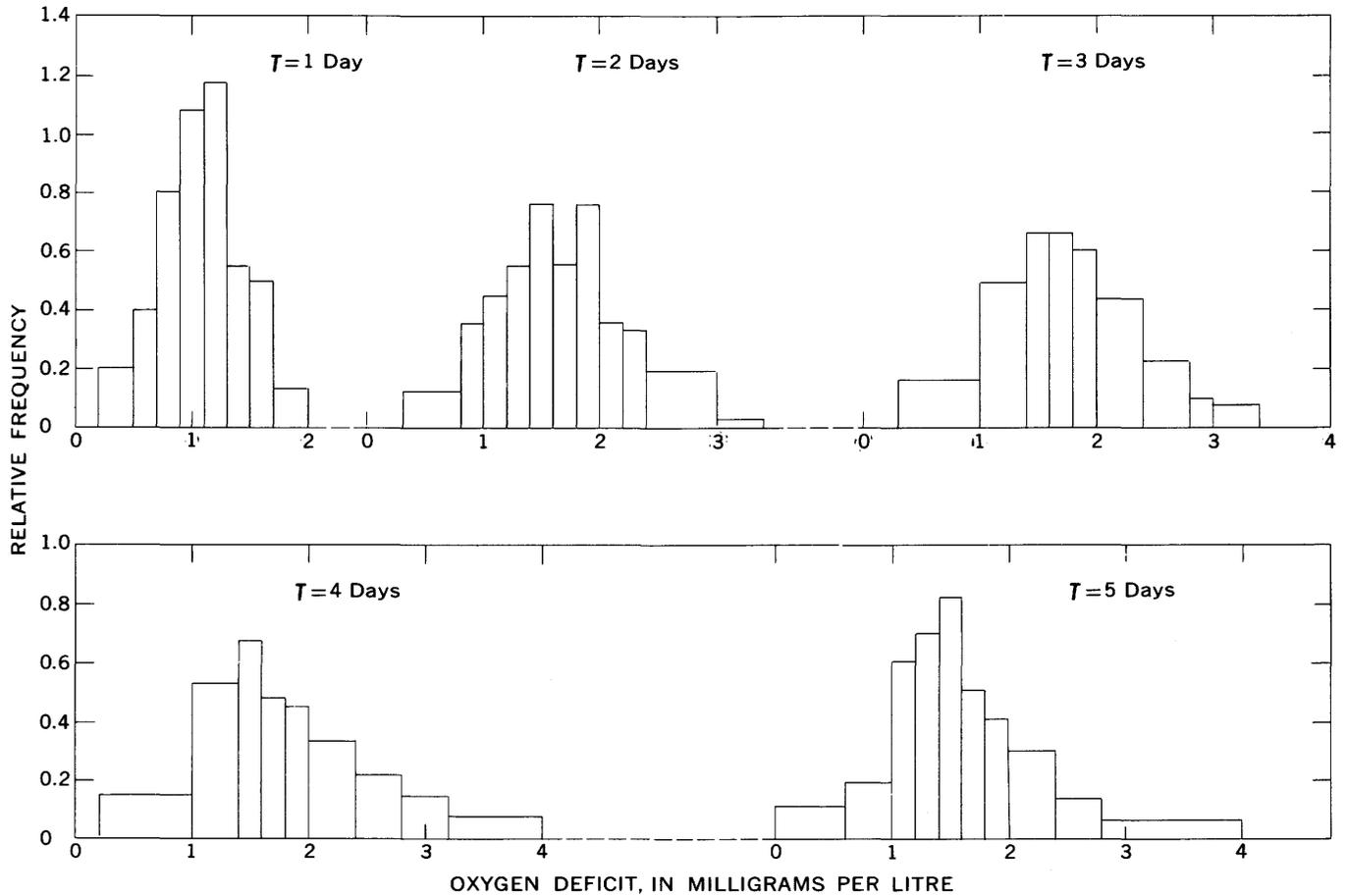
FIGURE 2.—Distribution of the oxygen deficit estimated for the conditions of case 2.
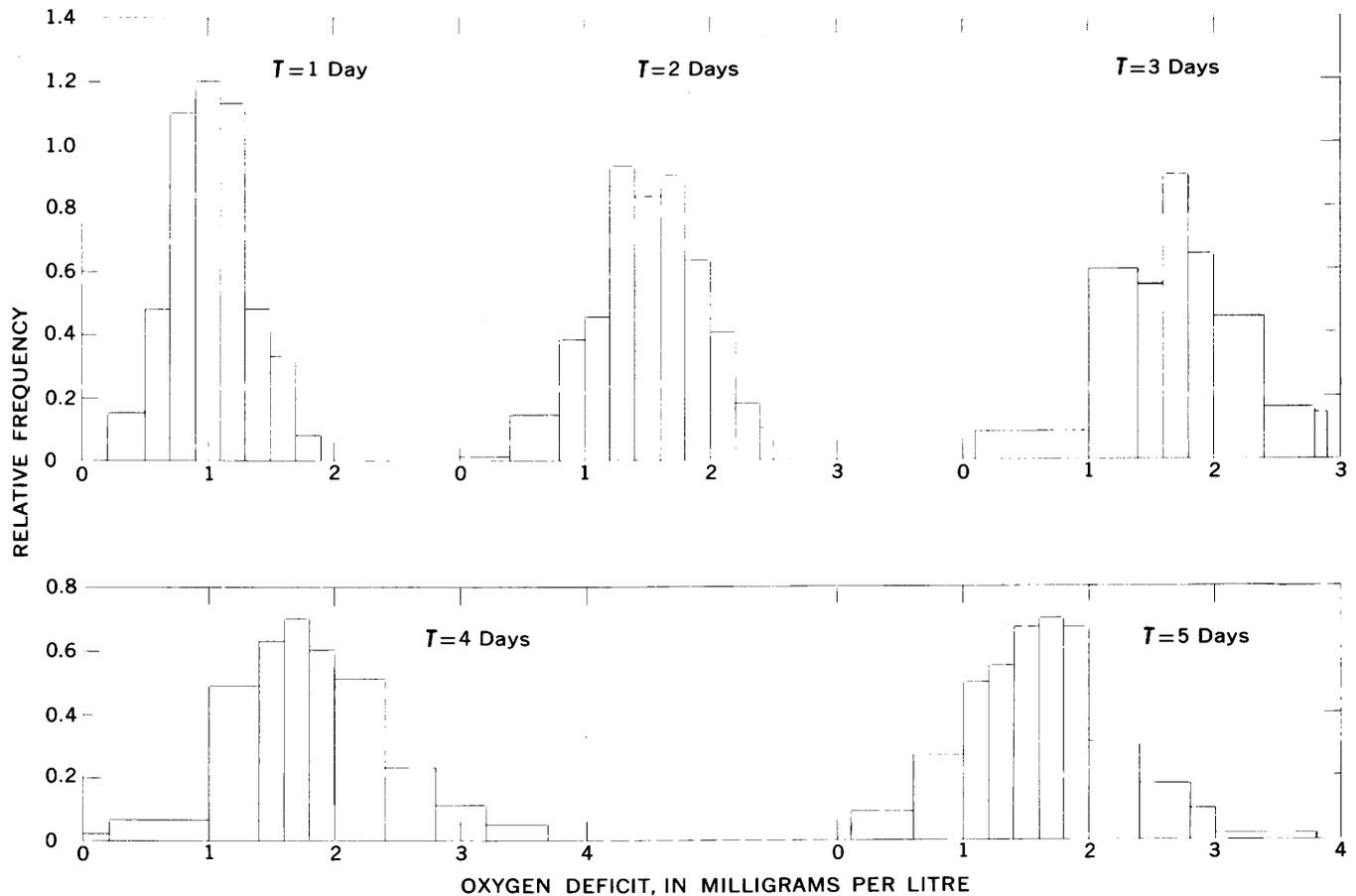
FIGURE 3.—Distribution of the oxygen deficit estimated for the conditions of case 3.

The values of the variance of the oxygen deficit given in table 1 show that all three cases give comparable results for small times of travel. However, as time increases, the variances of the deficit distributions estimated for cases 1 and 2 become larger than the variance estimated for case 3. Case 3 is preferred in the present study because of the expected positive correlation between K_1 and K_2. Recall that $r(K_1, K_2)$, the correlation coefficient between K_1 and K_2, was assumed to be $+0.5$ for case 3. At the present time, there is no experimental data to assist in the assigning of an exact value or function to the correlation coefficient. Inspection of the results presented in table 1 shows that the effect of the correlation coefficient on the oxygen-deficit distribution increases as travel time T increases; however, a time dependent $r(K_1, K_2)$ whose average value is $+0.5$ probably would not appreciably change the deficit distribution at large T.

To determine if $r(K_1, K_2)$ could be estimated from $r(\text{BOD}, D)$, the correlation coefficient between BOD and the oxygen deficit, a correlation study between BOD and the deficit for case 1 $[r(K_1, K_2)=0]$ and

case 3 $[r(K_1, K_2)=0.5]$ was made. The results are presented in table 2 and show that $r(\text{BOD}, D)$ is rather insensitive to $r(K_1, K_2)$. Hence, $r(K_1, K_2)$ cannot be efficiently estimated from a knowledge of $r(\text{BOD}, D)$ for the conditions considered in the present study.

The frequency distributions of the oxygen deficit presented in figures 1, 2, and 3 show that the deficit distributions become flatter and skewed to the right as time of travel increases. This type of skewness is especially favorable in the determination of probabilistic stream standards because the percentile limits of the oxygen deficit will be less sensitive to errors in the values estimated for the coefficients of variation of K_1 and K_2 and the correlation coefficient

TABLE 2.—*Correlation coefficients between BOD and oxygen deficit*

Time, T (days)	Correlation coefficient (BOD, D)	
	Case 1 $r(K_1, K_2)=0$	Case 3 $r(K_1, K_2)=0.5$
1 -----------	—0.94	—0.96
2 -----------	—.66	—.64
3 -----------	—.40	—.26
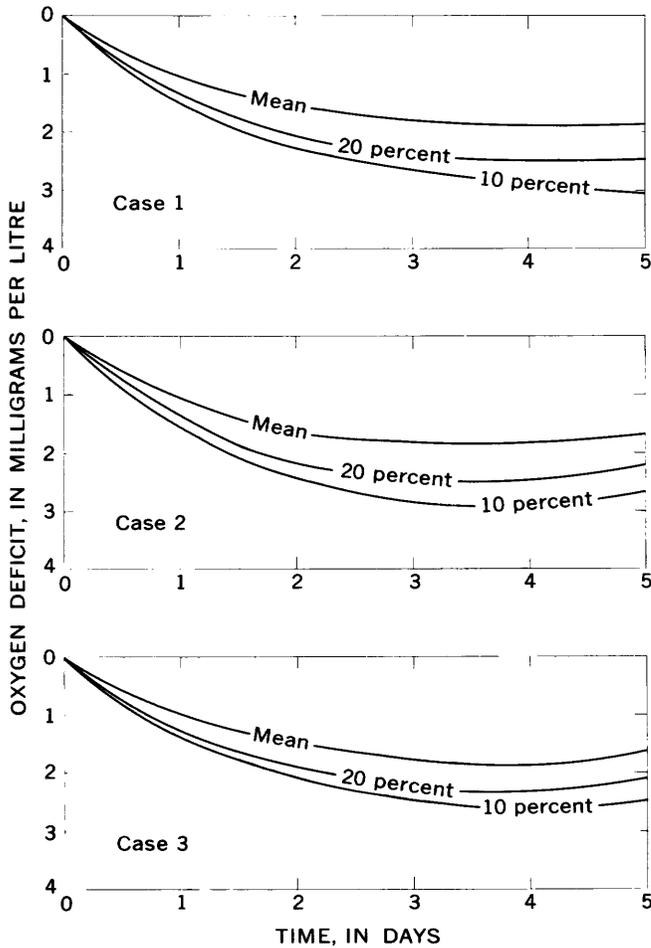
FIGURE 4.—Mean oxygen deficit and 10 and 20 percentile limits as a function of time for the oxygen-deficit distributions estimated for the conditions of cases 1, 2, and 3.

between K_1 and K_2. On the other hand, the most probable deficit values would not be very informative because of the flatness of the deficit distribution. Kothandaraman (1968) concluded that the most probable oxygen-deficit values of the deficit distributions were closer to the observed deficits than deficits predicted by deterministic equations at all times. At small times of travel, this conclusion is probably valid because of the high peaks in the frequency distributions. However, the largest time of travel that Kothandaraman (1968) considered was 0.73 day and for larger times of travel, the deficit distributions became flatter and the most probable deficit value cannot be estimated accurately.

The deterministic critical time of travel, that is, the travel time at which the deficit, computed from the Streeter and Phelps (1925) equation, is maximum, was 3.44 days for the conditions of the hypothetical example. The oxygen-deficit values pre-

sented in table 1 and the deficit profiles plotted in figure 4 show that critical time of travel for case 1 was not reached for the 5-day period considered. For cases 2 and 3, the critical time of travel was at some point between 3 and 5 days. Additional data points would be necessary to determine the critical time more exactly. In general, however, as will be discussed in later sections, the stochastic critical time of travel was larger than the deterministic time of travel.

SACRAMENTO RIVER DATA

The stochastic model developed in previous sections was also applied to data from the Sacramento River, Calif. A detailed description of the reach of the river where the BOD and oxygen-deficit data were collected is given by Thayer and Krutchkoff (1966) and the dissolved-oxygen concentration data for $T < 6$ days are presented in table 13 in "Supplemental Data." In addition to the dissolved-oxygen data, Thayer and Krutchkoff (1966) give the following parameters for the reach:

$$\overline{K}_1 = 0.35 \text{ days}^{-1}$$

$$\overline{K}_2 = 0.75 \text{ days}^{-1}$$

$$L_a = 0.20 \text{ mg/l per day}$$

$$D_B = 0.10 \text{ mg/l per day}$$

$$K_3 = 0.20 \text{ days}^{-1}$$

$$f(\zeta) = L_0 = 6.8 \text{ mg/l} = \text{constant}$$

$$C_s = 9 \text{ mg/l}$$

$$C_s - F(\zeta) = D_0 = 0.3 \text{ mg/l} = \text{constant}$$

The effects of longitudinal dispersion and of the addition of dissolved oxygen along the reach were neglected.

On the basis of these data, the variance of B in equation 41 was considered negligible, and the value B at any time of travel T was determined from equation 40 as

$$B = \frac{\overline{K}_1 L_a}{(K_2 - K_1 - K_3)(K_1 + K_3)} \left[e^{-\overline{K}_2 T} - e^{-(\overline{K}_1 + K_3) T} \right]$$

$$+ [C_s - F(\zeta)] e^{-\overline{K}_2 T} + \left[\frac{D_B}{\overline{K}_2} + \frac{\overline{K}_1 L_a}{\overline{K}_2 (\overline{K}_1 + K_3)} \right] \cdot [1 - e^{-\overline{K}_2 T}]$$

For the above values, equation 40 becomes

$$B = e^{-0.75T} [0.300 + 0.636 (1 - e^{0.20T}) + 0.303 (e^{0.75T} - 1)]$$

Following essentially the procedure used in case 3 of the previous section, and using the values of $C_v(K_1), C_v(K_2)$, and $r(K_1, K_2)$ given by equations 12, 16, and 20, respectively, and the value of B deter-

mined above, equation 41 was solved 200 times where $T/\Delta T$ was taken as 100. The values of the deficit computed from equation 39 with mean values of K_1 and K_2 and the mean, variance, and 10 percentile and 20 percentile limits of the oxygen deficit computed from the stochastic model are presented in table 3. The observed concentration of dissolved oxygen, that is, $C = C_s - D = 9 - D$, is plotted in figure 5 together with the dissolved-oxygen concentration profile predicted by equation 39 and the mean, 10 percentile, and 20 percentile limits of the stochastic deficit distribution. The frequency distributions of the oxygen deficit are presented in figure 6. The variances of the experimental data are presented in figure 7 as a function of the time of travel; also shown in figure 7 are the variances predicted by the stochastic model of Thayer and Krutchkoff (1966) and the stochastic model developed in the present study.

The dissolved-oxygen concentrations and the 10 and 20 percentile limit lines plotted in figure 5 can be used to check approximately the predicted deficit distribution. If the predicted distribution is correct,

TABLE 3.—*Significant parameters of the oxygen-deficit distribution for the Sacramento River data*

Time, T (days)	Deficit (eq 39) (mg/l)	Mean deficit (mg/l)	Variance of deficit (mg/l)^2	10 percentile limit (mg/l)	20 percentile limit (mg/l)
1 -------	1.48	1.40	0.107	1.78	1.68
2 -------	1.54	1.46	.119	1.87	1.71
3 -------	1.28	1.34	.159	1.84	1.63
4 -------	1.00	1.02	.166	1.54	1.28
5 -------	.76	.85	.127	1.25	1.08

then the 10 percentile and 20 percentile lines should have about 10 percent and 20 percent of the data points below them, respectively. The number of data points below each of these lines was counted for times between 0.2 and 5 days. It was found that 9.5 and 23.5 percent of the points lay below the lines, in good agreement with the expected percentages of 10 and 20 percent.

The frequency distributions for the oxygen deficit presented in figure 6 do not flatten and become skewed to the right as time of travel increases, as do the frequency distributions for the hypothetical ex-
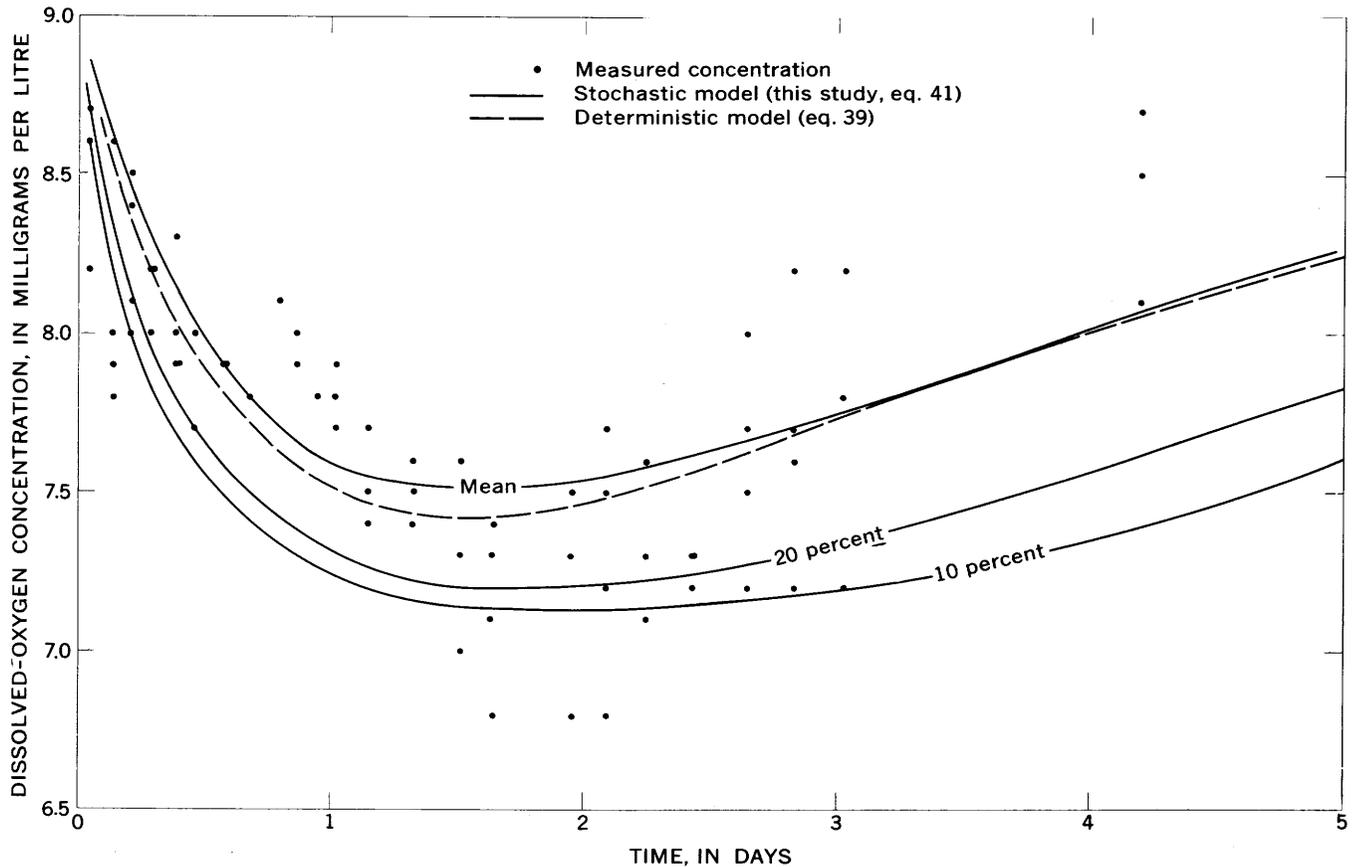
FIGURE 5.—Experimental dissolved-oxygen concentrations, mean dissolved-oxygen concentrations predicted by the deterministic model (eq 39), and mean, 10, and 20 percentile limits of the dissolved-oxygen concentrations predicted by the stochastic model (eq 41); Sacramento River data.
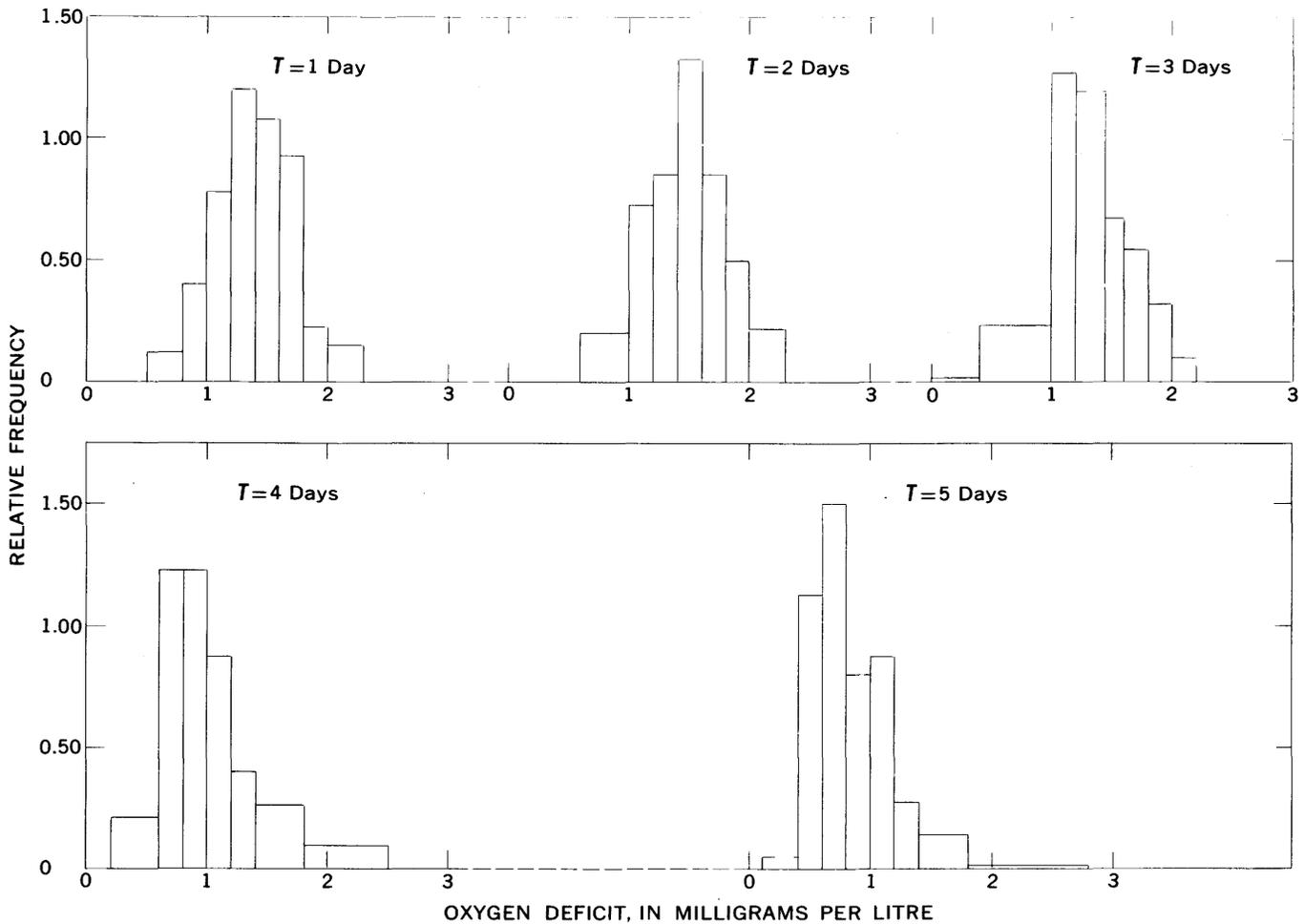
FIGURE 6.—Distribution of the oxygen deficit as estimated from equation 41; Sacramento River data.

ample (recall figs. 1, 2, and 3). This difference is apparently a result of differences in the values of the coefficients and parameters used in the computations.

There is considerable scatter in the oxygen-deficit variances plotted in figure 7, and as a result, it is difficult to decide which of the two models best fits the data. A disadvantage of the Thayer and Krutchkoff (1966) model is that the parameter Δ must be determined from the measured variance at some time of travel. They used a time near zero; however, because of the small variances and considerable scatter in the data at this time, it seems likely that Δ cannot be estimated very accurately. On the other hand, the stochastic model developed in the present study does not depend on a measurement of the deficit variance for predicting the deficit variance as a function of travel time.

Thayer and Krutchkoff (1966) concluded that the oxygen-deficit variance is a maximum at the travel time when the deficit is maximum. For the Sacramento River data (see fig. 5), the maximum deficit occurred between 1 and 2 days. On the other hand, figure 7 suggests that the deficit variance is a maximum for a time of about 3 days. The stochastic model predicted a similar behavior in that the variance increased rapidly for 3 days, then increased little between 3 and 4 days (see table 3). The occurrence of the maximum deficit variance at a time larger than the critical time of travel did not appear to cause T_c to shift downstream relative to the deterministic critical time of travel. However, the 10 and 20 percentile limits of dissolved-oxygen concentration become rather flat between the second and third days.

ESTIMATION OF THE ACCURACY OF THE MONTE CARLO PROCEDURE

The accuracy obtained from the Monte Carlo simulation procedure depends on the values chosen for m and n. Recall that m is the number of times the computations are repeated and that n, the number of steps in the random walk process, appeared in several equations where integrals were approxi-
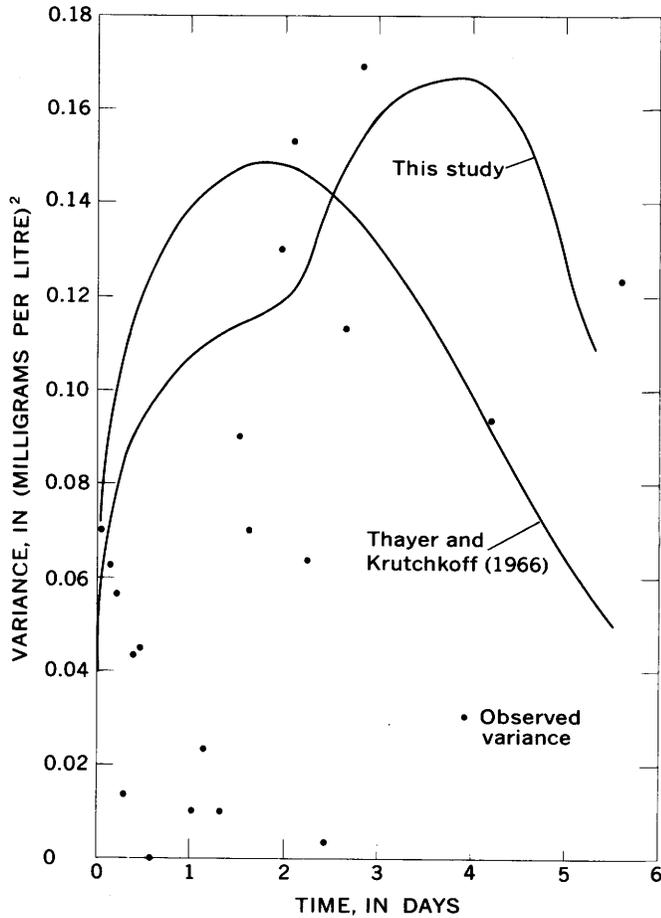
FIGURE 7.—Observed and predicted variance of the oxygen-deficit distributions as a function of time; Sacramento River data.

mated by finite sums. The accuracy obtained in these equations depends on the functional form of K_1 and K_2; if they are constants, then the accuracy will be 100 percent; if they are functions of time, then some error will result from the approximation of the integrals. Although the accuracy of the simulation procedure increases as m and n are increased, the computation time and resultant computer costs also increase. The development of equations for estimating the accuracy of the Monte Carlo method is described in the following paragraphs.

The comparison of the central moments of K_1T with the central moments of the limiting normal distribution was used as the criterion for estimating the accuracy of the Monte Carlo procedure. Moments of K_1T rather than K_1 were compared because determining the value of K_1T as the result of a random walk process is physically more meaningful.

The probability that the diffusing particle of a simple random walk will be at some location after n

steps is distributed according to the binomial distribution (Bailey, 1964). Hence, it follows analogously that the probability density function of K_1T is

$$P[K_1T = \sum_{j=1}^{n} \overline{K}_{1j}\Delta T - (n-2j)\Delta K_1] = \binom{n}{j}(p_1)^j(q_1)^{n-j}$$

(46)

Previously it was established, in the limit as $\Delta T \to 0$, $\Delta K_1 \to 0$, that K_1 was normally distributed with mean $\overline{K}_1 - \alpha_1\beta_1$ and variance β_1 for constant $\overline{K}_1$. Analogously, it can be shown that K_1T is normally distributed with mean $\overline{K}_1T - \alpha_1\beta_1T$ and variance β_1T^2.

Equation 46 gives exact values for the mean and variance of K_1T even for very large values of ΔT because the mean and variance depend only on α_1 and β_1 computed from equations 30 and 31. The third central moment of the limiting normal distribution is zero and therefore the error in the third central moment of K_1T could not be computed as a percentage difference. As an alternative, the difference between the fourth central moments of K_1T obtained from the limiting normal distribution, $\mu_4(K_1T, N)$, and from the probability density function defined by equation 46, $\mu_4(K_1T, \Delta T)$, was used as an indicator of the effect of the value of n on the accuracy of the Monte Carlo simulation procedure.

The fourth central moment of the normal distribution is

$$\mu_4 = 3\sigma^4$$

hence

$$\mu_4(K_1T, N) = 3\beta_1^2T^4$$

Similarly from equation 46,

$$\mu_4(K_1T, \Delta T) = (2\Delta K_1)^4\mu_4(j)$$

(47)

where j has a binomial distribution with parameters n and p_1. The fourth central moment of a binomial distribution is

$$\mu_4 = np_1q_1[1+3(n-2)p_1q_1]$$

hence

$$\mu_4(K_1T, \Delta T) = (2\Delta K_1)^4[3(np_1q_1)^2 + np_1q_1 - 6np_1^2q_1^2]$$

(48)

For small $\Delta T/T$, it follows from equation 30 and 31 that

$$p_1q_1 = \frac{1 - \dfrac{\alpha_1^2\beta_1\Delta T}{T}}{4} \approx \frac{1}{4}\exp(-\alpha_1^2\beta_1\Delta T/T).$$

(49)

Substituting equation 49 into equation 48 and noting that $n = T/\Delta T$ we obtain

$$\mu_4(K_1T, \Delta T \approx 3\beta_1^2 T^4 \exp(-2\alpha_1^2\beta_1\Delta T/T) + \beta_1^2 T^3 \Delta T$$
$$[4 \exp(-\alpha_1^2\beta_1\Delta T/T) - 6 \exp(-2\alpha_1^2\beta_1\Delta T/T)]. \quad (50)$$

The error, E, in the fourth central moment of K_1T is defined as

$$E = \frac{\mu_4(K_1T, \Delta T) - \mu_4(K_1T, N)}{\mu_4(K_1T, N)}. \quad (51)$$

Then, using equations 47 and 50,

$$E \approx \frac{1}{3\beta_1^2 T^4}\{3\beta_1^2 T^4[\exp(-2\alpha_1^2\beta_1\Delta T/T - 1] + \beta_1^2 T^3 \Delta T$$
$$[4 \exp(-\alpha_1^2\beta_1\Delta T/T) - 6 \exp(-2\alpha_1^2\beta_1\Delta T/T)]\}$$

Expanding the exponentials and taking the first two terms gives

$$E \approx \frac{-\dfrac{6\alpha_1^2\beta_1\Delta T}{T} - 2\beta_1^2 T^3 \Delta T\left(1 - \dfrac{8\alpha_1^2\beta_1\Delta T}{T}\right)}{3\beta_1^2 T^4}.$$

For the special case when $\alpha_1 = 0$, the expression for E becomes exact, and

$$E = -\frac{2}{3}\frac{\Delta T}{T} \quad (52)$$

Thus, if $n = T/\Delta T = 100$, the error in the fourth central moment of K_1T is less than 1 percent. The same arguments are also valid for the fourth central moment of K_2T.

The effect of the choice of the integer m on the accuracy of the Monte Carlo procedure was estimated on the basis of the weak law of large numbers (Mood and Graybill, 1963), or

$$P[|\bar{Z}_m - \mu| < \varepsilon] \geq 1 - \delta$$

In this expression, $\bar{Z}_m$ is the sample mean of a random sample of size m from a probability density with a mean of μ and a variance of σ^2, m is any integer greater than $\sigma^2/\varepsilon^2\delta$, and $\varepsilon > 0$ and $0 < \delta < 1$.

Chebyshev's inequality states that $P[g(Z) > s] < E[g(Z)]/s$ for a random variable Z and non-negative function $g(\cdot)$, where $E[g(Z)]$ indicates the expected value of $g(Z)$. Now, if we let $g(Z) = (\bar{Z}_m - \mu)^2$ and $s = \varepsilon^2$, then

$$P[|\bar{Z}_m - \mu|^2 < \varepsilon^2] \geq 1 - E[(\bar{Z}_m - \mu)^2]/\varepsilon^2;$$

but

$$E[(\bar{Z}_m - \mu)^2] = \frac{1}{m}\sigma^2 \text{ and } P[|\bar{Z}_m - \mu| < \varepsilon] \geq 1 - \delta$$

Therefore,

$$1 - \frac{1}{m}\frac{\sigma^2}{\varepsilon^2} \geq 1 - \delta \quad (53)$$

for

$$m > \sigma^2/\varepsilon^2\delta.$$

Thus, the error in an answer is inversely proportional to the square root of the sample size m. For example, m has to be > 160 in order that one is 90 percent certain that $\bar{Z}_m$ is within $\sigma/4$ of μ, that is

$$[m > \sigma^2/(0.25\sigma)^2(0.10) = 160]$$

In the hypothetical example and the treatment of the Sacramento River data described in the previous section, $n = T/\Delta T$ was taken as 100. Therefore, from equation 52 it follows that there is less than 1 percent error in the fourth central moments of K_1T and K_2T for an infinitely large sample size. The computations were repeated 200 times, that is, $m = 200$. From equation 53, we are 92 percent certain that the estimated mean value of the oxygen deficit will be within $\sigma/4$ of the true mean where σ is the standard deviation of the oxygen deficit.

ESTIMATION OF THE VARIANCE OF THE OXYGEN-DEFICIT DISTRIBUTION

The random walk model and Monte Carlo simulation procedure described in previous sections can be used to predict the variance of the dissolved-oxygen deficit at any time of travel for a stream for which L_0, K_1, K_2, the coefficients of variation of K_1 and K_2, and the correlation coefficient between K_1 and K_2 are known. In practice, however, it may not be feasible to compute the oxygen-deficit variance for all possible combinations of these variables because of the computer expenses involved in the simulation process. Hence, an approximate procedure for estimating the variance was developed. This development is described in the following paragraphs.

The basic procedure consists of expanding in a Taylor series the random part of the fundamental equation of the stochastic model, equation 41, and determining the expected values of the first few terms of the series.

From equation 41, it follows that

$$C_s - C - B = g(K_{1_1}, K_{1_2}, ..., K_{1_n}, K_{2_1}, K_{2_2}, ..., K_{2_n}) \quad (54)$$

Expanding g in a Taylor series gives

$$g = g(\bar{K}_1, \bar{K}_2) + \sum_{i=1}^{n}(K_{1_i} - \bar{K}_1)\frac{\partial g}{\partial K_{1_i}}\bigg|_{K_{1_i} = \bar{K}_1}$$
$$+ \sum_{i=1}^{n}(K_{2_i} - \bar{K}_2)\frac{\partial g}{\partial K_{2_i}}\bigg|_{K_{2_i} = \bar{K}_2}$$
$$+ \sum_{i=1}^{n}\frac{(K_{1_i} - \bar{K}_1)^2}{2}\frac{\partial^2 g}{\partial K_{1_i}^2}\bigg|_{K_{1_i} = \bar{K}_1}$$
$$+ \sum_{i=1}^{n}\frac{(K_{2_i} - \bar{K}_2)^2}{2}\frac{\partial^2 g}{\partial K_{2_i}^2}\bigg|_{K_{2_i} = \bar{K}_2}$$

$$+ \sum_{i=1}^{n} (K_{1_i} - \overline{K}_1)(K_{2_i} - \overline{K}_2) \frac{\partial^2 g}{\partial K_{1_i} \partial K_{2_i}} \Big|_{\substack{K_{1_i} = \overline{K}_1 \\ K_{2_i} = \overline{K}_2}} + \cdots$$

$$(55)$$

The last term on the right hand side of equation 55 has only a single summation because the terms $E[\,(K_{1_i} - \overline{K}_1)(K_{1_j} - \overline{K}_1)\,]$, $E[\,(K_{2_i} - \overline{K}_2)(K_{2_j} - \overline{K}_2)\,], E[K_{1_i} - \overline{K}_1)(K_{2_j} - \overline{K}_2)\,]$ are zero for $i \neq j$.

The mean value of g, that is $E[g]$, is obtained from equation 55 as

$$E[g] \approx E\left\{ g(\overline{K}_1, \overline{K}_2) + \sum_{i=1}^{n} \frac{(K_{1_i} - \overline{K}_1)^2}{2} \frac{\partial^2 g}{\partial K_{1_i}^2} \Big|_{K_{1_i} = \overline{K}_1} \right.$$

$$+ \sum_{i=1}^{n} \frac{(K_{2_i} - \overline{K}_2)^2}{2} \frac{\partial^2 g}{\partial K_{2_i}^2} \Big|_{K_{2_i} = \overline{K}_2} + \sum_{i=1}^{n}$$

$$\left. (K_{1_i} - \overline{K}_1)(K_{2_i} - \overline{K}_2) \frac{\partial^2 g}{\partial K_{1_i} \partial K_{2_i}} \Big|_{\substack{K_{1_i} = \overline{K}_1 \\ K_{2_i} = \overline{K}_2}} \right\}. \quad (56)$$

Computing the terms in equation 56 with g given by equations 41 and 54, and taking the limit as $n \to \infty$ gives

$$E[g] \approx g(\overline{K}_1, \overline{K}_2) + G_1 + G_2 + G_3 \quad (57)$$

where

$$g(\overline{K}_1, \overline{K}_2) = f(\zeta) e^{-\overline{K}_2 T} [\frac{\overline{K}_1}{a}(e^{aT} - 1)] \quad (58)$$

$$G_1 = \frac{\beta_1 T f(\zeta)}{2} e^{-\overline{K}_2 T} [\frac{\overline{K}_2}{a^2}(1 - e^{aT}) + \frac{\overline{K}_1 T}{a} e^{aT}] \quad (59)$$

$$G_2 = \frac{\beta_2 T f(\zeta)}{2} e^{-\overline{K}_2 T} [\frac{\overline{K}_1}{a^2}(e^{aT} - 1) - \frac{\overline{K}_1 T}{a}] \quad (60)$$

$$G_3 = r(K_1, K_2) \sqrt{\beta_1 \beta_2} T f(\zeta) e^{-\overline{K}_2 T}$$

$$[\frac{\overline{K}_1}{2a^2}(c^{aT} - 1) + \frac{\overline{K}_2}{2a^2}(1 - e^{aT})] \quad (61)$$

$$a = \overline{K}_2 - \overline{K}_1 - K_3 \quad (62)$$

and $r(K_1, K_2)$ is the correlation coefficient between K_1 and K_2. Note that

$$E[\,(K_{1_i} - K_1)^2\,] = \beta_1 T / \Delta T$$

and

$$E[\,(K_{2_i} - K_2)^2\,] = \beta_2 T / \Delta T.$$

The mean value of $C_s - C$, that is $E[C_s - C]$, is

$$E[C_s - C] = B + E[g] \quad (63)$$

Then by definition, the variance of g, or equivalently the variance of $C_s - C$, is

$$\text{var}(g) = E[\,(g - E(g))^2\,];$$

hence, from equation 55, it follows that

$$\text{var}(g) \approx E\left\{ \sum_{i=1}^{n} (K_{1_i} - \overline{K}_1)^2 (\frac{\partial g}{\partial K_{1_i}})^2 \Big|_{K_{1_i} = \overline{K}_1} + \sum_{i=1}^{n} \right.$$

$$(K_{2_i} - \overline{K}_2)^2 (\frac{\partial g}{\partial K_{2_i}})^2 \Big|_{K_{2_i} = \overline{K}_2} + 2 \sum_{i=1}^{n} (K_{1_i} - \overline{K}_1)$$

$$\left. (K_{2_i} - \overline{K}_2)(\frac{\partial g}{\partial K_{1_i}})(\frac{\partial g}{\partial K_{2_i}}) \Big|_{\substack{K_{1_i} = \overline{K}_1 \\ K_{2_i} = \overline{K}_2}} \right\} \quad (64)$$

Computing the terms in equation 64 and taking the limit as $n \to \infty$ gives

$$\text{var}(g) \approx G_4 + G_5 + G_6 \quad (65)$$

where

$$G_4 = \beta_1 T [f(\zeta)]^2 e^{-2\overline{K}_2 T} \left[\frac{\overline{K}_2^2}{2a^3}(e^{2aT} - 1) \right.$$

$$\left. + \frac{2\overline{K}_1 \overline{K}_2}{a^3}(e^{aT} - e^{2aT}) + \frac{\overline{K}_1^2 T}{a^2} e^{2aT} \right] \quad (66)$$

$$G_5 = \beta_2 T [f(\zeta)]^2 e^{-2\overline{K}_2 T}$$

$$\left[\frac{\overline{K}_1^2}{2a^3}(e^{2aT} - 1) + \frac{2\overline{K}_1^2}{a^3}(1 - e^{aT}) + \frac{\overline{K}_1^2 T}{a^2} \right] \quad (67)$$

$$G_6 = 2r(K_1, K_2)(\beta_1 \beta_2)^{1/2}[f(\zeta)]^2 T\, e^{-2\overline{K}_2 T} \left[-\frac{\overline{K}_1 \overline{K}_2}{2a^3} \right.$$

$$(e^{2aT} - 1) + \frac{\overline{K}_1 \overline{K}_2}{a^3}(e^{aT} - 1) + \frac{\overline{K}_1^2}{a^3}(e^{2aT} - e^{aT}) - \frac{\overline{K}_1^2 T}{a^2} e^{aT} \bigg] \quad (68)$$

and a is defined by equation 62.

Equations 63 and 65 for estimating the mean and variance of the oxygen-deficit distribution are based on only the first and second order terms in the Taylor series expansion, hence the estimates of the variance are less than the actual variance. No attempt was made in the present study to improve the accuracy of the estimated variance by considering higher order terms of the expansion; this is possible but tedious. Qualitative inspection of the next higher order terms in equation 65 shows that they are of the order and form

$$\beta_1^2 T^2 [f(\zeta)]^2 \frac{\overline{K}_1^2}{a^6} e^{-2K_2 T}$$

which indicates that the accuracy obtained in the variance of the deficit estimated by equation 65 decreases as β_1, β_2, and T increase and $\overline{K}_2 - \overline{K}_1 - K_3$ decreases.

For constant values of $C_v(K_1)$, $C_v(K_2)$, $r(K_1, K_2)$, and K_3, equation 65 can be put into the form

$$\text{var}(C_s - C) \approx [f(\zeta)]^2 \Psi(\overline{K}_1, \overline{K}_2, T). \qquad (69)$$

To facilitate obtaining first estimates of the variance of the oxygen-deficit distribution, Ψ-values were computed from equations 66, 67, and 68 for $\overline{K}_1$ values ranging from 0 to 0.5 days^{-1} and $\overline{K}_2$ values ranging from 0 to 2.5 days^{-1} for times of travel of 1, 2, and 3 days. In these computations, it was assumed that $C_v(K_1) = 0.35$, $C_v(K_2) = 0.3$, $r(K_1, K_2) = 0.5$, and $K_3 = 0$. Values of Ψ are plotted as a function of $\overline{K}_1$ and $\overline{K}_2$ in figures 8, 9, and 10 for travel times of 1, 2, and 3 days, respectively. The dashed line indicates the discontinuity at the point where $\overline{K}_1 = \overline{K}_2$.

The Ψ-values are underestimated if $C_v(K_1)$ and (or) $C_v(K_2)$ are less than the true values and conversely. At small times of travel, $C_v(K_1)$ is most important in the determination of Ψ and at large times

of travel, $C_v(K_2)$ is most important. On the other hand, Ψ decreases at all times as $r(K_1, K_2)$ increases. Computations were not carried out for travel times of 4 and 5 days because equation 65 becomes less reliable at large T.

The oxygen-deficit variances at $T = 1$, 2, and 3 days for cases 1 and 3 of the hypothetical example described previously were compared with the variances estimated from equation 65 and figures 8, 9, and 10. For case 1, equation 65 was used with $G_6 = 0$, that is, $r(K_1, K_2) = 0$; for case 3 the variance of the deficit was estimated from figures 8, 9, and 10 and the relation

$$\text{var}(D) = [f(\zeta)]^2 \Psi = 100\Psi.$$

The results presented in table 4 show that the estimated variances are comparable to those determined from the stochastic model for the first 2 days. For $T = 3$ days, however, the estimated variances are
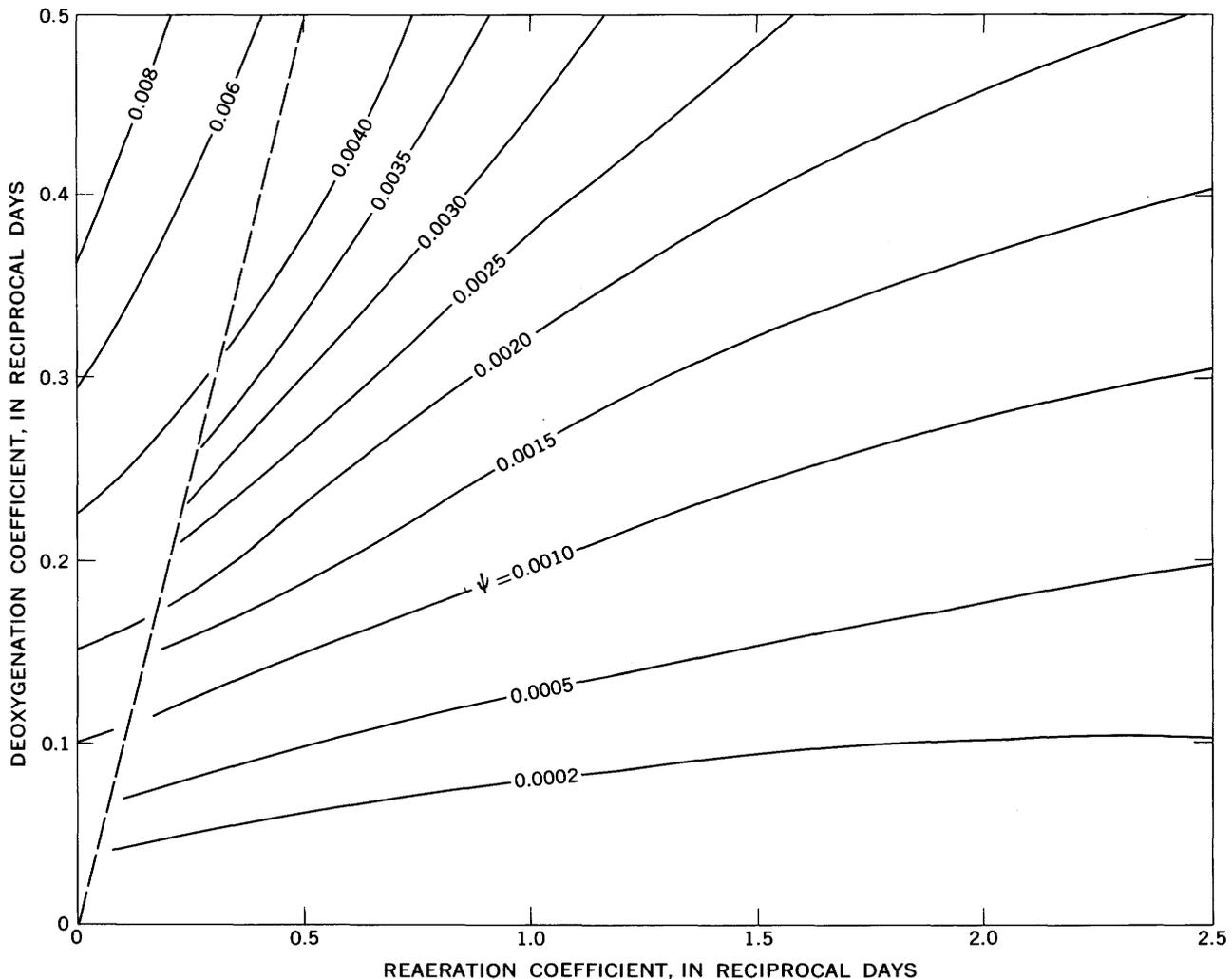
FIGURE 8.—Ψ-parameter as a function of the reaeration and deoxygenation coefficients; time of travel of 1 day.
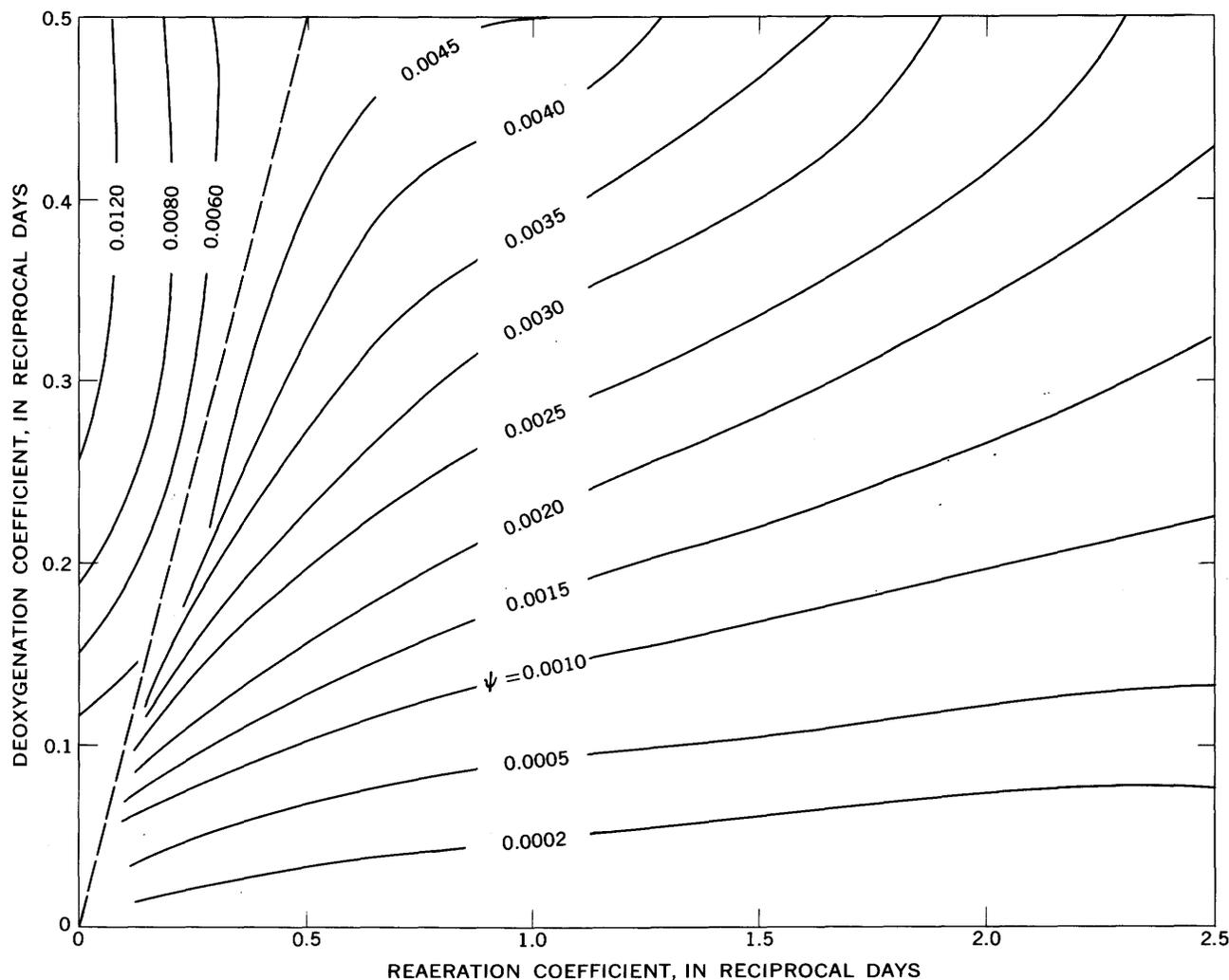
FIGURE 9.—Ψ-parameter as a function of the reaeration and deoxygenation coefficients; time of travel of 2 days.

TABLE 4.—*Variances of the oxygen-deficit distributions determined from the stochastic model and the Taylor series approximation (eq 65)*

Time, T (days)	Variance, in $(mg/l)^2$	
	Stochastic model	Equation 65
Case 1		
1 ------------	0.121	0.135
2 ------------	.315	.307
3 ------------	.528	.442
Case 3		
1 ------------	0.101	0.105
2 ------------	.191	.186
3 ------------	.266	.230

smaller than the variances from the stochastic model. This is to be expected because the higher order terms of the Taylor series expansion were neglected in the development of equation 65, as discussed previously. Despite this restriction, however, figures 8, 9, and 10 can be used to obtain a first estimate of the variance of the deficit distribution for those situations in which $T_c \leq 3$ days and when the deficit profile obtained by deterministic procedures is changing rapidly around $T = T_c$.

EXTENSION OF THE MODEL

In the previous sections, a stochastic model based on Monte Carlo simulation was developed for predicting the variance of the distribution of the dissolved-oxygen deficit at points downstream of a waste source for a stream system in which K_1 and K_2 are random variables. In addition, a simplified procedure based on Taylor series expansion of the equation for the stochastic model was developed for estimating the variance of the oxygen-deficit distribution. In this section, the model was extended such that four parameters could be considered as random variables: the BOD of the waste at the upstream end
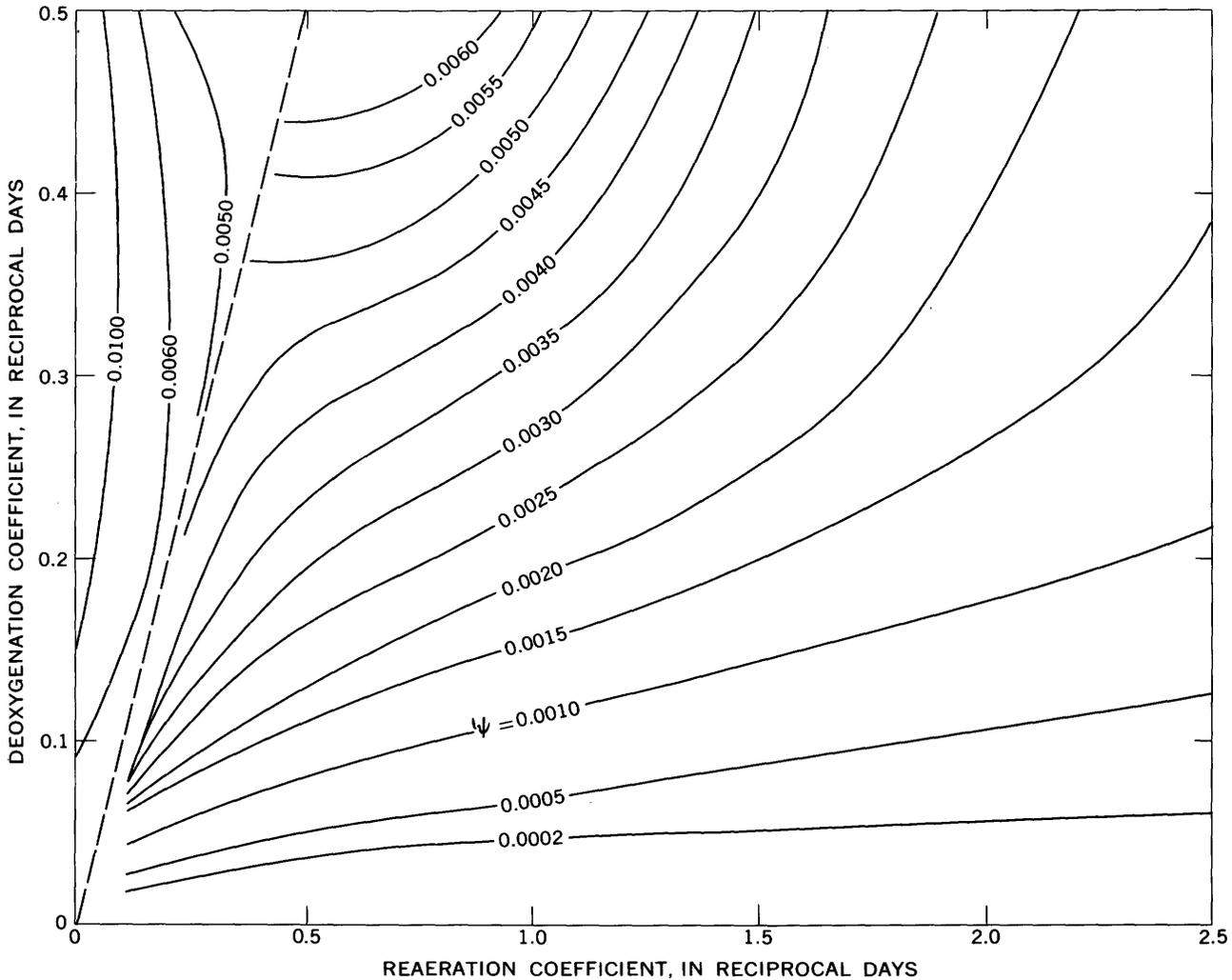
FIGURE 10.—Ψ-parameter as a function of the reaeration and deoxygenation coefficients; time of travel of 3 days.

of the reach (L_0), the deoxygenation coefficient at the upstream end of the reach (K_1), the deoxygenation coefficient along the reach (k), and the reaeration coefficient (K_2). In addition, correlation was assumed between K_2 and k and between K_1 and L_0.

The fundamental equation of the stochastic model (eq 41) was used with three changes: B was assumed negligible, K_3 was assumed zero, and the variation of K_1 along the reach was separated from the variation of the initial K_1 values by the addition of the k parameter. These changes are for the purpose of simplification only; they do not in any way change the basic procedures used in the computations. With these changes, equation 41 becomes

$$D = L_0 \exp\left(-\sum_{i=1}^{n} K_{2_i}\Delta T\right)\left\{-\frac{K_1+k_1}{(K_{2_1}-K_1-k_1)}\right.$$

$$+\sum_{i=1}^{n-1}\frac{(K_1+k_i)(K_{2_{i+1}})-(K_{2_i})(K_1+k_{i+1})}{(K_{2_i}-K_1-k_i)(K_{2_{i+1}}-K_1-k_{i+1})}$$

$$\exp\left[\sum_{j=1}^{i}(K_{2_j}-K_1-k_j)\Delta T\right]$$

$$\left.+\frac{K_1+k_n}{K_{2_n}-K_1-k_n}\exp\left[\sum_{j=1}^{n}(K_{2_j}-K_1-k_j)\Delta T\right]\right\}. \quad (70)$$

Equation 70 permits the computation of the oxygen deficit for a specific time T where $n\Delta T = T$ and n is the number of steps assumed for the random walk. Repetition of the computations m times then yields m values for the oxygen deficit from which the mean and the variance of the deficit are computed. In a practical situation, however, the time of travel of most importance is the critical time of travel or the time when the deficit is maximum. This is the time at which violations of the quality standards are most likely to occur and hence, the time for which estimates of the variance are most necessary. Therefore, in contrast to previous sections where computations were for general times of 1, 2, 3, 4, or 5 days,

calculations in this section will be specifically for the critical time of travel.

However, the critical time of travel for the stochastic model cannot be determined explicitly from equation 70, in contrast to the deterministic model for which T_c can be computed. In theory, T_c could be determined by applying the Monte Carlo simulation procedure to sequentially increasing times until the maximum deficit is found. In practice, however, the computer expenses would be prohibitive. Therefore, the procedure used was to compute the variances for the distributions of the deficit for the deterministic T_c, given by

$$T_{c_D} = \frac{1}{\overline{K}_2 - \overline{K}_1} ln \left[\frac{\overline{K}_2}{\overline{K}_1} \left(1 - \frac{D_0(\overline{K}_2 - \overline{K}_1)}{\overline{L}_0} \right) \right] \quad (71)$$

and for a stochastic critical time of travel, T_{c_s}, obtained from the Taylor series expansion approximation of equation 70. These critical times of travel in general will not be the same; results presented previously suggested that the critical time of travel for the stochastic model was larger than the deterministic critical time of travel. The procedure used to estimate the stochastic critical time of travel will be described shortly.

MONTE CARLO COMPUTATIONS

The basic Monte Carlo simulation procedure was as described previously and as shown in figures 22, 23, and 24; however, because of the several changes in this section, the procedure will be outlined qualitatively here. Input data consisted of the coefficients of variation of the reaeration coefficient (K_2), of the deoxygenation coefficient along the reach (k), of the total deoxygenation coefficient ($K_1 + k$), and of the BOD at the upstream end of the reach (L_0); the correlation coefficients between K_1 and L_0 and between K_2 and k; and the critical time of travel. Steps in the procedure were:

1. Selection of L_0 from a normal distribution.
2. Selection of K_1 from a normal distribution with the specified correlation coefficient between L_0 and K_1.
3. Selection of $K_1 + k$ and K_2 to be used in the random walk computations (eq 70).
4. Computation of the oxygen deficit.
5. Repetition of the procedure m times.
6. Computation of the mean and the variance from the m values of the oxygen deficit.

The Monte Carlo computations were completed for 25 sets of K_1 and K_2 values with the following conditions:

$C_v(K_1 + k) = 0.35$

$C_v(k) = 0.19$

$C_v(K_2) = 0.30$

$C_v(L_0) = 0.20$

$r(K_1, L_0) = -0.67$

$r(K_2, k) = 0.50$

$\overline{L}_0 = 1.0$ mg/l

$D_0 = 0$

$n = 100$

$m = 800$ for most computations, 200 for some computations.

These conditions were also used in all computations in this section with the Taylor series approximation of the stochastic model.

ESTIMATION OF THE MEAN AND THE VARIANCE OF THE OXYGEN-DEFICIT DISTRIBUTION

As in previous sections, the mean and the variance of the distribution of the oxygen deficit may be estimated from a Taylor series approximation of the stochastic model. Basically it is assumed that

$$D = h(K_{1_1}, K_{1_2}, ...K_{1_n}; K_{2_1}, K_{2_2}, ... K_{2_n}; \\ L_{0_1}, L_{0_2}, ...L_{0_n}; k_1, k_2, ...k_n). \quad (72)$$

Expanding in a Taylor series, taking expected values, and assuming correlation only between K_{1_i} and L_{0_i} and between K_{2_i} and k_i gives

$$\text{Mean } D = h(\overline{K}_1, \overline{K}_2, \overline{L}_0, \overline{k}) + \frac{\text{var } L_0}{2} \frac{\partial^2 h}{\partial L_0^2}$$

$$+ \frac{\beta_1''}{2} \sum_{i=1}^{n} \frac{T}{\Delta T} \frac{\partial^2 h}{\partial k_i^2} + \frac{\beta_2}{2} \sum_{i=1}^{n} \frac{T}{\Delta T} \frac{\partial^2 h}{\partial K_{2_i}^2}$$

$$+ r(K_2, k)(\beta_1'' \beta_2)^{1/2} \sum_{i=1}^{n} \frac{T}{\Delta T} \frac{\partial^2 h}{\partial K_{2_i} \partial k_i} + \frac{\beta_1'}{2} \frac{\partial^2 h}{\partial K_1^2}$$

$$+ r(K_1, L_0)(\beta_1' \text{ var } L_0)^{1/2} \frac{\partial^2 h}{\partial L_0 \partial K_1} \quad (73)$$

where var implies variance and $\beta_1' = \beta_1 - \beta_1''$.

Determining the derivatives from equation 70, evaluating at the mean values of the variables, and substituting into equation 72 gives

$$\text{Mean } D = h + H_1 + H_2 + H_3 + H_4 + H_5 \quad (74)$$

where

$$h = \frac{\overline{L}_0 \overline{K}_1}{\overline{K}_2 - \overline{K}_1} [e^{-\overline{K}_1 T} - e^{-\overline{K}_2 T}] \quad (75)$$

$$H_1 = \frac{\beta_1'' T \bar{L}_0 e^{-\bar{K}_2 T}}{2}\left[\frac{\bar{K}_2}{(\bar{K}_2-\bar{K}_1)^2}(1-e^{(\bar{K}_2-\bar{K}_1)T}) + \frac{\bar{K}_1 T}{\bar{K}_2-\bar{K}_1}\right.$$
$$\left. e^{(\bar{K}_2-\bar{K}_1)T}\right] \quad (76)$$

$$H_2 = \frac{\beta_2 T \bar{L}_0 e^{-\bar{K}_2 T}}{2}\left[\frac{\bar{K}_1}{(\bar{K}_2-\bar{K}_1)^2}(e^{(\bar{K}_2-\bar{K}_1)T}-1) - \frac{\bar{K}_1 T}{\bar{K}_2-\bar{K}_1}\right] \quad (77)$$

$$H_3 = r(K_2,k)(\beta_1''\beta_2)^{1/2}T\bar{L}_0 e^{-\bar{K}_2 T}\left[\frac{\bar{K}_1}{2(\bar{K}_2-\bar{K}_1)^2}\right.$$
$$\left.(e^{(\bar{K}_2-\bar{K}_1)T}-1) + \frac{\bar{K}_2}{2(\bar{K}_2-\bar{K}_1)^2}(1-e^{(\bar{K}_2-\bar{K}_1)T})\right] \quad (78)$$

$$H_4 = \frac{\beta_1' \bar{L}_0 e^{-K_2 T}}{2}\left[\frac{2\bar{K}_2}{(\bar{K}_2-\bar{K}_1)^3}(e^{(\bar{K}_2-\bar{K}_1)T}-1) - \frac{2\bar{K}_2 T}{(\bar{K}_2-\bar{K}_1)^2}\right.$$
$$\left. e^{(\bar{K}_1-\bar{K}_1)T} + \frac{\bar{K}_1 T^2}{(\bar{K}_2-\bar{K}_1)}e^{(\bar{K}_2-\bar{K}_1)T}\right] \quad (79)$$

$$H_5 = r(K_1,L_0)(\beta_1')^{1/2}\bar{L}_0 C_v(L_0) e^{-\bar{K}_2 T}\left[\frac{\bar{K}_2}{(\bar{K}_2-\bar{K}_1)^2}\right.$$
$$\left.(e^{(\bar{K}_2-\bar{K}_1)T}-1) - \frac{\bar{K}_1 T}{\bar{K}_2-\bar{K}_1}e^{(\bar{K}_2-\bar{K}_1)T}\right]. \quad (80)$$

The terms H_1, H_2, and H_3 represent the effects on the deficit of the variation of the deoxygenation coefficient along the reach, the variation of the reaeration coefficient, and the correlation between these coefficients, respectively; allowing for slight changes in nomenclature and the assumption that $K_3 = 0$ in the present section, H_1, H_2, and H_3 are equivalent to G_1, G_2, and G_3 (eqs 59, 60, and 61), respectively. The terms H_4 and H_5 represent the effect on the deficit of the variation of the initial value of the deoxygenation coefficient and the correlation between the initial deoxygenation coefficient and upstream BOD, respectively. Variations in the upstream BOD do not affect estimates of the mean deficit because $\partial^2 h/\partial L_0^2 = 0$. When the variables K_{1i}, K_{2i}, L_{0i}, and k_i are assumed independent of time of travel, then H_1, H_2, H_3, H_4, and H_5 are zero and equation 73 reduces to

$$\text{Mean } D = h \quad (81)$$

or the classical oxygen sag equation of Streeter and Phelps (1925) (see eq 4).

By similar arguments, it may be shown that

$$\text{Var } D = \beta_1'' \sum_{i=1}^{n}\frac{T}{\Delta T}\left(\frac{\partial h}{\partial k_1}\right)^2 + \beta_2 \sum_{i=1}^{n}\frac{T}{\Delta T}\left(\frac{\partial h}{\partial K_{2i}}\right)^2$$
$$+ 2r(K_2,k)(\beta_1''\beta_2)^{1/2}\sum_{i=1}^{n}\frac{T}{\Delta T}\left(\frac{\partial h}{\partial K_{2i}}\right)\left(\frac{\partial h}{\partial k_i}\right)$$
$$+ \text{Var } L_0\left(\frac{\partial h}{\partial L_0}\right)^2 + \beta_1'\left(\frac{\partial h}{\partial K_1}\right)^2$$

$$+ 2r(K_1,L_0)(\beta_1' \text{ var } L_0)^{1/2}\left(\frac{\partial h}{\partial K_1}\right)\left(\frac{\partial h}{\partial L_0}\right). \quad (82)$$

Evaluating derivatives as before, it follows that

$$\text{Var } D = H_6 + H_7 + H_8 + H_9 + H_{10} + H_{11} \quad (83)$$

where

$$H_6 = \beta_1'' T\bar{L}_0^2 e^{-2\bar{K}_2 T}\left[\frac{\bar{K}_2^2}{2(\bar{K}_2-\bar{K}_1)^3}(e^{(2\bar{K}_2-\bar{K}_1)T}-1)\right.$$
$$+ \frac{2\bar{K}_1\bar{K}_2}{(\bar{K}_2-\bar{K}_1)^3}(e^{(\bar{K}_2-\bar{K}_1)T}-e^{2(\bar{K}_2-\bar{K}_1)T}) + \frac{\bar{K}_1^2 T}{(\bar{K}_2-\bar{K}_1)^2}$$
$$\left. e^{2(\bar{K}_2-\bar{K}_1)T}\right] \quad (84)$$

$$H_7 = \beta_2 T\bar{L}_0^2 e^{-2\bar{K}_2 T}\left[\frac{\bar{K}_1^2}{2(\bar{K}_2-\bar{K}_1)^3}(e^{2(\bar{K}_2-\bar{K}_1)T}-1)\right.$$
$$\left.+ \frac{2\bar{K}_1^2}{(\bar{K}_2-\bar{K}_1)^3}(1-e^{(\bar{K}_2-\bar{K}_1)T}) + \frac{\bar{K}_1^2 T}{(\bar{K}_2-\bar{K}_1)^2}\right] \quad (85)$$

$$H_8 = 2r(\bar{K}_2,k)(\beta''\beta)^{1/2}\bar{L}^2 Te^{-2\bar{K}_2 T}\left[-\frac{\bar{K}_2\bar{K}_1}{2(\bar{K}_2-\bar{K}_1)^3}\right.$$
$$(e^{2(\bar{K}_2-\bar{K}_1)T}-1) + \frac{\bar{K}_2\bar{K}_1}{(\bar{K}_2-\bar{K}_1)^3}e^{(\bar{K}_2-\bar{K}_1)T}-1)$$
$$\frac{\bar{K}_1^2}{(\bar{K}_2-\bar{K}_1)^3}\left(e^{2(\bar{K}_2-\bar{K}_1)T}-e^{(\bar{K}_2-\bar{K}_1^2)T}\right) - \frac{\bar{K}_1^2 T}{(\bar{K}_2-\bar{K}_1)^2}$$
$$e^{(\bar{K}_2-\bar{K}_1)T}\right] \quad (86)$$

$$H_9 = \bar{L}_0^2[C_v(L_0)]^2 e^{-2\bar{K}_2 T}\left[\frac{\bar{K}_1^2}{(\bar{K}_2-\bar{K}_1)^2}(e^{2(\bar{K}_2-\bar{K}_1)T}\right.$$
$$\left.-2e^{(\bar{K}_2-\bar{K}_1)T}+1)\right] \quad (87)$$

$$H_{10} = \beta_1'\bar{L}_0^2 e^{-2\bar{K}_2 T}\left[\frac{\bar{K}_2^2}{(\bar{K}_2-\bar{K}_1)^4}(e^{2(\bar{K}_2-\bar{K}_1)T}-2e^{(\bar{K}_2-\bar{K}_1)T}+1)\right.$$
$$- \frac{2\bar{K}_2\bar{K}_1 T}{(\bar{K}_2-\bar{K}_1)^3}e^{2(\bar{K}_2-\bar{K}_1)T}-e^{(\bar{K}_2-\bar{K}_1)T}) + \frac{\bar{K}_1^2 T^2}{(\bar{K}_2-\bar{K}_1)^2}$$
$$e^{2(\bar{K}_2-\bar{K}_1)T}\right] \quad (88)$$

$$H_{11} = 2r(\bar{K}_1,L_0)(\beta_1')^{1/2}\bar{L}_0^2 C_v(L_0) e^{-2\bar{K}_2 T}\left[\frac{\bar{K}_2\bar{K}_1}{(\bar{K}_2-\bar{K}_1)^3}\right.$$
$$(e^{2(\bar{K}_2-\bar{K}_1)T}-2e^{(\bar{K}_2-\bar{K}_1)T}+1) - \frac{\bar{K}_1^2 T}{(\bar{K}_2-\bar{K}_1)^2}$$
$$(e^{2(\bar{K}_2-\bar{K}_1)T}-e^{(\bar{K}_2-\bar{K}_1)T})\right] \quad (89)$$

The terms H_6, H_7, and H_8 represent the effects on the variance of the distribution of the deficit of the

variation of the deoxygenation coefficient along the reach, the variation of the reaeration coefficient, and the correlation between these coefficients, respectively; again allowing for slight changes in nomenclature and the assumption that $K_3 = 0$ in the present section, H_6, H_7, and H_8 are equivalent to G_4, G_5, and G_6 (eqs 66, 67, and 68), respectively. The terms H_9, H_{10}, and H_{11} represent the effects on the variance of the variation of the initial BOD, the variation of the initial value of the deoxygenation coefficient, and the correlation between these variables, respectively.

ESTIMATION OF A STOCHASTIC CRITICAL TIME OF TRAVEL

As mentioned previously, it would be very difficult to determine explicitly the critical time of travel for the stochastic model; also the results of previous sections suggest that in general the stochastic critical time is larger than the deterministic critical time. To get an approximate estimate of T_{c_s}, equation 74 was used. Recall that this equation is the result of a Taylor series expansion of the equation for the stochastic model; because higher order terms of the expansion were neglected, the resultant T_{c_s} will be approximate only.

In theory, equation 74 can be differentiated with respect to T and the result set equal to zero and solved to determine T_c. When this is tried, however it is found that a complicated expression containing exponentials and quadratic terms must be solved by a trial-and-error iterative type of solution to determine T_c. Instead of this procedure, equation 74 was programed for a desk-top computer so that the oxygen deficit was computed as a function of T. After each computation, the time was incremented by 0.01 day and the computation was repeated. Because of the print-out feature of the computer, it was possible to follow the computations and to stop the program after the deficit passed through a maximum. In this manner, T_{c_s} could be estimated to the nearest 0.01 day for any combination of K_1 and K_2 values.

PRESENTATION AND DISCUSSION OF RESULTS

The results presented and discussed in this section are directed toward the practical determination of a first estimate of the variance of the distribution of the oxygen deficit at the critical time of travel. All graphs are for a critical time of travel (T_{c_s} or T_{c_D}) and are for a mean BOD at the upstream end of the reach ($\bar{L}_0$) of 1.0 mg/l. For other values of $\bar{L}_0$, the estimated mean deficit is a simple multiple of the value of $\bar{L}_0$ (recall eqs 70 and 74) and the variance

is a multiple of $\bar{L}_0^2$ (recall eqs 70 and 83). Values assumed for the coefficients of variation and the correlation coefficients were presented previously. Also recall that the oxygen deficit at the upstream end of the reach was zero.

CRITICAL TIME OF TRAVEL

The critical time of travel for the deterministic model was computed from equation 71 with $D_0 = 0$ for the same 25 sets of K_1 and K_2 values used in the Monte Carlo calculations, and the results are presented in table 5. Figure 11 is a plot of the deterministic critical time of travel as a function of the deoxygenation and reaeration coefficients. The dashed line indicates the discontinuity at the point where $K_1 = K_2$.

The critical time of travel for the stochastic model was estimated from the Taylor series expression for the oxygen deficit by the procedure described previously. The T_{c_s} values for the 25 sets of K_1 and K_2 values are presented in table 5 and figure 12 is a plot of the stochastic critical time of travel as a function of the deoxygenation and reaeration coefficients.

The critical times presented in table 5 show that the stochastic T_c was greater than the deterministic T_c for all combinations of K_1 and K_2 considered, in agreement with previous results. The difference, however, was not constant but was largest for the large K_2/K_1 ratios and in general decreased as K_2/K_1 decreased for most of the range of conditions considered. The percentage difference, defined as $(T_{c_s} - T_{c_D})100/T_{c_D}$, is presented in table 5 and plotted in figure 13 as a function of the ratio of the reaeration and deoxygenation coefficients. Where more

TABLE 5.—*Critical time of travel for the deterministic and stochastic models and percentage difference*

K_2 (days⁻¹)	K_1 (days⁻¹)	Ratio K_2/K_1	T_c, in days Deterministic	T_c, in days Stochastic	Percentage difference
0.60	0.025	24.0	5.53	8.21	+48.5
1.20	.050	24.0	2.76	4.10	48.6
1.80	.075	24.0	1.84	2.74	48.9
2.40	.100	24.0	1.38	2.05	48.6
.60	.045	13.3	4.67	5.90	26.3
1.20	.095	12.6	2.30	2.87	24.8
1.80	.140	12.9	1.54	1.93	25.3
2.40	.190	12.6	1.15	1.44	25.2
.60	.080	7.50	3.87	4.50	16.3
1.20	.160	7.50	1.94	2.25	16.0
1.80	.240	7.50	1.29	1.50	16.3
2.40	.320	7.50	.97	1.13	16.5
.40	.075	5.33	5.15	5.78	12.2
2.00	.375	5.33	1.03	116	12.6
.40	.100	4.00	4.62	5.07	9.7
2.05	.500	4.10	.91	1.00	9.9
.50	.150	3.33	3.44	3.73	8.4
1.65	.500	3.30	1.04	1.13	8.7
1.00	.370	2.70	1.58	1.69	7.0
1.00	.445	2.25	1.46	1.55	6.2
.50	.259	1.93	2.73	2.89	5.9
.80	.415	1.93	1.70	1.80	5.9
.30	.375	.800	2.98	3.10	4.0
.20	.325	.615	3.88	4.05	4.4
.10	.250	.400	6.10	6.41	5.1
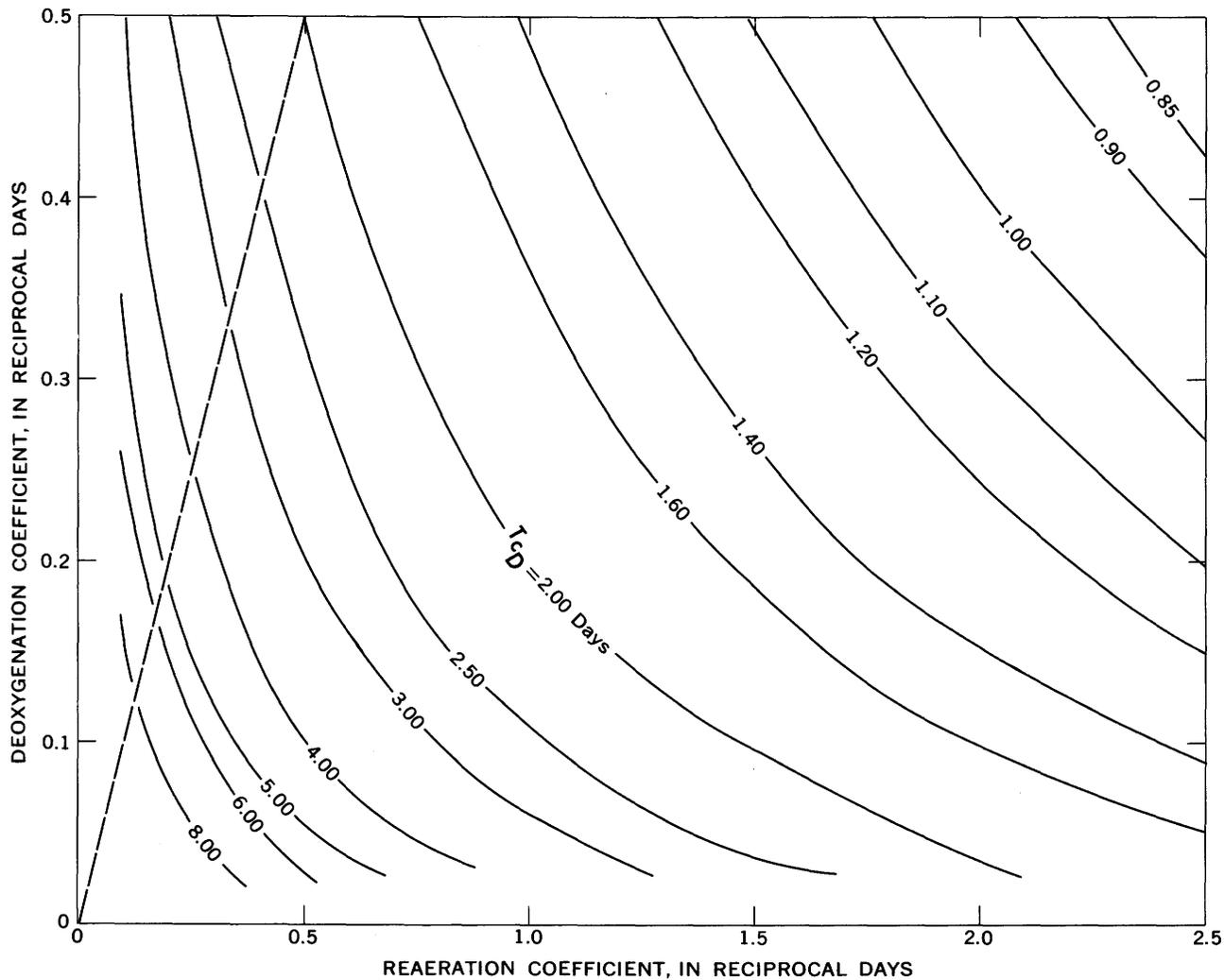
FIGURE 11.—Critical time of travel (deterministic model) as a function of the deoxygenation and reaeration coefficients.

than 1 percent difference exists for a ratio, the average is plotted. The percentage difference decreases as K_2/K_1 decreases for ratios larger than one, and conversely, increases as K_2/K_1 decreases for ratios smaller than one. This behavior suggests that the percentage difference between the critical times of travel is a minimum when the reaeration and deoxygenation coefficients are approximately equal. The K_1 and K_2 values used in the computations were chosen so as to cover the range of coefficients expected for natural streams and rivers; figure 13 can be used to estimate the percentage difference in the critical times for any ratio of coefficients for a specific situation.

Figures 11 and 12 are graphs for estimating the critical time of travel for specific values of K_1 and K_2; figure 11 is for T_{c_D} and figure 12 is for T_{c_s}. Comparison of these graphs shows in a general manner that T_{c_s} is larger than T_{c_D}; these graphs also show the large dependence of T_c on one of the coefficients when the other coefficient is small.

MEAN OXYGEN DEFICIT

The oxygen deficit at the critical time of travel was computed from the deterministic model for T_{c_D} and from the stochastic model for both T_{c_D} and T_{c_s} and the results are presented in table 6. The oxygen deficit is referred to as the mean oxygen deficit because the values reported for the stochastic model are the arithmetic means of the results of the repetitions of the Monte Carlo procedure for each set of conditions. For the T_{c_s} computations, the number of repetitions was 800; for the T_{c_D} computations, the number was either 200 or 800. Use of 200 repetitions for the 24.0 and 7.50 ratios probably explains why the variability of these means is larger than the variability of the means for these ratios when 800 repetitions were used.
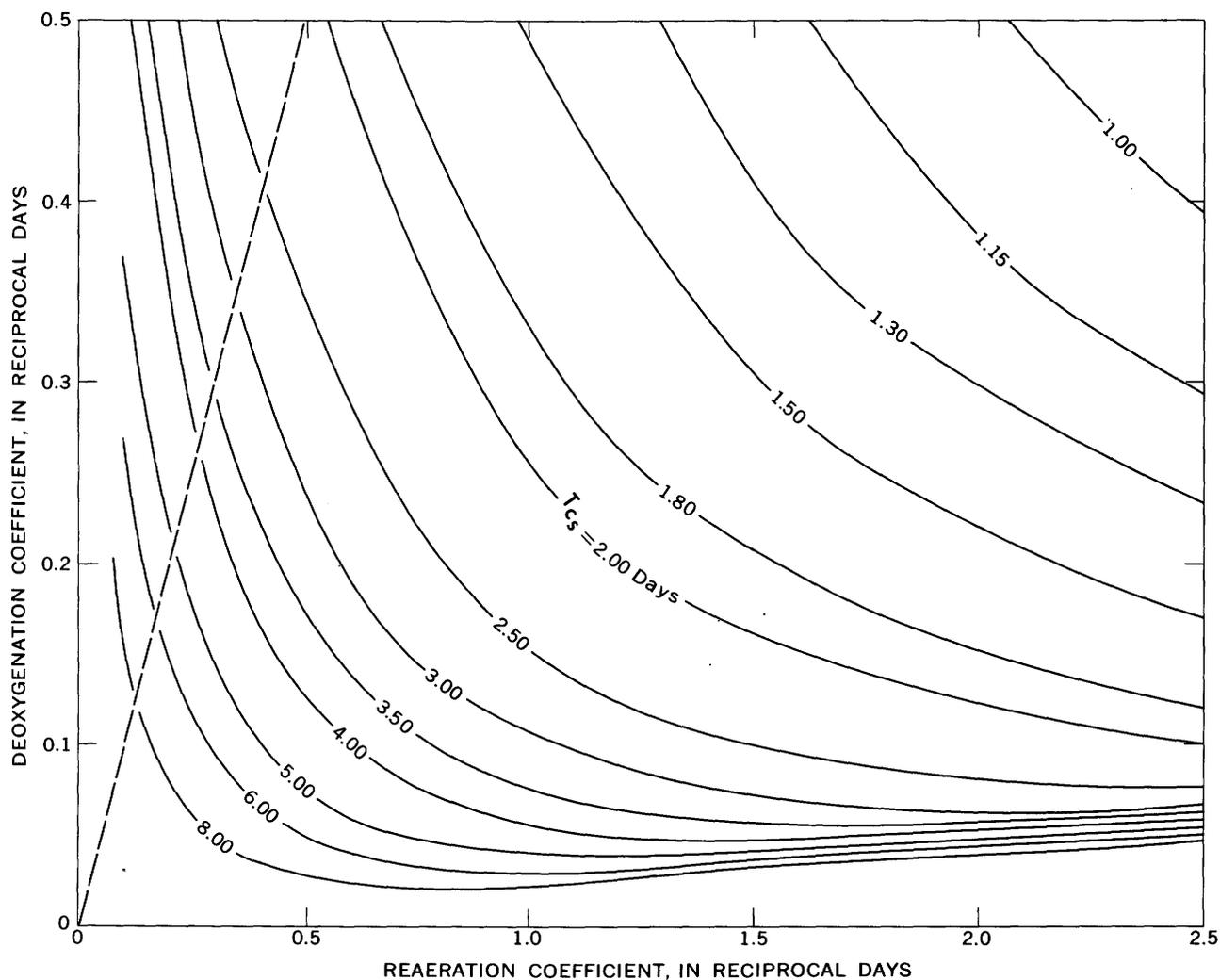
FIGURE 12.—Critical time of travel (stochastic model) as a function of the deoxygenation and reaeration coefficients.

The oxygen deficits are plotted in figure 14 as a function of the ratio of the reaeration and deoxygenation coefficients. The results in figure 14 and table 6 show that the oxygen deficit for the deterministic model tends to be slightly larger than the deficit for the stochastic model for small ratios, but in general the differences among the oxygen deficits are negligible. Also, the relatively large difference in T_{c_s} and T_{c_D} for large ratios had relatively little effect on the computation of the deficit by the stochastic model.

The oxygen deficit was also computed from the Taylor series approximation (recall eq 73) for T_{c_s} and the results are presented in table 7 together with stochastic model results for T_{c_s}. The oxygen deficits are plotted in figure 15 as a function of the ratio of the reaeration and deoxygenation coefficients. The results show that the oxygen deficits computed for

TABLE 6.—*Mean oxygen deficit for deterministic and stochastic models*

K_2 (days^{-1})	K_1 (days^{-1})	Ratio K_2/K_1	Mean oxygen deficit, in mg/l		
			Deterministic	Stochastic	Stochastic
			T_{c_D}	T_{c_D}	T_{c_s}
0.60	0.025	24.0	0.0363	0.0343	0.0380
1.20	.050	24.0	.0363	.0362	.0380
1.80	.075	24.0	.0363	.0377	.0380
2.40	.100	24.0	.0363	.0388	.0380
.60	.045	13.3	.0608	.0632	.0625
1.20	.095	12.6	.0637	.0644	.0647
1.80	.140	12.9	.0627	.0651	.0628
2.40	.190	12.6	.0637	.0641	.0653
.60	.080	7.50	.0978	.0962	.0970
1.20	.160	7.50	.0978	.0968	.0975
1.80	.240	7.50	.0978	.0944	.0996
2.40	.320	7.50	.0978	.0898	.100
.40	.075	5.33	.127	.122	.127
2.00	.375	5.33	.127	.125	.121
.40	.100	4.00	.158	.153	.152
2.05	.500	4.10	.155	.148	.153
.50	.150	3.33	.179	.162	.172
1.65	.500	3.30	.180	.171	.176
1.00	.370	2.70	.206	.195	.200
1.00	.445	2.25	.232	.219	.223
.50	.259	1.93	.256	.225	.245
.80	.415	1.93	.256	------	.242
.30	.375	.800	.410	.382	.386
.20	.325	.615	.460	------	.436
.10	.250	.400	.543	.506	.507
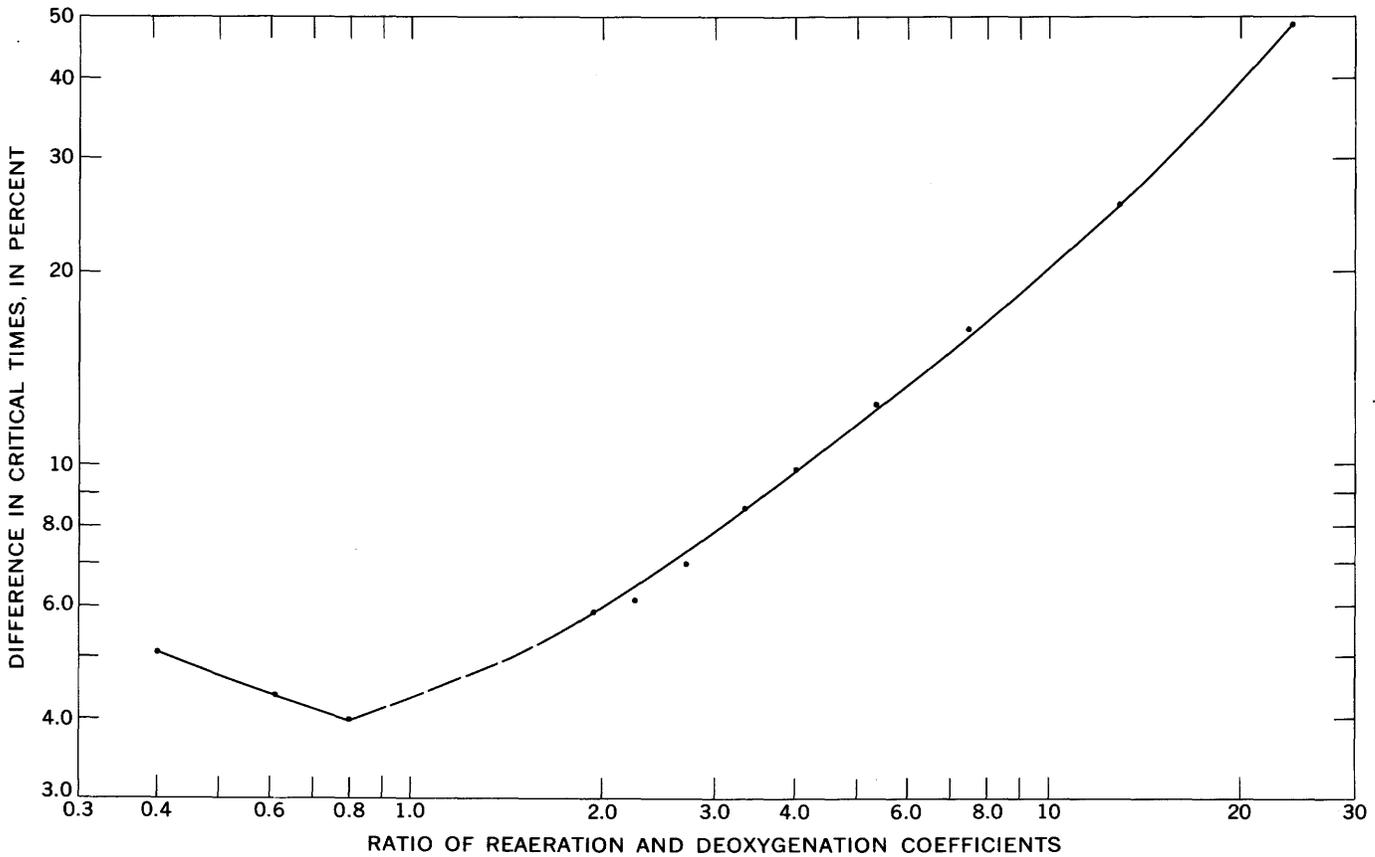
FIGURE 13.—Percentage difference in deterministic and stochastic critical times of travel as a function of the ratio of the re-aeration and deoxygenation coefficients.

TABLE 7.—*Mean oxygen deficit for stochastic model and Taylor series approximation, with T_{c_s}*

K_2 (days⁻¹)	K_1 (days⁻¹)	Ratio K_2/K_1	Mean oxygen deficit, in mg/l	
			Sto-chastic	Taylor series
0.60	0.025	24.0	0.0380	0.0387
1.20	.050	24.0	.0380	.0387
1.80	.075	24.0	.0380	.0387
2.40	.100	24.0	.0380	.0387
.60	.045	13.3	.0625	.0628
1.20	.095	12.6	.0647	.0655
1.80	.140	12.9	.0628	.0646
2.40	.190	12.6	.0653	.0655
.60	.080	7.50	.0970	.0983
1.20	.160	7.50	.0975	.0983
1.80	.240	7.50	.0996	.0983
2.40	.320	7.50	.100	.0983
.40	.075	5.33	.127	.126
2.00	.375	5.33	.121	.126
0.40	.100	4.00	.152	.155
2.05	.500	4.10	.153	.152
.50	.150	3.33	.172	.175
1.65	.500	3.30	.176	.176
1.00	.370	2.70	.200	.200
1.00	.445	2.25	.223	.224
.50	.259	1.93	.245	.246
.80	.415	1.93	.242	.246
.30	.375	.800	.386	.389
.20	.325	.615	.436	.436
.10	.250	.400	.507	.515

T_{c_s} from the stochastic model and the Taylor series approximation are essentially the same over the entire range of ratios considered.

VARIANCE OF THE OXYGEN DEFICIT

The variance of the oxygen deficit at the critical time of travel was computed from the stochastic model and from the Taylor series approximation of the model for both T_{c_D} and T_{c_s} and the results are presented in table 8. The effect of using 800 repeti-

TABLE 8.—*Variance of the oxygen deficit from the stochastic model and the Taylor series approximation*

K_2 (days⁻¹)	K_1 (days⁻¹)	Ratio K_2/K_1	Variance, in (mg/l)²			
			Stochastic		Taylor series	
			T_{c_D}	T_{c_s}	T_{c_D}	T_{c_s}
0.60	0.025	24.0	0.00024	0.00055	0.00020	0.00026
1.20	.050	24.0	.00023	.00055	.00020	.00026
1.80	.075	24.0	.00039	.00055	.00020	.00026
2.40	.100	24.0	.00034	.00055	.00020	.00026
.60	.045	13.3	.00074	.00100	.00047	.00055
1.20	.095	12.6	.00073	.00093	.00051	.00059
1.80	.140	12.9	.00072	.00094	.00049	.00058
2.40	.190	12.6	.00097	.00105	.00051	.00059
.60	.080	7.50	.00126	.00160	.00100	.00112
1.20	.160	7.50	.00170	.00161	.00101	.00112
1.80	.240	7.50	.00115	.00165	.00100	.00112
2.40	.320	7.50	.00118	.00182	.00100	.00112
.40	.075	5.33	.00205	.00211	.00150	.00164
2.00	.375	5.33	.00206	.00216	.00150	.00164
.40	.100	4.00	.00274	.00265	.00205	.00220
2.05	.500	4.10	.00271	.00273	.00200	.00214
.50	.150	3.33	.00292	.00356	.00245	.00261
1.65	.500	3.30	.00338	.00322	.00248	.00264
1.00	.370	2.70	.00408	.00411	.00298	.00314
1.00	.445	2.25	.00482	.00424	.00349	.00365
.50	.259	1.93	.00469	.00480	.00393	.00411
.80	.415	1.93	------	.00462	.00393	.00410
.30	.375	.800	.00940	.00959	.00696	.00719
.20	.325	.615	------	.01135	.00799	.00828
.10	.250	.400	.0147	.01322	.00992	.0104
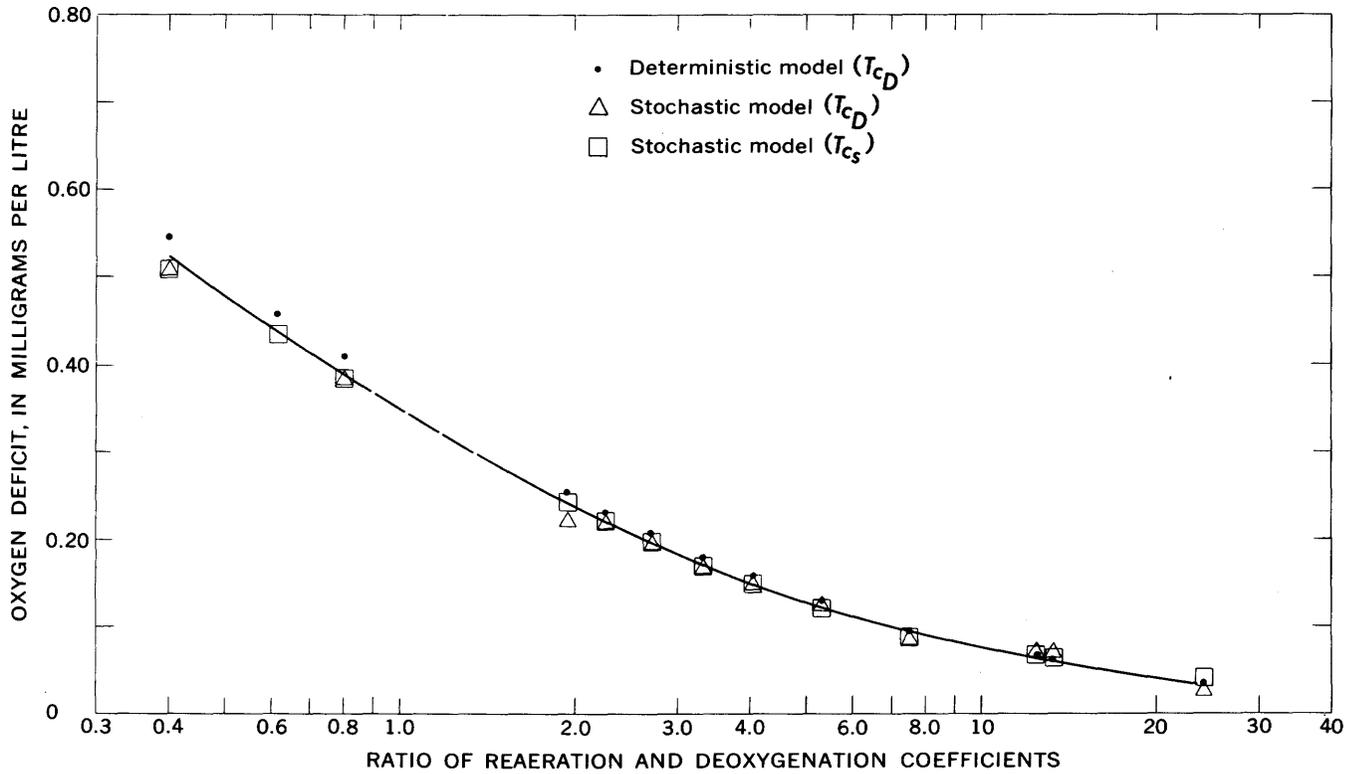
FIGURE 14.—Oxygen deficit as a function of the ratio of the reaeration and deoxygenation coefficients; deterministic and stochastic models.
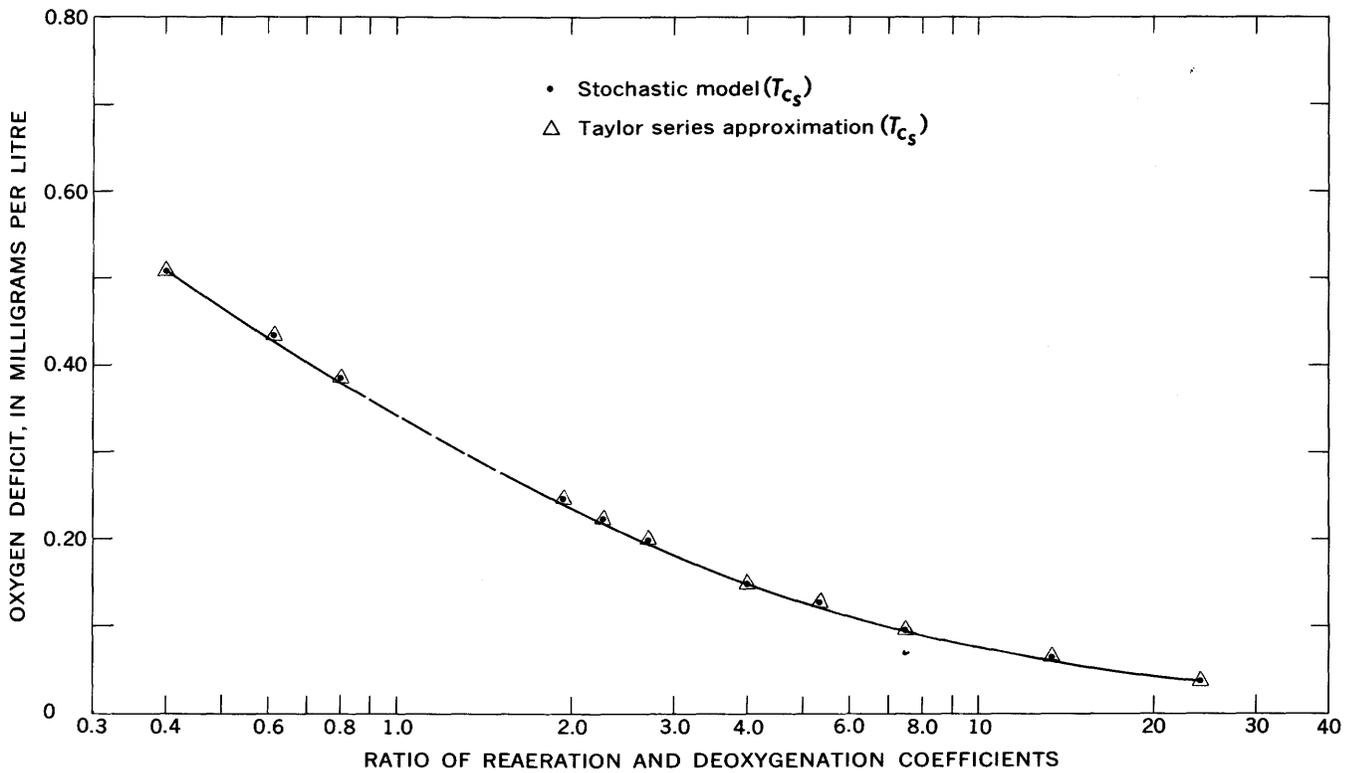
FIGURE 15.—Oxygen deficit as a function of the ratio of the reaeration and deoxygenation coefficients; stochastic model and Taylor series approximation.

tions in the computations for the stochastic model with T_{c_g} as compared with 200 repetitions for T_{c_D} is again noticeable, particularly for the 24.0 and 7.50 ratios. The variability of the variances within a group is much larger for the calculations with 200 repetitions. It is also interesting that the largest variances for the 7.50 ratio occurred for different combinations of K_2 and K_1. This probably occurs because of the random nature of the simulation process.

The variances computed from the stochastic model and the Taylor series approximation with the deterministic critical time of travel are plotted in figure 16 as a function of the ratio of the reaeration and deoxygenation coefficients. The average variance was plotted for those ratios where several variances were computed. The variance from the stochastic model is larger than the variance from the Taylor series approximation over the entire range of ratios considered. This difference undoubtedly occurred because higher order terms in the Taylor series expansion were neglected. Figure 16 also shows the random nature of the simulation process in that the points for the stochastic model tend to scatter about the trend line whereas the points for the Taylor series approximation all tend to lie on the trend line.

The variances computed with the stochastic criti-
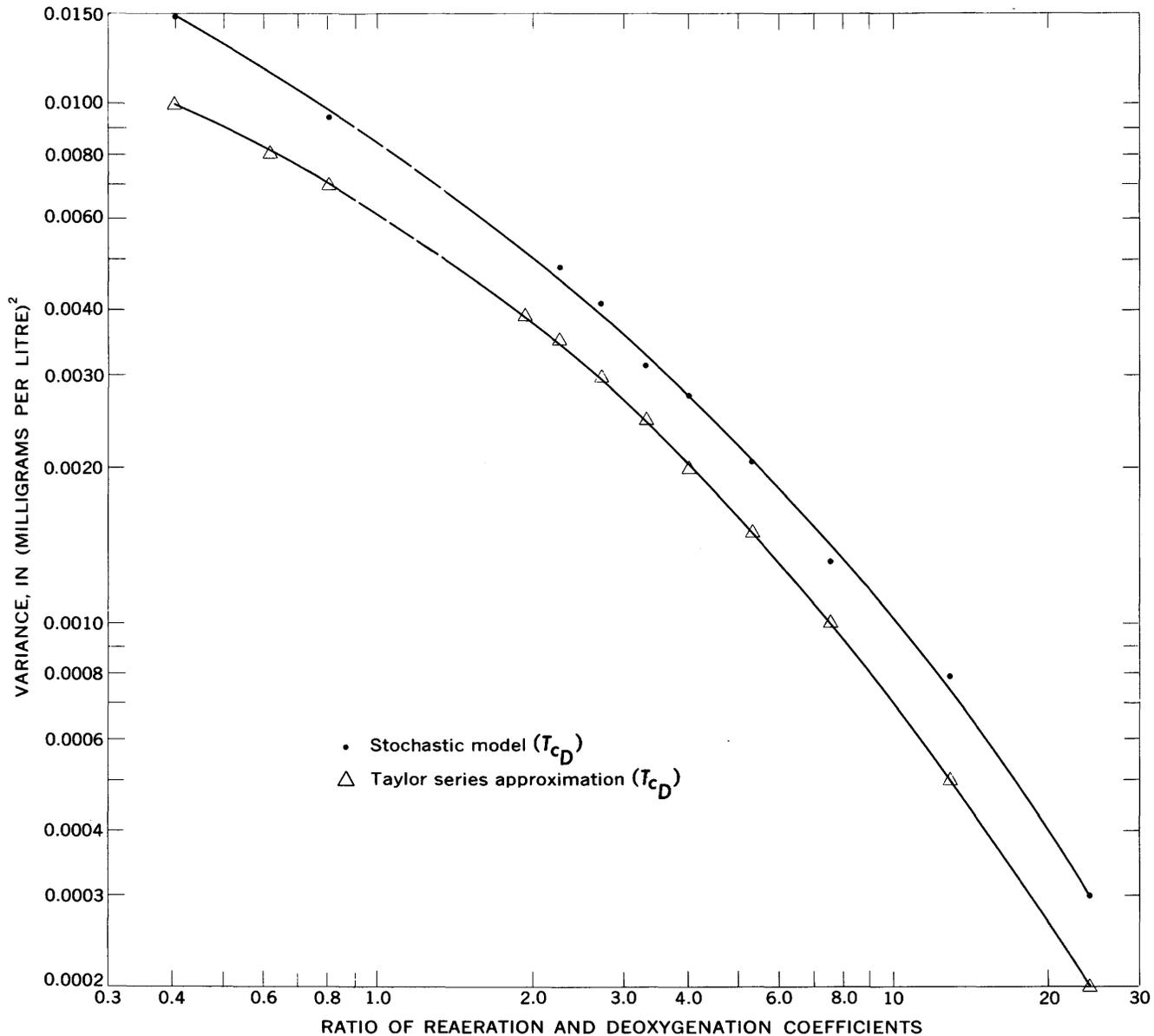
FIGURE 16.—Variance of the oxygen deficit as a function of the ratio of the reaeration and deoxygenation coefficients; stochastic model and Taylor series approximation with the deterministic critical time of travel.

cal time of travel are plotted in figure 17 as a function of the ratio of the reaeration and deoxygenation coefficients. The dependence of the variance on the ratio is essentially identical to the behavior for the deterministic critical time of travel shown in figure 16, with the exception that the curve for the Taylor series approximation falls away more rapidly than the stochastic model curve for large ratio values.

The effect on the variance of the type of critical travel time used in the computations is shown in figures 18 and 19. Figure 18 shows the variation

with the ratio of the reaeration and deoxygenation coefficients of the variance computed from the stochastic model; figure 19 shows the variation with the ratio of the coefficients of the variance computed from the Taylor series approximation. Figure 18 shows that the variances are essentially identical for the two critical times up to a ratio of about 4 at which point the variances computed for T_{c_D} drop off more rapidly than the variances computed for T_{c_s}. Figure 19 shows essentially the same behavior for the Taylor series approximation except that there is less scatter of the points around the curves. Also the
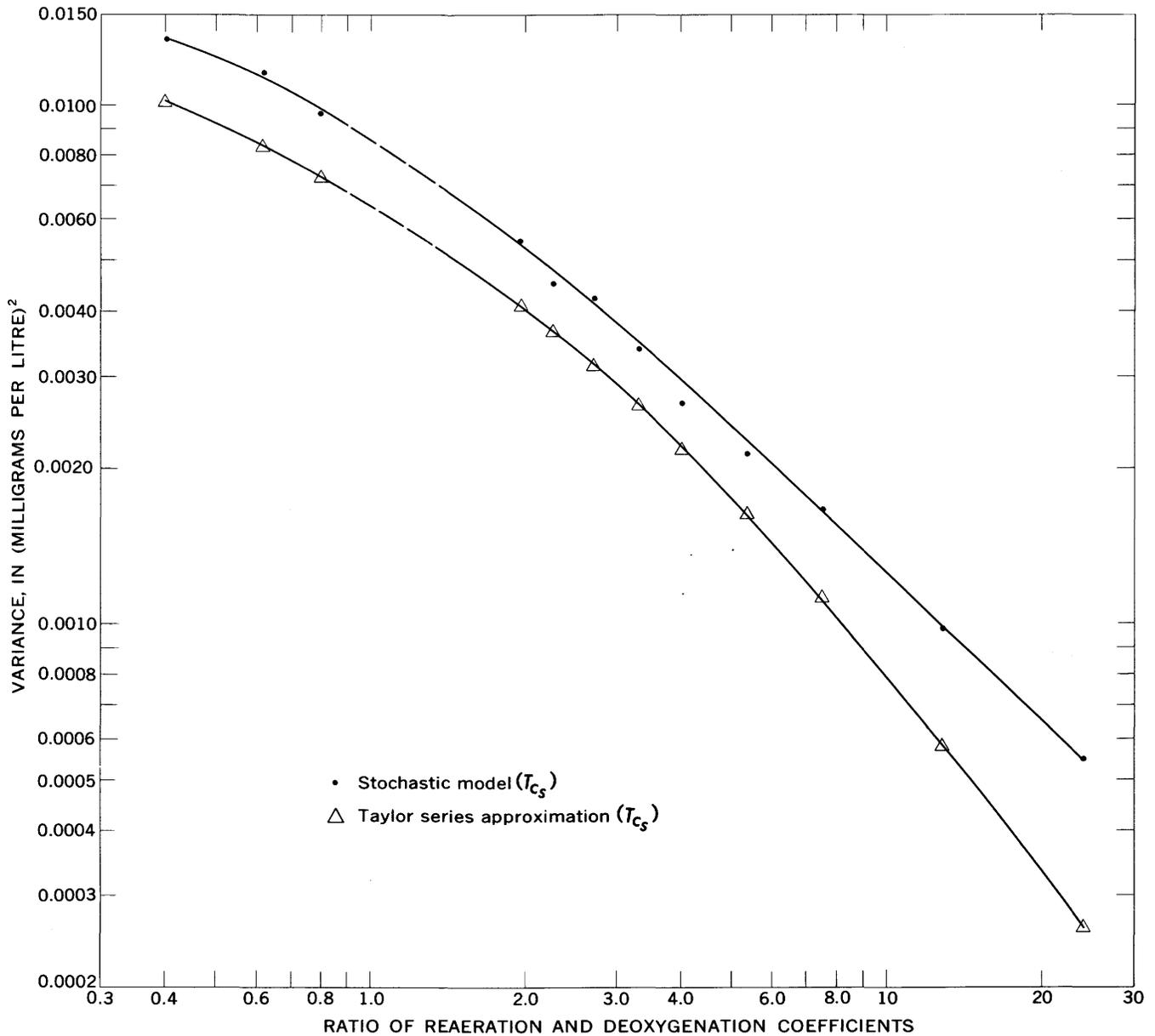
FIGURE 17.—Variance of the oxygen deficit as a function of the ratio of the reaeration and deoxygenation coefficients; stochastic model and Taylor series approximation with the stochastic critical time of travel.
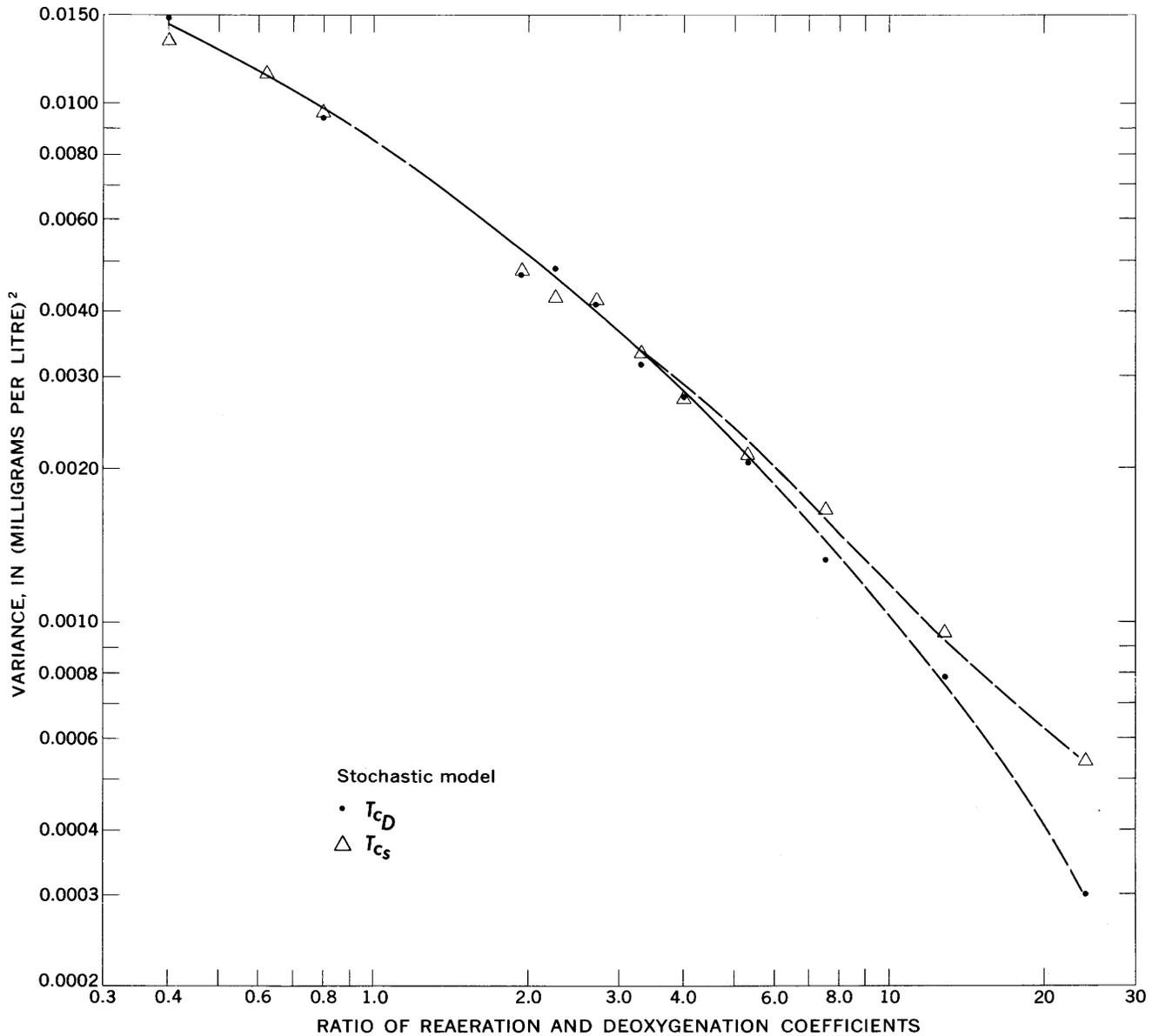
FIGURE 18.—Variance of the oxygen deficit as a function of the ratio of the reaeration and deoxygenation coefficients; stochastic model with the deterministic and stochastic critical times of travel.

T_{c_s} results are always larger than the T_{c_D} results and the difference increases as the ratio of the reaeration and deoxygenation coefficients increases, in agreement with the dependence of the difference of the critical times of travel on the ratio (recall fig. 13). Consideration of figures 13, 18, and 19 suggests that the variance for a specific set of conditions tends to increase as the estimate of the critical time of travel increases and the amount of the increase increases as the ratio of the coefficients increases. This observation is qualitatively in agreement with the results of the analysis of the Sacramento River data where it was concluded that the variance was a maxi-

mum for some time larger than the critical time of travel.

The Taylor series expression for the variance (recall eq 83) may be used to determine the relative contributions of the various terms to the total variance. The terms H_6, H_7, H_8, H_9, H_{10}, and H_{11} were computed for each of the sets of reaeration and deoxygenation coefficients (see table 5) with T_{c_s} and the other coefficients given previously. The results are presented in table 9. Because the results were the same for each ratio of K_2 and K_1, only one result for each ratio is given in table 9. Recall that H_6, H_7, and H_8 represent the effects on the variance of the varia-
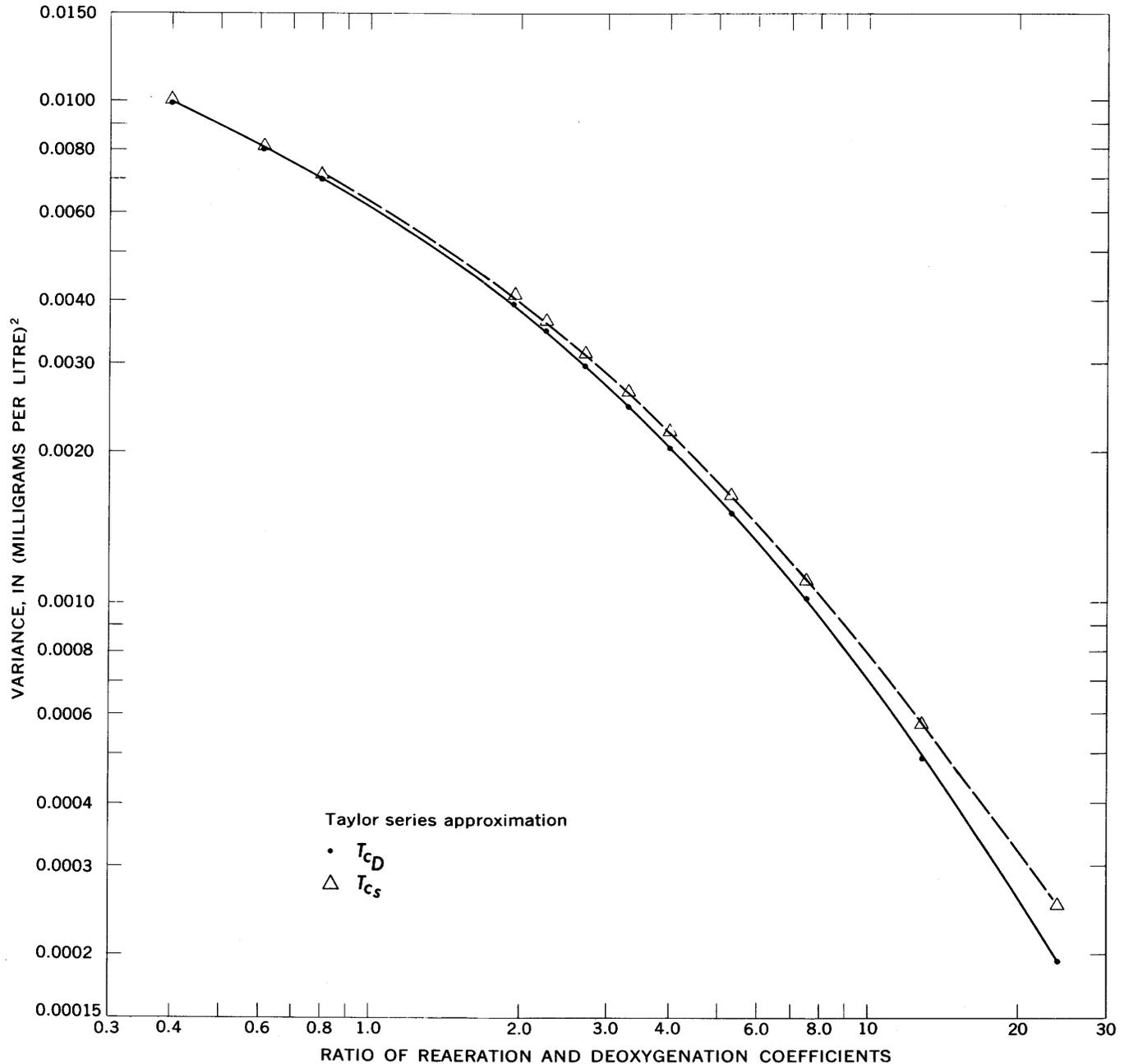
FIGURE 19.—Variance of the oxygen deficit as a function of the ratio of the reaeration and deoxygenation coefficients; Taylor series approximation with the deterministic and stochastic critical times of travel.

tion of the deoxygenation coefficient along the reach, the variation of the reaeration coefficient, and the correlation between these coefficients, respectively. Also the terms H_9, H_{10}, and H_{11} represent the effects on the variance of the variation of the initial BOD, the variation of the initial value of the deoxygenation coefficient, and the correlation between these coefficients, respectively.

The H terms are plotted in figure 20 as a function of the ratio of the reaeration and deoxygenation co-

efficients. The terms H_8 and H_{11} were negative; hence, the negative of these two were plotted in figure 20. The variation of the total variance, that is, the sum of H_6, H_7, H_8, H_9, H_{10}, and H_{11}, with the ratio of the deoxygenation and reaeration coefficients was presented previously in figure 19.

Figure 20 and table 9 show that H_7 changes the largest amount over the range of ratios considered; that is, H_7 is largest for the small ratios and smallest for the large ratios. Recall that H_7 gives the effect on

TABLE 9.—*Distribution of the variance among the terms making up the variance estimated by the Taylor series approximation of the stochastic model*

Ratio (K_2/K_1)	H_6	H_7	H_8	H_9	H_{10}	H_{11}	Variance $(mg/l)^2$
			H terms, in (milligrams per litre)^2				
24.0	0.0000744	0.0000493	−0.0000812	0.0000990	0.000280	−0.000166	0.000256
12.9	.000212	.000153	−.000241	.000208	.000599	−.000352	.000579
7.50	.000464	.000376	−.000560	.000383	.00111	−.000650	.00112
5.33	.000718	.000642	−.000910	.000544	.00157	−.000921	.00164
4.00	.000997	.000985	−.00133	.000710	.00203	−.00120	.00219
3.33	.00120	.00128	−.00166	.000827	.00236	−.00140	.00261
2.70	.00147	.00170	−.00211	.000971	.00275	−.00163	.00315
2.25	.00171	.00215	−.00257	.00110	.00310	−.00184	.00365
1.93	.00191	.00260	−.00299	.00120	.00340	−.00202	.00410
.800	.00282	.00670	−.00582	.00161	.00460	−.00272	.00719
.615	.00288	.00844	−.00660	.00162	.00470	−.00275	.00829
.400	.00269	.0118	−.00753	.00148	.00454	−.00259	.0104
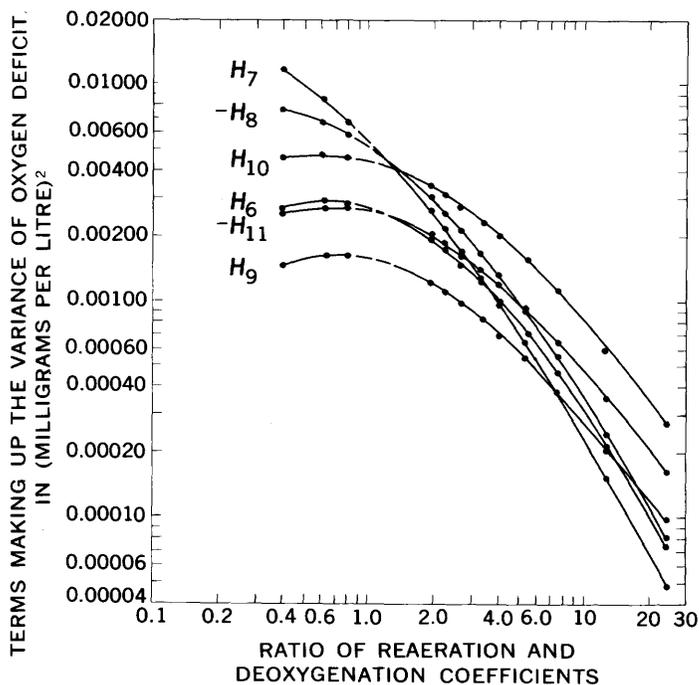
FIGURE 20.—Distribution of the variance among the terms making up the variance as a function of the ratio of the reaeration and deoxygenation coefficients.

the variance of random variations in the reaeration coefficient. Thus, in a deep, slow-flowing stream with a small reaeration coefficient, random variations in K_2 make the largest contribution to the variance of the oxygen deficit. On the other hand, for a shallow, rapidly flowing stream with a large reaeration coefficient and the same deoxygenation coefficient, random variations in K_2 make the smallest contribution to the variance of the oxygen deficit. This difference in contribution may also be seen by computing the percentage of the total variance that each of the six terms contributes. For $K_2/K_1=24.0$, the percentages are 29.1, 19.3, −31.7, 38.7, 109.4, and −64.8 for H_6, H_7, H_8, H_9, H_{10}, and H_{11}, respectively; for $K_2/K_1=$

0.400, the percentages are 25.9, 113.6, −72.5, 14.2, 43.7, and −24.9, respectively. Thus, the contribution of H_7 increases from 19.3 percent to 113.6 percent as K_2/K_1 decreases from 24.0 to 0.400.

Figure 20 and table 9 show that H_6, H_9, H_{10}, and H_{11} all have maximum contributions to the variance at a ratio of 0.615 for the specific values of ratios considered. However, for ratios between 0.4 and about 2, these four factors do not change appreciably. Also, as the ratio decreases, the dependence of the total variance on the ratio decreases (see fig. 19). Recall that H_6, H_9, H_{10}, and H_{11} give the effect on the variance of random variations in k, random variations in L_0, random variations in K_1, and correlation between L_0 and k, respectively.

Figure 20 shows that H_9 contributes the least to the total variance over most of the ratio range considered. On the other hand, H_{10} contributes the most to the total variance for ratios larger than 1.0. Thus, of the two factors describing the waste at the upstream end of the reach, the variations in the deoxygenation coefficient, K_1, contribute the greater amount to the total variance.

Figure 20 shows that the six terms are most nearly equal in the middle part of the range of ratios considered, that is, between ratios of about 2 and 6. For larger and smaller ratios, the values of the different terms diverge. However, for all ratios considered, all the terms contribute significantly to the total variance with the minimum contribution being the 19.3 percent of H_7 for a ratio of 24.0.

In considering the results presented in table 9 and figure 20, it should be remembered that these results are specifically only for the values of the coefficients of variation and the correlation coefficients used in the computations. For other values of these coefficients, the distribution of the total variance among the six terms would undoubtedly be different.

Further work is needed to determine the sensitivity of the equation to these coefficients.

The variances computed from the stochastic model with T_{c_g} were used to prepare a graph showing the variance as a function of the reaeration coefficient and the critical time of travel. The result is presented in figure 21. Because the mean value of the BOD at the upstream end of the reach was assumed to be 1.0 mg/l, figure 21 is similar to the graphs presented previously for the Ψ-parameter (recall figs. 8, 9, and 10). The one important difference is that figure 21 is specifically for the critical time of travel whereas figures 8, 9, and 10 are for general times of 1, 2, and 3 days, respectively. Note the difference in the figures, however. The additional constraint of a critical time results in the deoxygenation coefficient varying linearly with the reaeration coefficient for a specific value of the variance; on the other hand, the

dependence shown in figures 8, 9, and 10 in general is not linear but depends on the values of the variables under consideration.

EVALUATION OF THE MODEL

The problem of predicting the response of the dissolved-oxygen concentration of a stream to the addition of biodegradable wastes has been of much interest ever since the pioneering work of Streeter and Phelps (1925). Much of this interest has been directed toward predicting the dissolved-oxygen concentration or the oxygen deficit at downstream points as a function of the hydraulic properties of the stream and the deoxygenation coefficient and BOD of the waste. With the development of computers, however, the use of mathematical techniques that previously would not have been practical has be-
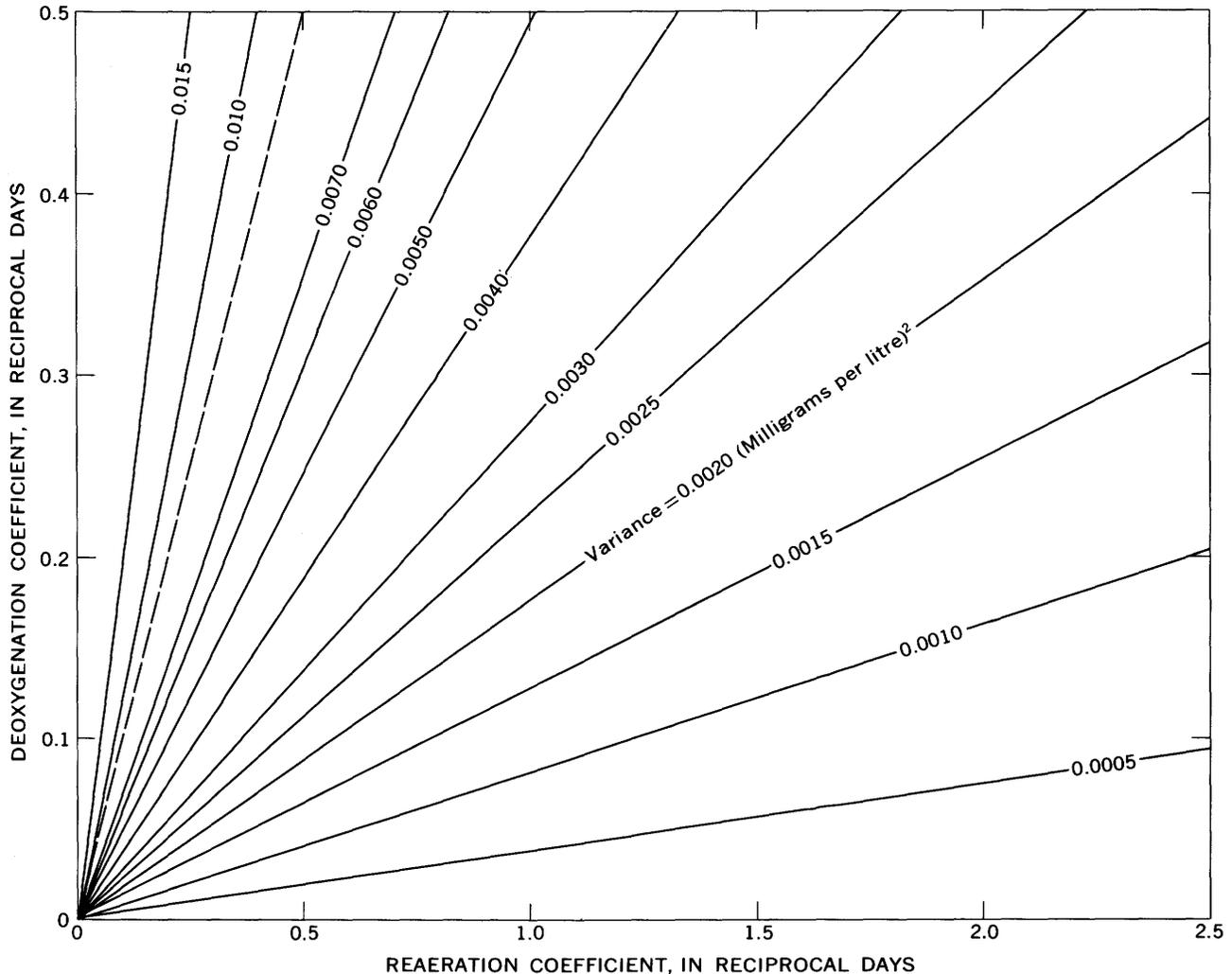

FIGURE 21.—Variance of the oxygen deficit as a function of the deoxygenation and reaeration coefficients; stochastic model with the stochastic critical time of travel.

come common. With these techniques, it has become possible to predict the variances of the oxygen deficit and the biochemical-oxygen-demand, in addition to the mean values, at downstream points. This ability adds a new dimension to water-quality standards in that it is possible to estimate the confidence limits for the mean oxygen deficit and hence the probability of failing to meet the standard by any specified amount.

The stochastic model of this study uses Monte Carlo simulation in which a complex system with random components is operated by random numbers chosen so that they simulate the physical behavior of the components. The distribution of the dissolved-oxygen deficit and consequently the mean and variance of the distribution are estimated by repeating the process a large number of times. The model has the capability of considering the reaeration and deoxygenation coefficients and the BOD at the upstream end of the reach as random variables; in addition, correlation between pairs of these variables can be considered in the model. The model does not require actual measurements of the variance for prediction of other variances, as does the model of Thayer and Krutchkoff (1966). However, considerable information is needed. For each coefficient considered to be a random variable, the variance and the mean value, or equivalently, the coefficient of variation, must be known. For each pair of variables for which correlation is assumed, the correlation coefficient must be known.

Information of this type for the most part is not available; it was necessary to assume values for the coefficients of variation of the reaeration and deoxygenation coefficients and for the correlation coefficient between these variables for the test of the model with the Sacramento River data. The values assumed, however, were derived from data on other rivers; the coefficient of variation of the deoxygenation coefficient was determined from the Ohio River data and the coefficient of variation of the reaeration coefficient was estimated from data for Tennessee Valley streams. The value used for the correlation coefficient was arbitrarily assumed. Consideration of the physical processes involved in the reaeration and deoxygenation process suggests that a positive correlation would be expected but gives no indication of the magnitude of the coefficient. It is well known that the deoxygenation and reaeration coefficients determined for one stream or one particular reach of a stream should be used for other streams or other reaches only with caution because of the sensitivity of these coefficients to changes in conditions. But

note that in this instance it is the variability of the coefficients with respect to the mean values that is transferred between streams rather than the mean values. Thus, the assumption is that variations in the coefficients of variation for different streams will be considerably less than variations in the mean values. This assumption is supported by the data for the Tennessee Valley rivers presented in table 11. Note that the reaeration coefficients vary over about a 22-fold range, whereas the coefficients of variation vary only over about a 6-fold range.

The tests of the stochastic model with the hypothetical examples, the Sacramento River data, and the computations of the variances at the critical time of travel for the most part gave reasonable values for the variance of the oxygen deficit. This result suggests that the values assumed for the coefficients of variation and the correlation coefficients were reasonable.

As discussed in the previous section, the stochastic model also has the capability of handling the division of random variations in the deoxygenation coefficient between those of the coefficient at the upstream end of the reach and those of the coefficient along the reach. The variances of these two coefficients may be significantly different, as the analysis of the Ohio River data showed; furthermore this difference may significantly affect the calculation of the variance of the deficit distribution. The stochastic model in general can consider any coefficient as a normally distributed random variable if the mean and variance are known; additionally, correlation between any two variables can be treated if the correlation coefficient is known. The problem is the determination of the coefficients of variation and the correlation coefficients for the variables of interest. Further work in this area is needed.

Comprehensive studies of water quality usually require segmenting the stream when conditions change appreciably with distance downstream. In segmenting, the conditions at the downstream end of each segment serve as the input conditions to the next segment. The stochastic model can be used in this situation, however, a small modification is needed. Recall that in the development of the basic equation of the stochastic model, equation 41, it was assumed that the variance of the initial dissolved-oxygen deficit, $C_s - F(\zeta)$, was small. With segmenting, this assumption is probably valid only for the first segment because in reality it is the variance of this term, $C_s - F(\zeta)$, which the stochastic model estimates for the downstream end of each reach. Hence, the input deficit to the next reach (or segment) is a

random variable with some variance that cannot be neglected. This modification requires that $C_s - F(\zeta)$ be removed from the B factor (recall equation 40) and added to the right hand side of equation 41 as the term $[C_s - F(\zeta)] \exp\left(-\sum_{i=1}^{n} K_{2_i} \Delta T\right)$.

Another approach that could be used in applying the stochastic model to a segmented stream is to use equations 42, 43, 44, and 45. These difference equations segment the stream at ΔT intervals and therefore could be used recurrently in the computation procedure, as described previously. Similarly, the Taylor series approximations of the stochastic model could be applied readily to a segmented stream system.

The basic equation of the stochastic model can be expanded in a Taylor series to give an approximate equation for the variance. This equation involves sums and differences of exponentials and can be solved without a computer; the Monte Carlo computations, on the other hand, can be done efficiently only on a computer. It is possible to prepare charts for estimating the variance (recall figs. 8, 9, 10, and 21); these, however, are for specific conditions. Thus, in the event that a preliminary estimate of the variance is desired for conditions not covered by the available charts, the approximate equation may be used. The approximate equation may also be used to indicate how the variance is divided among the various terms describing the total variance, as described previously.

The stochastic model is a step toward the fulfillment of the need for a procedure for predicting the variances, in addition to the mean values, of the biochemical-oxygen-demand and the dissolved-oxygen deficit at points in a stream downstream from a waste source. Further work is needed, however, on both the experimental and theoretical aspects of the problem. Experimental aspects needing further study are the:

1. Coefficients of variation of the various coefficients and the dependence, if any, on the type of stream.
2. Correlation coefficients between the reaeration and deoxygenation coefficients and possible dependence on the time of travel.
3. Possible correlation between BOD and the oxygen deficit and between BOD and the deoxygenation coefficient.
4. Possible variations in K_3, the rate constant for removal of BOD by sedimentation to the stream bottom.

An experimental problem fundamental to all aspects of dissolved-oxygen balance studies is the time interval required for the BOD determinations; for reaches where the input BOD varies with time, the standard procedure is not satisfactory. Work is needed in the area of correlating BOD with some quantity that can be measured rapidly, for example, total organic carbon. Some work of this type has been done, however, considerable additional effort is needed.

Theoretical aspects needing further study are:

1. Possible analytical methods for determining the probability distribution of the oxygen deficit.
2. Methods for considering lateral variations in the rate coefficients and the concentrations of BOD and dissolved oxygen.
3. Sensitivity analysis of the stochastic model to determine which of the coefficients of variation and correlation coefficients are most important in the determination of the variance.
4. Accelerated Monte Carlo techniques to reduce the computer time required in the determination of the oxygen-deficit distributions.

SUMMARY

A random walk model was developed for predicting the distribution of the biochemical-oxygen-demand for points downstream from a waste source for a stream system in which the deoxygenation coefficient is a normally distributed random variable. The model has the capability of considering both the mean and variance of the deoxygenation coefficient as functions of the time of travel through the reach.

A stochastic model using a Monte Carlo technique for simulating a random walk process was developed for estimating the distribution of the dissolved-oxygen deficit for points downstream from a waste source for a stream system in which both the deoxygenation and reaeration coefficients are normally distributed random variables. The model has the capability of considering the mean and variance of the two coefficients as functions of the time of travel through the reach. The model has the additional capabilities of considering the biochemical-oxygen-demand at the upstream end of the reach as a normally distributed random variable and of dividing random variations of the deoxygenation coefficient into variations of the deoxygenation coefficient at the upstream end of the reach and of the deoxygenation coefficient along the reach.

Equations for approximating the mean oxygen deficit and the variance of the oxygen deficit were

developed by expanding the basic equation of the stochastic model in a Taylor series.

Principal conclusions for the range of conditions considered are:

1. Random variations in the type and concentration of wastes discharged into a stream and random variations in the hydraulic conditions within a stream provide ample basis for considering the deoxygenation and reaeration coefficients and the biochemical-oxygen-demand at the up-stream end of the reach as random variables.

2. The distribution function for the biochemical-oxygen-demand derived from the random walk model shows that the BOD is distributed according to the lognormal distribution.

3. The random walk model simulated by the Monte Carlo technique efficiently estimates the variance of the dissolved-oxygen deficits for points in a stream downstream from a waste source. The error in the simulation process was found to be inversely proportional to the number of steps used in the random walk process and inversely proportional to the square root of the number of times the process is repeated.

4. The predicted frequency distributions of the oxygen deficit became flatter and skewed to the right as time of travel increased. This type of skewness is favorable in the determination of probabilistic water-quality standards because the percentile limits of the oxygen deficit will be less sensitive to errors in the values estimated for the coefficients of variation of the deoxygenation and reaeration coefficients and the correlation coefficient between these two coefficients.

5. The critical time of travel estimated from the stochastic model was always larger than the critical time of travel computed from the deterministic model; the difference decreased as the ratio of the reaeration and deoxygenation coefficients decreased.

6. The critical time of travel for both the stochastic and deterministic models was extremely sensitive to the reaeration coefficient when the deoxygenation coefficient was small and extremely sensitive to the deoxygenation coefficient when the reaeration coefficient was small, for the range of coefficients considered.

7. Differences among the mean oxygen deficits computed from the deterministic model, the stochastic model, and the Taylor series approximation of the stochastic model with both the

deterministic and stochastic critical time of travel were essentially negligible.

8. The variance of the oxygen deficit seems to be maximum for some time of travel larger than the critical time of travel. The Sacramento River data, although containing considerable scatter, tend to demonstrate this effect.

9. The variance estimated from the Taylor series approximation of the stochastic model was comparable to the variance obtained from the stochastic model for small times of travel; as the time of travel increased, the Taylor series approximation underestimated the variance because of the neglecting of higher order terms.

10. The variance at the critical time of travel estimated from the Taylor series approximation was less than the variance of the stochastic model over the entire range of deoxygenation and reaeration coefficients considered. The same behavior was found for both the deterministic and the stochastic critical times of travel.

11. The variance at the critical time of travel showed the greatest dependence on the ratio of the reaeration and deoxygenation coefficients at large values of the ratio and the smallest dependence at small values of the ratio. On the other hand, the variance was largest for the small ratios and smallest for the large ratios.

12. The distribution of the variance among the six terms making up the Taylor series approximation of the stochastic model shows that the term giving the effect of random variations in the reaeration coefficient varies the largest amount over the range of conditions considered. Of the two factors describing the waste at the upstream end of the reach, the deoxygenation coefficient contributes the greater amount to the variance.

13. The stochastic model developed in this study requires estimates of the coefficient of variation of the deoxygenation and reaeration coefficients and of the correlation coefficient between these variables for estimating the variance of the oxygen deficit as a function of the time of travel. In comparison, the model of Thayer and Krutchkoff (1966) requires the measurement of the variance at some time of travel for predicting the variance at other times of travel.

14. There is considerable need for a procedure for predicting the variances, in addition to the mean values, of the biochemical-oxygen-de-

mand and the dissolved-oxygen deficit at points in a stream downstream from a waste source. The stochastic model developed in this study is a step toward the fulfillment of this need.

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SUPPLEMENTAL DATA

TABLE 10.—*Biochemical-oxygen-demand data for the Ohio River (from Kothandaraman, 1968) and results of the data analysis*

Observation No.	T (days)	BOD (mg/l)	K_1 (days^{-1})	L_0 (mg/l)	$\overline{K}_1$ (days^{-1})	var (K_1) (days^{-2})
1	0.99	2.42	0.286	9.8	0.280	0.0013
	1.86	4.19	.315	---	----	-----
	2.95	5.60	.266	---	----	-----
	4.85	7.02	.217	---	----	-----
2	.85	2.01	.317	8.5	.284	.0009
	1.52	3.18	.298	---	----	-----
	2.60	4.39	.239	---	----	-----
	4.51	6.06	.273	---	----	-----
3	1.02	2.49	.281	10.0	.240	.0011
	2.21	4.21	.219	---	----	-----
	3.12	5.24	.215	---	----	-----
	4.91	6.63	.193	---	----	-----
4	.95	2.44	.298	9.9	.277	.0009
	1.96	4.35	.293	---	----	-----
	2.88	5.40	.228	---	----	-----
	4.83	7.12	.247	---	----	-----
5	.68	1.73	.286	9.8	.254	.0013
	1.68	3.55	.256	---	----	-----
	2.59	4.84	.254	---	----	-----
	4.89	6.58	.188	---	----	-----
6	.96	2.35	.267	10.4	.239	.0007
	1.88	3.91	.234	---	----	-----
	2.86	5.18	.222	---	----	-----
	4.81	6.80	.191	---	----	-----
7	0.97	2.62	.317	9.9	.275	.0013
	1.84	4.01	.244	---	----	-----
	2.93	5.54	.276	---	----	-----
	4.85	7.05	.221	---	----	-----
8	.86	1.49	.227	8.4	.216	.0007
	1.53	2.53	.243	---	----	-----
	2.59	3.52	.174	---	----	-----
	4.52	5.18	.215	---	----	-----
9	1.02	2.25	.229	10.8	.201	.0006
	2.21	4.03	.196	---	----	-----
	3.12	5.02	.174	---	----	-----
	4.91	6.49	.164	---	----	-----
10	.96	2.53	.269	11.1	.218	.0014
	1.96	3.96	.183	---	----	-----
	2.88	5.03	.176	---	----	-----
	4.83	6.98	.199	---	----	-----
11	.67	1.48	.268	9.0	.233	.0010
	1.67	3.08	.239	---	----	-----
	2.58	4.09	.206	---	----	-----
	4.60	5.63	.186	---	----	-----
12	1.03	2.04	.259	8.7	.226	.0011
	1.95	3.30	.228	---	----	-----
	2.94	4.13	.169	---	----	-----
	4.89	5.62	.202	---	----	-----
13	.96	2.29	.230	9.7	.241	.0009
	1.83	3.66	.235	---	----	-----
	2.92	4.84	.199	---	----	-----
	4.82	6.52	.223	---	----	-----
14	.88	1.52	.230	8.3	.207	.0005
	1.54	2.35	.198	---	----	-----
	2.62	3.35	.170	---	----	-----
	4.53	4.91	.198	---	----	-----
15	1.02	1.79	.178	10.8	.151	.0005
	2.21	3.07	.129	---	----	-----
	3.12	4.09	.156	---	----	-----
	4.91	5.39	.120	---	----	-----
16	.96	2.35	.292	9.6	.254	.0018
	1.96	4.01	.260	---	----	-----
	2.88	4.84	.175	---	----	-----
	4.83	6.64	.244	---	----	-----
17	.66	1.23	.210	9.5	.186	.0006
	1.66	2.67	.191	---	----	-----
	2.57	3.72	.183	---	----	-----
	4.59	5.15	.141	---	----	-----
18	.98	1.73	.232	8.5	.197	.0008
	1.90	2.81	.189	---	----	-----
	2.89	3.64	.159	---	----	-----
	4.83	4.99	.168	---	----	-----

TABLE 10.—*Biochemical-oxygen-demand data for the Ohio River (from Kothandaraman, 1968) and results of the data analysis*—Continued

Observation No.	T (days)	BOD (mg/l)	K_1 (days^{-1})	L_0 (mg/l)	$\overline{K}_1$ (days^{-1})	var (K_1) (days^{-2})
19	.98	1.76	.237	8.5	.218	.0003
	1.85	2.84	.201	---	----	-----
	2.79	3.96	.235	---	----	-----
	4.84	5.49	.200	---	----	-----
20	.89	1.47	.242	7.6	.211	.0006
	1.55	2.23	.201	---	----	-----
	2.64	3.37	.219	---	----	-----
	4.54	4.56	.174	---	----	-----
21	1.02	1.32	.174	8.1	.164	.0002
	2.21	2.33	.136	---	----	-----
	3.12	3.14	.166	---	----	-----
	4.91	4.30	.149	---	----	-----
22	.96	1.84	.217	9.8	.181	.0007
	1.96	3.01	.159	---	----	-----
	2.88	3.86	.145	---	----	-----
	4.90	5.59	.170	---	----	-----
23	.64	1.23	.224	9.2	.193	.0012
	1.64	2.72	.207	---	----	-----
	2.55	3.45	.131	---	----	-----
	4.57	5.34	.197	---	----	-----
24	1.01	1.26	.141	9.5	.127	.0004
	1.93	2.16	.126	---	----	-----
	2.92	2.81	.094	---	----	-----
	4.86	4.40	.140	---	----	-----
25	1.00	1.55	.204	8.4	.195	.0001
	1.88	2.60	.189	---	----	-----
	2.96	3.73	.201	---	----	-----
	4.86	5.05	.175	---	----	-----
26	.90	1.13	.127	10.5	.105	.0002
	1.56	1.71	.097	---	----	-----
	2.65	2.62	.100	---	----	-----
	4.55	3.88	.092	---	----	-----
27	1.02	1.22	.152	8.5	.135	.0002
	2.21	2.20	.121	---	----	-----
	3.10	2.85	.122	---	----	-----
	4.91	4.13	.142	---	----	-----
28	.96	1.34	.160	9.4	.142	.0003
	1.96	2.37	.137	---	----	-----
	2.88	3.20	.137	---	----	-----
	4.90	4.44	.110	---	----	-----
29	.62	.82	.144	9.6	.120	.0002
	1.62	1.79	.117	---	----	-----
	2.54	2.62	.122	---	----	-----
	4.56	3.92	.102	---	----	-----
30	.96	1.11	.131	9.4	.122	.0001
	1.88	2.06	.132	---	----	-----
	2.86	2.78	.105	---	----	-----
	4.81	4.16	.120	---	----	-----
31	.87	1.61	.316	6.7	.207	.0052
	1.87	2.41	.171	---	----	-----
	2.92	3.07	.159	---	----	-----
	4.75	3.84	.130	---	----	-----
32	.96	1.86	.241	9.0	.185	.0021
	1.56	2.62	.188	---	----	-----
	2.60	3.42	.129	---	----	-----
	4.44	4.64	.134	---	----	-----
33	1.19	2.06	.155	12.2	.128	.0003
	2.54	3.48	.118	---	----	-----
	3.23	4.17	.119	---	----	-----
	5.10	5.62	.106	---	----	-----
34	.83	1.59	.221	9.5	.154	.0022
	1.98	2.77	.140	---	----	-----
	2.87	3.42	.114	---	----	-----
	4.75	4.48	.102	---	----	-----
35	.46	.87	.190	10.4	.157	.0014
	1.60	2.74	.192	---	----	-----
	2.50	3.50	.116	---	----	-----
	4.38	4.86	.117	---	----	-----
36	1.13	1.82	.246	7.5	.196	.0018
	2.02	2.64	.175	---	----	-----
	2.85	3.29	.173	---	----	-----
	4.77	4.20	.129	---	----	-----

TABLE 10.—*Biochemical-oxygen-demand data for the Ohio River (from Kothandaraman, 1968) and results of the data analysis*—Continued

Observation No.	T (days)	BOD (mg/l)	K_1 (days^{-1})	L_0 (mg/l)	$\overline{K}_1$ (days^{-1})	var (K_1) (days^{-2})
37	.87	1.61	.416	5.3	.286	.0095
	1.87	2.34	.220	---	----	-----
	2.92	2.79	.157	---	----	-----
	4.75	3.62	.219	---	----	-----
38	.96	1.68	.264	7.5	.208	.0019
	1.56	2.33	.197	---	----	-----
	2.60	3.03	.140	---	----	-----
	4.44	4.36	.192	---	----	-----
39	1.19	1.70	.118	13.0	.090	.0005
	2.54	2.82	.077	---	----	-----
	3.23	3.40	.085	---	----	-----
	5.10	4.34	.055	---	----	-----
40	.83	1.44	.253	7.6	.174	.0037
	1.69	2.46	.157	---	----	-----
	2.87	3.13	.157	---	----	-----
	4.75	3.76	.081	---	----	-----
41	.46	.82	.221	8.5	.163	.0026
	1.60	2.23	.178	---	----	-----
	2.50	3.02	.150	---	----	-----
	4.38	4.07	.113	---	----	-----
42	1.13	1.79	.196	9.0	.168	.0014
	2.02	2.86	.180	---	----	-----
	2.85	3.63	.161	---	----	-----
	4.77	4.54	.097	---	----	-----
43	.87	1.57	.370	5.7	.274	.0044
	1.87	2.34	.206	---	----	-----
	2.92	3.10	.244	---	----	-----
	4.75	3.93	.210	---	----	-----
44	.96	1.39	.173	9.1	.119	.0012
	1.56	1.85	.103	---	----	-----
	2.60	2.45	.083	---	----	-----
	4.44	3.51	.094	---	----	-----
45	1.19	1.74	.185	8.8	.154	.0007
	2.54	2.98	.143	---	----	-----
	3.23	3.55	.149	---	----	-----
	5.10	4.52	.109	---	----	-----
46	.82	1.34	.320	5.8	.260	.0166
	1.98	2.48	.254	---	----	-----
	2.87	2.66	.063	---	----	-----
	4.75	4.36	.415	---	----	-----
47	.54	.57	.144	7.6	.119	.0005
	1.69	1.54	.129	---	----	-----
	2.58	1.96	.081	---	----	-----
	3.42	2.48	.115	---	----	-----
48	1.13	1.94	.226	8.6	.182	.0015
	2.02	2.89	.173	---	----	-----
	2.85	3.62	.165	---	----	-----
	4.77	4.63	.118	---	----	-----
49	.87	1.56	.315	6.5	.252	.0037
	1.87	2.68	.257	---	----	-----
	2.92	3.50	.230	---	----	-----
	4.75	4.20	.145	---	----	-----
50	.96	1.37	.234	6.8	.165	.0022
	1.56	1.74	.118	---	----	-----
	2.60	2.34	.121	---	----	-----
	4.44	3.36	.141	---	----	-----
51	.82	1.28	.164	10.2	.117	.0009
	1.98	2.40	.116	---	----	-----
	2.87	3.00	.090	---	----	-----
	4.75	4.11	.089	---	----	-----
52	.54	.60	.185	6.3	.130	.0021
	1.69	1.45	.140	---	----	-----
	2.58	1.83	.092	---	----	-----
	3.42	2.15	.088	---	----	-----
53	1.13	1.73	.147	11.3	.107	.0009
	2.02	2.52	.097	---	----	-----
	2.85	3.03	.072	---	----	-----
	4.77	4.13	.106	---	----	-----
54	.87	1.58	.351	6.0	.313	.0336
	1.87	2.52	.239	---	----	-----
	2.29	3.36	.658	---	----	-----
	4.75	4.32	.184	---	----	-----
55	.96	1.28	.201	7.3	.146	.0020
	1.56	1.60	.091	---	----	-----
	2.60	2.16	.099	---	----	-----
	4.44	3.46	.158	---	----	-----
56	1.19	1.40	.130	9.8	.097	.0005
	2.54	2.22	.076	---	----	-----
	3.23	2.66	.087	---	----	-----
	5.10	3.62	.077	---	----	-----
57	.83	1.28	.207	8.1	.165	.0013
	1.98	2.46	.165	---	----	-----
	2.87	3.21	.160	---	----	-----
	4.75	4.08	.104	---	----	-----
58	.54	.59	.199	5.8	.152	.0018
	1.69	1.58	.183	---	----	-----
	2.58	1.90	.089	---	----	-----
	3.42	2.36	.149	---	----	-----
59	1.13	1.44	.164	8.5	.125	.0009
	2.02	2.16	.121	---	----	-----
	2.85	2.66	.099	---	----	-----
	4.77	3.54	.085	---	----	-----
60	.88	2.10	.119	21.1	.093	.0005
	1.88	3.42	.072	---	----	-----
	2.75	4.72	.088	---	----	-----
	4.88	6.75	.062	---	----	-----
61	.90	4.09	.186	26.6	.122	.0001
	1.81	6.19	.108	---	----	-----
	2.40	7.20	.086	---	----	-----
	4.52	10.43	.086	---	----	-----
62	2.04	4.05	.101	21.8	.084	.0003
	3.23	5.43	.068	---	----	-----
	5.00	6.95	.055	---	----	-----
	6.96	9.07	.079	---	----	-----
63	1.04	2.60	.199	13.9	.191	.0030
	1.79	4.61	.261	---	----	-----
	2.98	5.88	.124	---	----	-----
	4.75	7.60	.136	---	----	-----
64	.73	2.97	.244	18.2	.173	.0034
	1.35	4.88	.216	---	----	-----
	2.54	6.72	.125	---	----	-----
	4.31	8.71	.108	---	----	-----
65	1.04	3.42	.137	25.7	.121	.0005
	2.23	6.32	.117	---	----	-----
	3.17	7.74	.081	---	----	-----
	4.98	11.57	.133	---	----	-----
66	.94	1.54	.052	32.4	.048	.0001
	1.88	2.60	.037	---	----	-----
	2.81	3.57	.036	---	----	-----
	4.62	6.50	.059	---	----	-----
67	.88	1.50	.071	24.8	.051	.0002
	1.88	2.65	.051	---	----	-----
	2.75	3.51	.046	---	----	-----
	4.88	5.07	.036	---	----	-----
68	2.04	6.03	.105	25.2	.116	.0004
	3.23	7.72	.078	---	----	-----
	5.00	11.18	.125	---	----	-----
	6.96	13.14	.077	---	----	-----
69	1.01	2.30	.228	11.2	.204	.0006
	1.76	3.59	.209	---	----	-----
	2.95	4.98	.169	---	----	-----
	4.72	6.65	.177	---	----	-----
70	.78	2.35	.180	17.9	.143	.0011
	1.35	3.67	.156	---	----	-----
	2.54	5.82	.138	---	----	-----
	4.31	7.61	.091	---	----	-----
71	1.04	2.76	.094	29.7	.084	.0003
	2.23	5.22	.080	---	----	-----
	3.17	6.42	.053	---	----	-----
	4.98	10.23	.099	---	----	-----
72	.94	1.18	.051	25.4	.053	.0004
	1.88	2.26	.049	---	----	-----
	2.81	3.22	.046	---	----	-----
	4.62	5.44	.095	---	----	-----

TABLE 10.—*Biochemical-oxygen-demand data for the Ohio River (from Kothandaraman, 1968) and results of the data analysis*—Continued

Observation No.	T (days)	BOD (mg/l)	K_1 (days^{-1})	L_0 (mg/l)	$\overline{K}_1$ (days^{-1})	var (K_1) (days^{-2})
73 ---	.59	1.38	.184	13.4	.142	.0008
	1.53	2.76	.130	---	----	-----
	2.47	3.78	.108	---	----	-----
	4.28	5.94	.140	---	----	-----
74 ---	.88	1.39	.086	19.0	.065	.0002
	1.88	2.56	.069	---	----	-----
	2.75	3.31	.054	---	----	-----
	4.88	4.79	.047	---	----	-----
75 ---	2.04	3.55	.110	17.7	.096	.0003
	3.23	5.03	.093	---	----	-----
	5.00	6.35	.062	---	----	-----
	6.96	8.19	.090	---	----	-----
76 ---	.94	2.01	.233	10.2	.226	.0137
	1.69	3.75	.318	---	----	-----
	3.88	5.32	.127	---	----	-----
	4.65	6.74	.467	---	----	-----
77 ---	.73	1.63	.113	20.6	.082	.0003
	1.35	2.65	.089	---	----	-----
	2.54	4.16	.074	---	----	-----
	4.31	5.92	.064	---	----	-----
78 ---	.59	1.06	.089	20.7	.061	.0005
	1.53	2.24	.066	---	----	-----
	2.47	2.70	.027	---	----	-----
	4.28	4.38	.054	---	----	-----
79 ---	.94	2.39	.372	8.1	.314	.0029
	1.69	3.26	.220	---	----	-----
	2.88	4.66	.287	---	----	-----
	4.65	6.06	.295	---	----	-----
80 ---	.73	1.20	.041	40.4	.036	.0000
	1.35	1.86	.027	---	----	-----
	2.54	3.29	.032	---	----	-----
	4.31	5.55	.035	---	----	-----
81 ---	.94	1.34	.187	8.3	.178	.0005
	1.69	2.30	.198	---	----	-----
	2.88	3.32	.157	---	----	-----
	4.65	4.42	.141	---	----	-----
82 ---	.73	1.32	.170	11.3	.159	.0014
	1.35	2.58	.218	---	----	-----
	2.54	3.98	.147	---	----	-----
	4.31	5.33	.115	---	----	-----
83 ---	1.04	2.96	.305	10.9	.295	.0045
	2.23	5.20	.279	---	----	-----
	3.17	6.38	.247	---	----	-----
	4.98	8.80	.424	---	----	-----

TABLE 11.—*Mean, variance, and coefficient of variation of the reaeration coefficient data of Churchill, Elmore, and Buckingham (1962)*

Experiment No.	Number of observations	K_2 (days^{-1})	Var (K_2) (days^{-2})	$C_v(K_2)$
1 ------	19	2.920	2.741	0.56
2 ------	19	1.449	.027	.11
3 ------	25	1.061	.152	.37
4 ------	29	.550	.051	.40
5 ------	29	.842	.139	.43
6 ------	30	1.170	.092	.26
7 ------	26	.315	.018	.43
8 ------	30	3.422	.414	.19
9 ------	16	2.819	.146	.14
10 ------	5	1.574	.023	.10
11 ------	31	.505	.043	.41
12 ------	26	.420	.023	.36
13 ------	27	.300	.015	.41
14 ------	26	.660	.114	.51
15 ------	18	.559	.017	.23
16 ------	7	.670	.085	.43
17 ------	20	1.309	.184	.33
18 ------	8	0.284	.006	.28
19 ------	8	0.261	.016	.49
20 ------	8	1.896	.063	.13
21 ------	8	.870	.051	.26
22 ------	6	.934	.101	.34
23 ------	7	.983	.135	.37
24 ------	7	1.006	.025	.16
25 ------	8	.557	.012	.20
26 ------	8	.903	.043	.23
28 ------	16	5.858	3.289	.31
29 ------	16	1.812	.336	.32
30 ------	19	3.265	.285	.16

Average $C_v(K_2) = 0.307$

TABLE 12.—*Correlation coefficient between the biochemical-oxygen-demand and the dissolved-oxygen concentration (from Moushegian and Krutchkoff, 1969)*

	Correlation coefficient, r(BOD, C)												
K_2	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6
K_1						$T = 1$ day							
0.1	.70	.59	.52	.49	.46	.45	.43	.43	.42	.42	.41	.41	.41
.2	--	.69	.61	.56	.53	.51	.50	.48	.47	.46	.46	.45	.44
.3	.80	.70	.63	.59	.56	.53	.51	.50	.48	.47	.46	.45	.44
.4	.81	--	.64	.60	.56	.53	.51	.50	.48	.47	.46	.44	.43
.5	.81	.71	.64	.59	.56	.53	.51	.49	.47	.46	.44	.43	.42
K_1						$T = 2$ days							
0.1	.67	.56	.51	.48	.46	.45	.44	.43	.42	.41	.40	.39	.38
.2	--	.60	.53	.50	.47	.44	.43	.41	.39	.38	.36	.35	.34
.3	.70	.58	.52	.47	.44	.41	.39	.37	.36	.34	.32	.31	.30
.4	.68	--	.49	.41	.41	.38	.36	.34	.32	.30	.29	.27	.26
.5	.65	.53	.45	.41	.37	.34	.37	.30	.28	.27	.25	.24	.23
K_1						$T = 3$ days							
0.1	.63	.53	.48	.45	.43	.41	.39	.37	.35	.34	.32	.31	.29
.2	--	.52	.46	.41	.38	.36	.33	.31	.29	.28	.26	.25	.24
.3	.60	.47	.41	.36	.33	.30	.28	.26	.24	.23	.22	.24	.20
.4	.54	--	.35	.31	.27	.25	.23	.21	.20	.19	.17	.17	.16
.5	.49	.36	.30	.23	.21	.19	.19	.17	.16	.15	.14	.13	.13
K_1						$T = 4$ days							
0.1	.60	.50	.44	.41	.38	.35	.33	.31	.29	.27	.26	.25	.24
.2	--	.44	.38	.34	.30	.27	.25	.23	.22	.21	.20	.19	.18
.3	.49	.37	.31	.27	.24	.21	.19	.18	.17	.16	.15	.14	.13
.4	.41	--	.24	.21	.18	.16	.15	.13	.12	.12	.11	.10	.10
.5	.35	.24	.19	.16	.14	.12	.11	.10	.09	.09	.08	.08	.07
K_1						$T = 5$ days							
0.1	.56	.46	.40	.36	.32	.30	.27	.25	.24	.23	.21	.20	.20
.2	--	.37	.31	.27	.24	.21	.19	.18	.17	.16	.15	.14	.14
.3	.40	.29	.23	.19	.17	.15	.14	.12	.12	.11	.10	.10	.09
.4	.31	--	.17	.14	.12	.10	.09	.09	.08	.07	.07	.07	.06
.5	.24	.16	.12	.10	.08	.07	.06	.06	.05	.05	.05	.04	.04

TABLE 13.—*Dissolved-oxygen concentration data for the Sacramento River (from Thayer and Krutchkoff, 1966)*

River mile [1]	T (days)	Dissolved oxygen concentration (mg/l)
46.3	0.00	8.3
		8.6
		8.9
45.1	.05	8.2
		8.7
		8.6
43.4	.14	8.0
		8.6
		8.3
		8.4
42.1	.22	8.1
		8.0
		8.4
		8.5
41.1	.29	8.2
		8.0
		8.2
39.8	.39	7.9
		7.9
		8.0
		8.3
38.6	.47	7.7
		8.0
37.2	.58	7.9
		7.9
35.9	.68	7.8
34.4	.80	8.1
33.5	.87	8.0
		7.9
32.5	.95	7.8
31.6	1.02	7.9
		7.7
		7.8
30.1	1.15	7.4
		7.5
		7.7

TABLE 13.—*Dissolved-oxygen concentration data for the Sacramento River (from Thayer and Krutchkoff, 1966)*—Continued

River mile [1]	T (days)	Dissolved oxygen concentration (mg/l)
28.4	1.33	7.4
		7.5
		7.6
27.4	1.52	7.3
		7.0
		7.6
26.8	1.63	7.1
		7.3
		6.8
		7.4
25.5	1.96	7.3
		7.5
		6.8
24.3	2.09	7.2
		7.5
		6.8
		7.7
23.3	2.25	7.6
		7.1
		7.3
22.3	2.43	7.2
		7.3
		7.3
21.1	2.65	7.2
		7.7
		8.0
20.1	2.83	7.5
		7.2
		7.7
		8.2
18.8	3.03	7.6
		7.2
		7.8
17.5	4.2	8.2
		8.1
		8.5
		8.7
15.1	5.6	8.1
		8.4
		8.8

[1] River kilometre=river mile × 1.609.
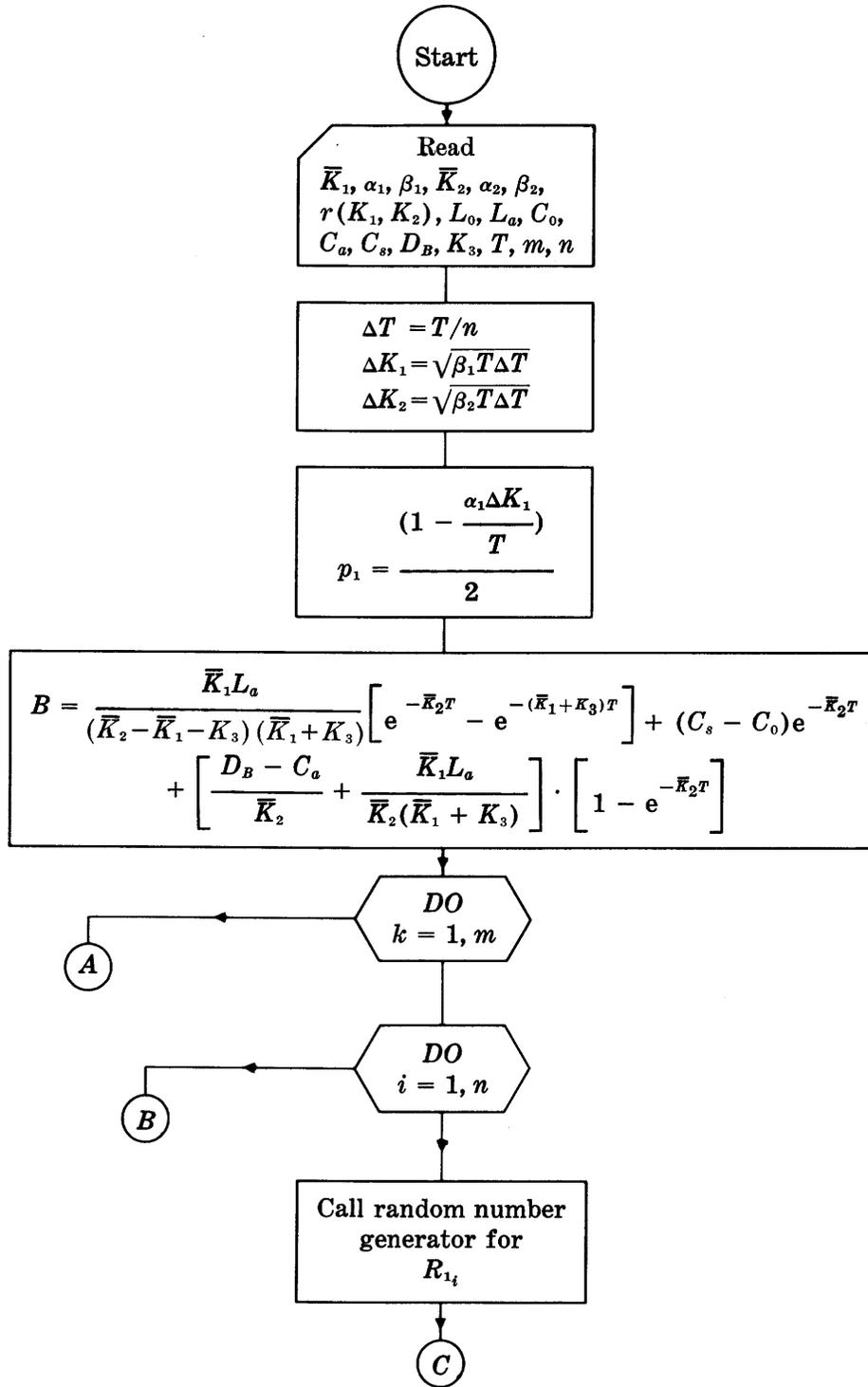
FIGURE 22.—Flow chart for the computer program for the stochastic model for estimating the variance of the oxygen deficit.

C

Call random number
generator for
R_{2_i}

$\geq$ $<$

$R_{1_i} : p_1$

$K_{1_i} \Delta T = \overline{K}_1 \Delta T - \Delta K_1$

$K_{1_i} \Delta T = \overline{K}_1 \Delta T + \Delta K_1$

B

$<$ $\geq$

$R_{2_i} : \dfrac{1 - r(K_1, K_2)}{2}$

$<$ $\geq$

$R_{2_i} : \dfrac{1 - r(K_1, K_2)}{2}$

$K_{2_i} \Delta T = \overline{K}_2 \Delta T + \Delta K_2$

$K_{2_i} \Delta T = \overline{K}_2 \Delta T - \Delta K_2$

$K_{2_i} \Delta T = \overline{K}_2 \Delta T - \Delta K_2$

$K_{2_i} \Delta T = \overline{K}_2 \Delta T + \Delta K_2$

2

$$D_k = B + L_0 \exp\left[- \sum_{i=1}^{n} K_{2i} \Delta T \right] \left\{ - \frac{K_{1_1}}{K_{2_1} - K_{1_1} - K_3} + \sum_{i=1}^{n-1} \frac{K_{1_i}(K_{2_{i+1}} - K_3) - (K_{2_i} - K_3) K_{1_{i+1}}}{(K_{2_i} - K_{1_i} - K_3)(K_{2_{i+1}} - K_{1_{i+1}} - K_3)} \cdot \right.$$

$$\left. \exp\left[\sum_{j=1}^{i} (K_{2_j} - K_{1_j} - K_3) \Delta T \right] + \frac{K_{1_n}}{K_{2_n} - K_{1_n} - K_3} \cdot \exp\left[\sum_{j=1}^{n} (K_{2_j} - K_{1_j} - K_3) \Delta T \right] \right\}$$

D
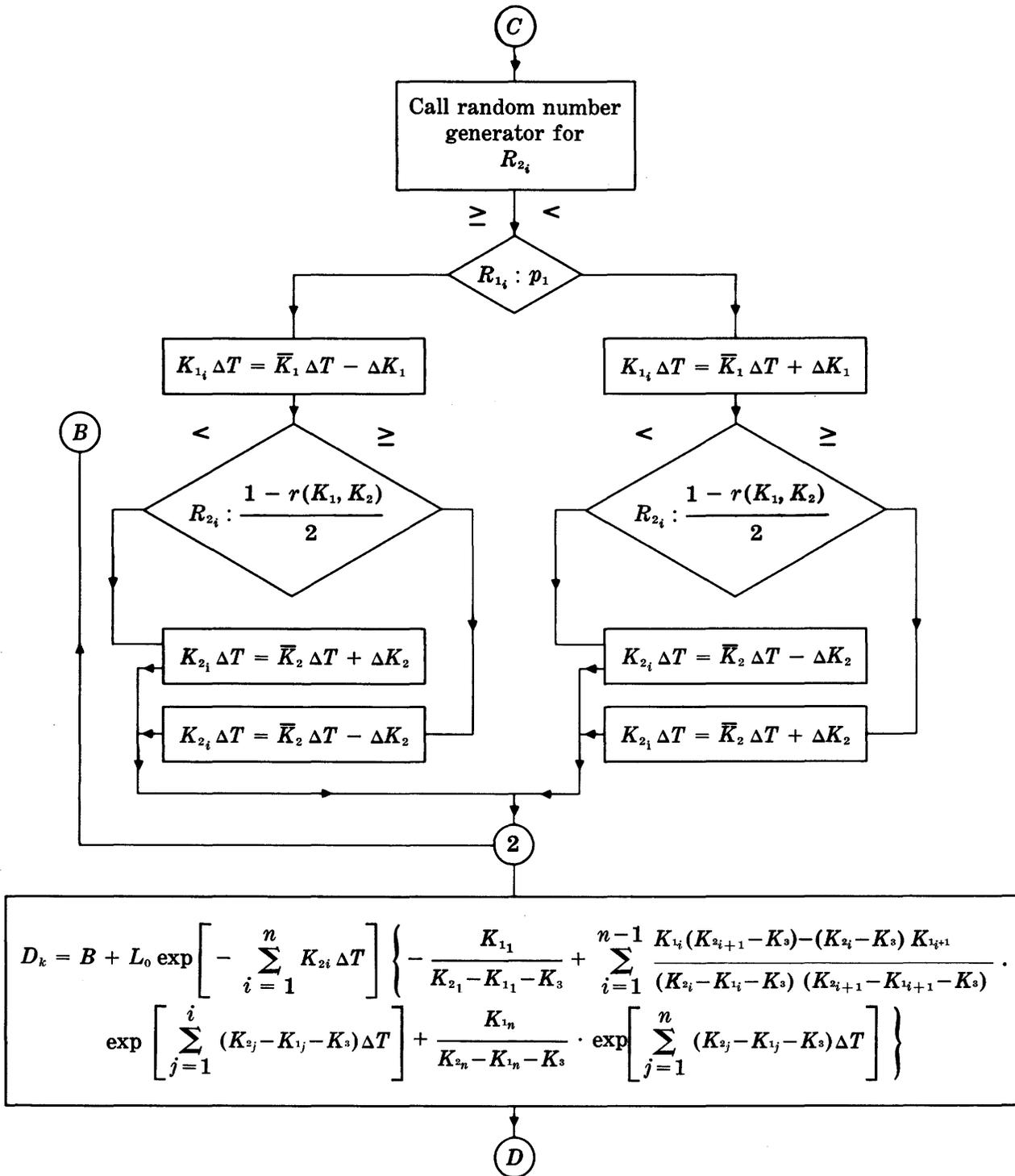
FIGURE 23.—Continuation of the flow chart for the computer program for the stochastic model for estimating the variance of the oxygen deficit.

$$\text{Mean } D = \sum_{k=1}^{m} \frac{D_k}{m}$$

$$\text{Var } (D) = \sum_{k=1}^{m} \frac{(D_k)^2}{m} - [\text{mean } D]^2$$

Order D_k such
that $D_k < D_{k+1}$ for
all k

Write mean D, var (D), and
D_k for $k = 1, m$

Stop
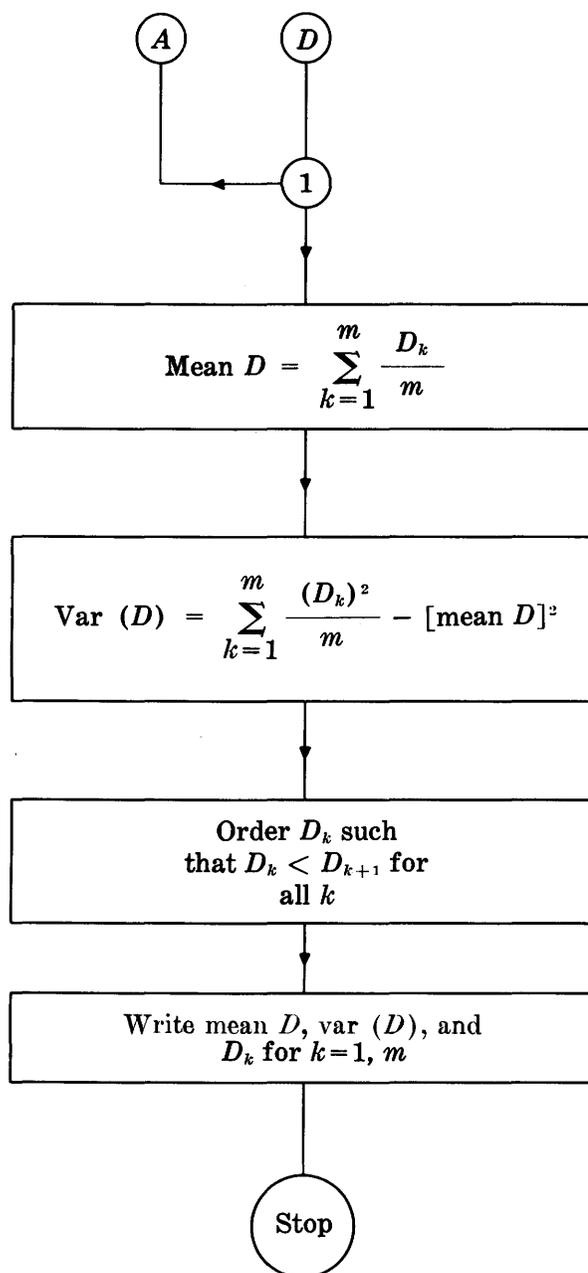
FIGURE 24.—Completion of the flow chart for the computer
program for the stochastic model for estimating the
variance of the oxygen deficit.