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# Contents

# Abstract

The Navier-Stokes equations for a Newtonian ideal gas are examined to determine the factorizable form of the equations relevant to the construction of a factorizable relaxation scheme. The principal linearization of the equations is found by examining the relative magnitude of the terms for short-wavelength errors. The principal part of the operator is then found. Comparison of the factors of the Navier-Stokes and Euler equations differ qualitatively because of the coupling of entropy and pressure through thermal diffusion. Special cases of the factorization are considered.

# 1 Introduction

Considerable progress has been made in recent years in the development of fast multigrid methods for the Euler and Navier-Stokes equations. In particular, the development of factorizable discretizations shows the promise of attaining ideal, or textbook, multigrid efficiency. A factorizable discretization is one in which the governing equations are partitioned into component elliptic, hyperbolic and parabolic subsystems. Each of these subsystems is then solved or relaxed by an appropriate numerical algorithm. By treating each partition optimally, the full system can be solved with the efficiency of the slowest converging subsystem. These ideas have been propounded by Achi Brandt for some time (Brandt, 1985) and are the subject of a recent review by Thomas, Diskin, and Brandt (2003).

In the multigrid literature, factorizability is described in terms of the properties of the determinant of the system of equations (Sidilkover, 1999; Ta'asan, 1993; Thomas et al., 2003). To relax the system of equations, the increment in the dependent variables is written in terms of a set of auxiliary variables, or "ghost" variables in Brandt's terminology. The transformation of variables is designed such that the system of equations in terms of the auxiliary variables is in triangular form, with the diagonal elements being the factors of the determinant. Increments in the auxiliary variables are used to update the original variables though the transformation. The triangularization of the system by means of the ghost variables is nothing more than decomposing the system into a set of nearly linearly independent modes. The factors of the determinant are the operators which act on each mode, and the auxiliary variables are the mode amplitudes. The key point is that the modes represent the short-wavelength components of the solution error. Only the terms in the equations that contribute strongly at the short wavelengths need to be considered in factoring the equations. These terms are called the principal part of the operator.

The factorizable property may be examined in a variety of ways. Thomas et al. (2003) view the construction of a factorizable discrete operator by treating the relaxation process as being an approximation to a Newton iteration. Starting with a linearization of the full system, they eliminate the subprincipal terms of the Jacobian and express the solution updates in terms of ghost variables. Ta'asan (1993) works directly with the full nonlinear Euler equations to derive a set of "canonical variables" corresponding to the different partitions of the equations. Using these variables he constructs a multigrid solver based on direct relaxation of the nonlinear system (Ta'asan, 1994). Sidilkover (1999) devises a factorizable scheme for the Euler equations by designing the difference operators such that the discrete potential, entropy and vorticity modes are preserved exactly in the constant coefficient case. This is generalized to a conservative discretization of the full nonlinear system in a straightforward way (Roberts, Sidilkover, and Thomas, 2000). As with Ta'asan, Sidilkover prefers to directly relax the full nonlinear system.

Although factorizable schemes have been constructed and demonstrated textbook multigrid efficiency for the incompressible Navier-Stokes equations (Brandt & Yavneh, 1992; Thomas, Diskin, and Brandt, 2001; Swanson, 2001) and for the incompressible and compressible Euler equations (Roberts et al., 2000; Roberts, Swanson, and Sidilkover, 1999; Ta'asan, 1994) in two dimensions, the development of such schemes for the compressible Navier-Stokes equations is still in its infancy. An initial attempt at such a scheme has been reported by Thomas, Diskin, and Brandt (1999). A difficulty arises because the nature of the factors in the viscous compressible case is qualitatively different than either the viscous incompressible or the inviscid compressible equations. Thermal conductivity has the effect of coupling the energy and continuity equations in compressible viscous flow. For certain values of the Prandtl number, described below, the two factors can be decoupled, and in general the coupling is weak. Thomas et al. (2003) present one possible approach to factorizing the compressible Navier-Stokes equations. Relaxation of their system will require a block relaxation of the two thermodynamic variables, which are pressure and internal energy in their case. A possible disadvantage of their formulation is that the energy and continuity modes do not decouple even for the appropriate value of the Prandtl number.

In the remainder of this paper, the principal linearization of the compressible Navier-Stokes is developed. The full Navier-Stokes equations are presented in Section 2. Following Thomas et al. (2003), in Section 3 the motivation is given by viewing relaxation methods as approximate Newton solvers. The principal linearization is derived from a perturbation expansion of the full equations. A discussion of various limits and

special cases of the factorization is presented in Section 4. The inviscid and incompressible limits are shown to lead to the appropriate factorization of the Euler and incompressible Navier-Stokes equation. With this factorization in mind, a decomposition into vortical, entropy and acoustic modes is developed, which may form the basis of a fast multigrid solver, in Section 5. These modes transform smoothly to the inviscid case as well. In addition, this decomposition has a counterpart in the study of compressible turbulent flow.

## 2 The Navier-Stokes equations

The analysis which follows is for a Newtonian fluid obeying an ideal gas law, with constant specific heats. Only steady flows are considered. For the purposes of examining the factors of the Navier-Stokes equations, the primitive variables are the most convenient. Letting $(u_i, p, s)$ be the state vector of velocity, pressure, and entropy, the Navier-Stokes equations may be written

$$u_j u_{i,j} + \frac{1}{\rho} p_{,i} = \frac{1}{\rho} \tau_{ij,j}, \tag{1a}$$

$$\rho c^2 u_{j,j} + u_j p_{,j} = (\gamma - 1)\left(\Phi - q_{j,j}\right), \tag{1b}$$

$$u_j s_{,j} = \frac{1}{\rho T}\left(\Phi - q_{j,j}\right), \tag{1c}$$

where $\rho$ is the density, $T$ is the static temperature and $\gamma$ is the ratio of the specific heats. The Einstein summation convention is used, and partial differentiation with respect to the $i$-th coordinate is indicated by the comma subscript notation. The viscous stress tensor is defined as $\tau_{ij} = \mu' d_{kk}\delta_{ij} + 2\mu d_{ij}$, where $\delta_{ij}$ is the Kronecker delta and $d_{ij}$ is the deviatoric strain tensor $d_{ij} = \left(u_{i,j} + u_{j,i}\right)/2$. The first and second coefficients of viscosity $\mu$ and $\mu'$ are related through the definition of the bulk viscosity, $3\kappa = 3\mu' + 2\mu \geq 0$. By Stokes' hypothesis $\kappa = 0$, which gives $\mu' = -2\mu/3$. However, it is not necessary to use this relation to determine the principal linearization of the equations.

The two terms on the right-hand side of the entropy equation contain the effects of viscous heating and thermal diffusion. The dissipation function $\Phi = d_{ij}\tau_{ij} \geq 0$ is the viscous work done on the fluid. The heat flux vector is given by $q_i = -\lambda T_{,i} = -\left(\mu c_p/\mathrm{Pr}\right) T_{,i}$, where $\lambda$ is the thermal conductivity, $c_p$ is the specific heat at constant pressure, and Pr is the Prandtl number. The temperature and density may be expressed in terms of the primary thermodynamic variables $s$ and $p$:

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma} e^{(s-s_0)/c_p},$$

$$\frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{1/\gamma} e^{-(s-s_0)/c_p}$$

where the subscript 0 denotes a reference value. The equation of state can also be written $p = \left(\left(\gamma - 1\right)/\gamma\right)\rho c_p T$, and the speed of sound is given by $c^2 = \gamma p/\rho = \left(\gamma - 1\right) c_p T$.

## 3 Relaxation and principal linearization

Relaxation methods are a class of iterative methods for the solution of a system of algebraic equations. An iterative method for solving a system of equations $A\varphi = b$ consists of generating a sequence of approximations $\{\varphi_0\ \varphi_1\ \varphi_2\ \ldots\}$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. The system is linearized about the $n$-th iterate to obtain an equation for the error, $A_\varphi(\varphi - \varphi_n) \approx b - A\varphi_n$ where $A_\varphi \equiv \partial(A\varphi)/\partial\varphi$ is the Jacobian of $A$ and $b - A\varphi_n$ is the residual at the $n$-th iteration. When the system of equations is linear, then $A_\varphi = A$, and the equation for the error is exact.

A Newton solver is obtained when all the terms of the linearization are kept and the resulting system for the error is solved exactly. For discretizations of partial differential equations, such as the Navier-Stokes

equations, this requires the inversion of a sparse, banded matrix. Newton methods have the outstanding property of quadratic convergence once the approximate solution is near the exact solution. Unfortunately, direct inversion is expensive and often impractical. Furthermore, the initial iterates of Newton's method can be far from the exact solution—for nonlinear problems, many iterations may be required before the region of quadratic convergence is reached.

Relaxation methods avoid direct inversion by replacing the Jacobian $A_\varphi$ with an approximation that is inexpensive to invert. Generally, this means throwing away all or some of the off-diagonal terms of the Jacobian. For example, keeping just the diagonal terms leads to Jacobi iteration; the Gauss-Seidel method results from throwing away the terms above the diagonal. Inverting the matrix involves only the cost of a matrix multiplication, which because of the sparsity of the matrix is $O(n)$.

Because it involves only local operations, relaxation is effective in eliminating the short-wavelength components of the error, where the short wavelengths are those on the order of $h$, the grid spacing. For this reason only those terms of the linearization that contribute strongly to the short-wavelength error components need be kept. These terms are referred to as the "principal part" of the linearization. Efficient multigrid methods are based on a relaxation method, or smoother, for reducing the short-wavelength component of the error, and a coarse-grid operator to reduce the long-wavelength error.

In Thomas et al. (2003) the authors describe the process of determining the principal part by reference to the operator alone. Taking the determinant, only those terms that contribute to the leading order are retained. In the following treatment, the principal part is found by expanding the equations in a small perturbation about a smooth state and keeping the dominant terms. This leads to the same result as Thomas, et al., but has the advantage of showing directly why certain terms are subprincipal. Furthermore, the procedure followed here is more familiar to fluid dynamicists and is very similar to a perturbation theory used in the study of compressible turbulence (Chu & Kovásznay, 1958; Kovásznay, 1953).

## 3.1 Principal linearization of inviscid terms

The principal linearization of the Navier-Stokes equations starts by considering a full linearization and keeping only those terms that are dominant on the scale of the grid spacing. The linearization is found by replacing the variables $(s, u_i, p)$ with

$$u_i \leftarrow u_i + \epsilon \hat{u}_i, \tag{2a}$$

$$p \leftarrow p + \epsilon \hat{p}, \tag{2b}$$

$$s \leftarrow s + \epsilon \hat{s} \tag{2c}$$

where $\epsilon \ll 1$. Equations (1) are expanded in $\epsilon$ and the linear terms are kept.

The inviscid terms of the Navier-Stokes equations (1) are those terms on the left-hand side. Let $U$ be the velocity scale and $L$ be the physical length scale of the flow. These are local scales, and can be different in different regions of the flow. For example, $L$ can be a chord length, leading-edge radius, or boundary-layer thickness. The grid spacing is $h$. It is assumed that $h \ll L$, or in other words, the grid is sufficiently fine to resolve the physical features of the flow. Making the substitutions Eqs. (2) into Eqs. (1) and keeping only the term of $O(\epsilon)$ yields the linearization of the inviscid terms

$$u_j \hat{u}_{i,j} + \hat{u}_j u_{i,j} + \frac{1}{\rho}\hat{p}_{,i} - \frac{\hat{p}}{\rho c^2}\frac{p_{,i}}{\rho} + \frac{\hat{s}}{c_p}\frac{p_{,i}}{\rho}, \tag{3a}$$

$$\rho c^2 \hat{u}_{j,j} + \gamma \hat{p} u_{j,j} + u_j \hat{p}_{,j} + p_{,j}\hat{u}_j, \tag{3b}$$

$$u_j \hat{s}_{,j} + \hat{u}_j s_{,j}. \tag{3c}$$

The scaling of the variables is

$$u_i, \hat{u}_i \sim U \qquad p, \hat{p} \sim \rho U^2 \qquad s, \hat{s} \sim c_p M^2.$$

Because only the short-wavelength components of the error are of interest, the derivatives of the perturbation quantities scale as

$$\hat{u}_{i,j} \sim \frac{U}{h} \qquad \hat{p}_{,j} \sim \frac{\rho U^2}{h} \qquad \hat{s}_{,j} \sim \frac{c_p M^2}{h}$$

but the derivatives of the solution values scale as

$$u_{i,j} \sim \frac{U}{L} \qquad p_{,j} \sim \frac{\rho U^2}{L} \qquad s_{,j} \sim \frac{c_p M^2}{L}.$$

Using the assumption that $h \ll L$, it is seen that terms involving the derivatives of solution quantities can be neglected compared to the terms involving derivatives of the perturbation quantities. This gives the principal linearization of the inviscid terms

$$u_j \hat{u}_{i,j} + \frac{1}{\rho}\hat{p}_{,i}, \tag{4a}$$

$$\rho c^2 \hat{u}_{j,j} + u_j \hat{p}_{i,j}, \tag{4b}$$

$$u_j \hat{s}_{,j}. \tag{4c}$$

Also, note that the ratio of the second term to the first term of Eq. (4b) is $O(M^2)$. Thus for low Mach number flows, $M^2 \ll 1$, the pressure advection term in the continuity equation is subprincipal.

### 3.2 Principal linearization of the viscous terms

The right-hand sides of Eqs. (1c) and (1b), the entropy and continuity equations, consist of the heat diffusion and viscous dissipation terms. The principal part of the linearization of each term is considered. The scalings of the viscosity and temperature are

$$\mu, \mu' \sim \frac{\rho U L}{\mathrm{Re}} \qquad T, \hat{T} \sim \frac{U^2}{c_p}.$$

The Reynolds number Re is based on the relevant local velocity and length scales. Also, it is assumed that $\partial\mu/\partial T \ll \mu/T$, $\partial^2\mu/\partial T^2 \ll \mu/T^2$, so the variations of the viscosity can be neglected when linearizing.

Examining the scaling of the deviatoric strain tensor and its derivatives gives

$$d_{ij} \sim \frac{U}{L}, \qquad d_{ij,j} \sim \frac{U}{L^2}, \qquad \hat{d}_{ij} \sim \frac{U}{h}, \qquad \hat{d}_{ij,j} \sim \frac{U}{h^2}.$$

As a consequence, the dominant term in the linearization of the dissipation function $\Phi$ is

$$2\mu' d_{ii}\hat{d}_{jj} + 4\mu d_{ij}\hat{d}_{ij}.$$

Considering the thermal diffusion term next, note that the temperature derivatives scale as

$$T_{,i} \sim \frac{U^2}{c_p L}, \qquad T_{,ii} \sim \frac{U^2}{c_p L^2}, \qquad \hat{T}_{,i} \sim \frac{U^2}{c_p h}, \qquad \hat{T}_{,ii} \sim \frac{U^2}{c_p h^2},$$

from which it follows that the dominant term in the heat-flux linearization is

$$\frac{\mu c_p}{\mathrm{Pr}}\hat{T}_{,ii}.$$

Using the equation of state to express this in terms of $\hat{s}$ and $\hat{p}$ gives

$$\frac{\nu}{\mathrm{Pr}}\left(\frac{\rho c^2}{\gamma - 1}\frac{\hat{s}_{,ii}}{c_p} + \hat{p}_{,ii}\right).$$

Examination of the relative magnitude of the two terms yields

$$2\mu' d_{ii}\hat{d}_{jj} + 4\mu d_{ij}\hat{d}_{ij} \sim \frac{\rho U^3 L}{\mathrm{Re}}\frac{1}{Lh},$$
$$\frac{\mu c_p}{\mathrm{Pr}}\hat{T}_{,ii} \sim \frac{\rho U^3 L}{\mathrm{Re}\,\mathrm{Pr}}\frac{1}{h^2}.$$

It is seen that that thermal diffusion term is larger by $O(\mathrm{Pr}^{-1}L/h)$, so the dissipation term is subprincipal and can be neglected in the linearization.

Linearizing the viscous stress terms $\tau_{ij,j}/\rho$ on the right-hand side of the momentum equation (1a) and comparing magnitudes of the terms yields

$$\frac{1}{\rho}\left(\mu'\hat{d}_{jj,i} + 2\mu\hat{d}_{ij,j}\right) \sim \frac{\mu}{\rho}\frac{U}{h^2},$$

all other terms of being at least $O(h/L)$ smaller.

In summary, the principal part of the viscous terms in the Navier-Stokes equations are

$$\frac{1}{\rho}\left(\mu'\hat{d}_{jj,i} + 2\mu\hat{d}_{ij,j}\right), \tag{5a}$$
$$(\gamma-1)\frac{\nu}{\mathrm{Pr}}\left(\frac{\rho c^2}{\gamma-1}\frac{\hat{s}_{,ii}}{c_p} + \hat{p}_{,ii}\right), \tag{5b}$$
$$(\gamma-1)\frac{c_p}{\rho c^2}\frac{\nu}{\mathrm{Pr}}\left(\frac{\rho c^2}{\gamma-1}\frac{\hat{s}_{,ii}}{c_p} + \hat{p}_{,ii}\right). \tag{5c}$$

in the momentum, continuity, and entropy equations, respectively.

### 3.3 The complete principal linearization

Having found the principal linearization of the inviscid and viscous terms in Eqs. (4) and (5), the principal linearization of the full equations is found by comparing the relative magnitude of the terms in each equation.

Starting with the entropy equation, a comparison of the magnitudes of the terms in Eqs. (4c) and (5c) yields the relative balance of advection and thermal diffusion

$$U\frac{c_p M^2}{h} \sim \frac{c_p}{\rho c^2}\frac{\mu}{\mathrm{Pr}}\frac{U^2}{h^2}.$$

This can be restated as the requirement $\mathrm{Re}_h\mathrm{Pr} = O(1)$ for both terms to be principal. If the Prandtl number is $O(1)$, then the cell Reynolds number must be $O(1)$ for the viscous terms in the entropy equation to be principal.

Examination of Eqs. (4a) and (5a) gives

$$\frac{U^2}{h} \sim \frac{\mu}{\rho}\frac{U}{h^2}$$

as the dominant inertial and shear stress balance, or $\mathrm{Re}_h = O(1)$.

Finally, the continuity equation terms from Eqs. (4b) and (5b) are comparable when

$$\rho c^2\frac{U}{h} \sim \frac{\mu}{\mathrm{Pr}}\frac{U^2}{h^2}.$$

The relative magnitude of the viscous terms compared to the inviscid terms is $M^2/(\mathrm{Re}_h\mathrm{Pr})$, which for moderate Mach numbers corresponds to the condition that the cell Reynolds number is order unity. However,

for low Mach numbers the terms on the right-hand side of the continuity equation are subprincipal. Once again, this is reflection that in the incompressible limit, Eq. (1b) reduces to the statement that $u_{i,i} = 0$.

To summarize, the principal linearization of the Navier-Stokes equations is

$$u_j \hat{u}_{i,j} + \frac{1}{\rho}\hat{p}_{,i} = \frac{1}{\rho}\left(\mu' \hat{d}_{jj,i} + 2\mu \hat{d}_{ij,j}\right), \tag{6a}$$

$$\rho c^2 \hat{u}_{j,j} + u_j \hat{p}_{,j} = (\gamma - 1)\frac{\nu}{\mathrm{Pr}}\left(\frac{\rho c^2}{\gamma - 1}\frac{\hat{s}_{,ii}}{c_p} + \hat{p}_{,ii}\right), \tag{6b}$$

$$u_j \hat{s}_{,j} = (\gamma - 1)\frac{c_p}{\rho c^2}\frac{\nu}{\mathrm{Pr}}\left(\frac{\rho c^2}{\gamma - 1}\frac{\hat{s}_{,ii}}{c_p} + \hat{p}_{,ii}\right). \tag{6c}$$

### 3.4 Interpretation of the principal linearization

As stated in the beginning of this section, the motivation for finding the principal linearization is that, because a relaxation method is effective at eliminating short wavelength error components, only those terms of the linearized operator that contribute strongly at short wavelengths are necessary. This property may be stated more formally. Let $\mathbf{L}_{\mathrm{FL}}$ be the full linearization of the operator, and $\mathbf{L}$ be the principal linearization. The principal linearization satisfies the property

$$\|\mathbf{I} - \mathbf{L}^{-1}\mathbf{L}_{\mathrm{FL}}\| = O\left(\frac{h}{L}\right). \tag{7}$$

In other words, the inverse of the principal linearization is a good approximation to the inverse of the full linear operator at the scale of the grid spacing.

An important point to emphasize here is that the key assumption used to derive the principal linearization is $h \ll L$, which implies that the flow is well-resolved on the grid. It was shown above that formally the viscous terms are principal when $\mathrm{Re}_h = O(1)$, which is quite restrictive. For example, a typical calculation of a laminar boundary layer flow over a flat plate will have on the order of 20–30 grid points across the boundary layer. The local Reynolds number in the middle of the boundary layer, based on the local flow speed and the normal grid spacing, is ∼3–10. When solving the system of equations using a full multigrid (FMG) process, the viscous region will be underresolved on the coarse grids. The principal linearization will not be strictly valid until the finest grid or two are reached. Also implicit in the condition $h \ll L$ is that the region of interest is well-removed from the boundaries and any discontinuities, such as shocks. The complications that arise near boundaries and discontinuies is discussed at length in Thomas et al. (2003).

For typical aerodynamic flows where the viscous effects are confined to thin shear layers, the variation in length scales over the domain is quite large. Not all the terms will be principal in the different regions of the flow. In practice, the variation in physical scales is handled by variation in grid spacing throughout the computational domain. As a consequence, the principal linearization will differ throughout the computational domain. In the next section, various limiting cases of the principal linearization for Eq. (6) are considered.

## 4 Factors of the Navier-Stokes equations

In Section 3 the two components of an efficient multigrid method are given: a relaxation method for smoothing the short-wavelength component of the error, and a coarse-grid operator to reduce the long-wavelength error. For multigrid to be fully effective, it is necessary that the solution error is well-represented on the coarse grid. This is the case for elliptic partial differential equations. However, for hyperbolic equations not all smooth components of the error are visible to the coarse grid operator. Those components normal to the advection direction are only represented at the level of the truncation error. The coarse-grid correction for these contains only part of this smooth error component. Multigrid is not effective for these types of equations.

The physics of fluid flow is a combination of advection and diffusion phenomena, and as such the Navier-Stokes equations are a combination of elliptic and hyperbolic-parabolic operators. Energy and vorticity are carried along streamlines and transported transversely by diffusion. Conservation of mass represents a balance between pressure and the divergence of the velocity field, which is elliptic behavior subsonically and hyperbolic supersonically. To construct an efficient multigrid solver for the Navier-Stokes equations, it is necessary to split the system into these component parts such that each partition may be treated in the most appropriate way.

The principal linearization in Eq. (6) can be used to identify and isolate the various subsystems of the Navier-Stokes equations. Introducing the notation $Q = u_i \partial_i$ and $Q_\nu = Q - \nu\Delta$ for convenience, the system may be written in the operator form $\mathbf{L}\,\epsilon\hat{\mathbf{q}} = \mathbf{r}$, or

$$\begin{pmatrix} Q_\nu - (\nu+\nu')\partial_i\nabla\cdot & \frac{1}{\rho}\partial_i & 0 \\ \rho c^2 \nabla\cdot & Q - \nu\frac{\gamma-1}{\mathrm{Pr}}\Delta & -\nu\frac{\rho c^2}{c_p \mathrm{Pr}}\Delta \\ 0 & -\nu\frac{c_p(\gamma-1)}{\rho c^2 \mathrm{Pr}}\Delta & Q - \frac{\nu}{\mathrm{Pr}}\Delta \end{pmatrix} \begin{pmatrix} \epsilon\hat{u}_i \\ \epsilon\hat{p} \\ \epsilon\hat{s} \end{pmatrix} = \mathbf{r}. \tag{8}$$

Because $\mathbf{L}$ is a matrix of operators, the formal eigenvalues of $\mathbf{L}$ are the "eigen-operators" of the system, in a generalization of characteristic analysis of hyperbolic equations. These eigenvalues are pseudo-differential operators. Unfortunately, not all these are true differential operators. However, taking the formal determinant of $\mathbf{L}$ gives an expression whose factors are identifiable differential operators. Some of these factors are the eigenvalues of $\mathbf{L}$, while the remaining factors are products of those eigenvalues which do not correspond to any real differential operator.

The determinant of Eq. (8) is

$$\det \mathbf{L} = Q_\nu^{d-1} \left\{ \left(Q - \frac{\nu}{\mathrm{Pr}}\Delta\right) \left[ Q\left(Q - \gamma\frac{\nu}{\mathrm{Pr}}\Delta\right) - c^2\Delta \right] + \left(\frac{\nu}{\mathrm{Pr}} - 2\nu - \nu'\right) Q \left(Q - \gamma\frac{\nu}{\mathrm{Pr}}\Delta\right)\Delta \right\}, \tag{9}$$

where $d$ is the number of space dimensions. The repeated advection-diffusion factors correspond to the transport of vorticity in two and three dimensions, and to the conservation of helicity in three dimensions (Ta'asan, 1993). The factor in the curly braces corresponds to energy and mass conservation. This factor is called the "core operator" by Brandt. Thermal diffusion couples temperature variations to pressure in Eq. (8). However, there are some special cases in which the core operator can be factorized, which will be examined below. Furthermore, some of these special cases will occur in regions of many flows of engineering interest. For this reason, these limits can be used when relaxing the full Navier-Stokes equations in these regions.

## 4.1 Inviscid equations

As shown in Section 3, the viscous terms are principal only where the cell Reynolds number is $O(1)$. For most aerodynamic flows, viscous effects are significant only in thin shear layers, and the bulk of the flow is inviscid. The inviscid limit of Eq. (8) is

$$\mathbf{L}_{\nu\to 0} = \begin{pmatrix} Q & \frac{1}{\rho}\partial_i & 0 \\ \rho c^2 \nabla\cdot & Q & 0 \\ 0 & 0 & Q \end{pmatrix}. \tag{10}$$

The advection-diffusion operator reduces to pure advection, and factors out of the core operator. The determinant of Eq. (10) is

$$\det \mathbf{L}_{\nu\to 0} = Q^d \left(Q^2 - c^2\Delta\right). \tag{11}$$

Two of the advection factors correspond to vorticity and helicity conservation, as before. The additional advection factor is the transport of entropy along streamlines, which can be seen by inspection of Eq. (10). The remaining second-order operator is the full potential operator, which corresponds to conservation of mass. The full potential operator connects the pressure field to the irrotational part of the velocity field.

Note that in the one-dimensional case, the full potential operator factors into two first-order operators, corresponding to the left- and right-running acoustic characteristics.

Although the discussion concerns the principal linearization of the equations, there is a correspondence of these terms in the full nonlinear case as well. When the flow is uniform at infinity, the vorticity remains zero and the entropy is constant. In such a case, all the advection terms are satisfied identically. The solution of the Euler equations is fully determined by the solution of the continuity equation alone. By writing the velocity as the gradient of a scalar potential function, the continuity equation reduces to the full potential operator acting on the potential function. Thus it is seen how the factors of the determinant of $\mathbf{L}$ in the inviscid limit correspond directly to a well-known property of inviscid, compressible flow.

## 4.2 Low speed flow

Another important limit is at low speed. Specifically, when $M^2 \ll 1$, then the pressure advection term in the continuity equation is subprincipal. With fluctuations in entropy scaling as given in Section 3.1, the entropy diffusion term in the continuity equation is also subprincipal. In the entropy equation, the diffusion of pressure and entropy are of comparable magnitude, and remain principal. The resulting reduced form of the principal linearization is

$$\mathbf{L}_{M\to 0} = \begin{pmatrix} Q_\nu - (\nu+\nu')\partial_i\nabla\cdot & \frac{1}{\rho}\partial_i & 0 \\ \rho c^2\nabla\cdot & 0 & 0 \\ 0 & -\nu\frac{c_p(\gamma-1)}{\rho c^2 \mathrm{Pr}}\Delta & Q - \frac{\nu}{\mathrm{Pr}}\Delta \end{pmatrix}. \tag{12}$$

This corresponds to a nearly constant-density fluid, and occurs when the boundaries of the domain are adiabatic.

If there is significant heating at the boundaries, then we can have that $s, \hat{s} \gg c_p M^2$. In this case, the temperature (or entropy) fluctuations are decoupled from the pressure fluctuations. If $M^2 \ll 1$ the pressure advection-diffusion term in the continuity equation and the pressure diffusion term in the entropy equation are subprincipal, leading to the principal linearization

$$\mathbf{L}_{M\to 0} = \begin{pmatrix} Q_\nu - (\nu+\nu')\partial_i\nabla\cdot & \frac{1}{\rho}\partial_i & 0 \\ \rho c^2\nabla\cdot & 0 & -\nu\frac{\rho c^2}{c_p \mathrm{Pr}}\Delta \\ 0 & 0 & Q - \frac{\nu}{\mathrm{Pr}}\Delta \end{pmatrix}. \tag{13}$$

For both Eqs. (12) and (13) the determinant of the operator is

$$\det \mathbf{L}_{M\to 0} = -c^2 Q_\nu^{d-1}(Q - \frac{\nu}{\mathrm{Pr}}\Delta)\Delta. \tag{14}$$

The decoupling of the temperature and pressure variations is reflected by the core operator factoring into an advection diffusion term and the Laplacian. The advection-diffusion factor acts on the entropy. The Laplacian correponds to conservation of mass.

These two low speed limiting cases are distinct—one corresponds to incompressible flow, the other to low Mach number but compressible flow. An important point to note here is that the determinant of the operator is the same for both limits discussed above. By the criterion presented in Thomas et al. (2003), one would neglect altogether the off-diagonal dissipative terms in the continuity and entropy equations in the incompressible limit, resulting in the principal linearization

$$\mathbf{L}_{M\to 0} = \begin{pmatrix} Q_\nu - (\nu+\nu')\partial_i\nabla\cdot & \frac{1}{\rho}\partial_i & 0 \\ \rho c^2\nabla\cdot & 0 & 0 \\ 0 & 0 & Q - \frac{\nu}{\mathrm{Pr}}\Delta \end{pmatrix}.$$

This form of the operator does not satisfy the property given by Eq. (7).

### 4.3 A special case of the Prandtl number

Examination of Eq. (9) shows the term in the curly braces can be factored when the Prandtl number is $1/(2+\mu'/\mu)$. Substituting into the equations yields

$$\mathbf{L}_{\mathrm{Pr}=1/(2+\mu'/\mu)} = \begin{pmatrix} Q_\nu - (\nu+\nu')\partial_i\nabla\cdot & \frac{1}{\rho}\partial_i & 0 \\ \rho c^2\nabla\cdot & Q-(\gamma-1)(2\nu+\nu')\Delta & -(2\nu+\nu')\frac{\rho c^2}{c_p}\Delta \\ 0 & -(2\nu+\nu')\frac{c_p(\gamma-1)}{\rho c^2}\Delta & Q-(2\nu+\nu')\Delta \end{pmatrix} \tag{15}$$

and this has the determinant

$$\det \mathbf{L}_{\mathrm{Pr}=1/(2+\mu'/\mu)} = Q_\nu^{d-1}\left[Q-(2\nu+\nu')\Delta\right]\left\{Q\left[Q-\gamma\left(2\nu+\nu'\right)\Delta\right]-c^2\Delta\right\}. \tag{16}$$

The new advection factor that appears reflects the transport of entropy along a streamline. The remaining term corresponds to continuity, but the diffusion terms reflect that the pressure is coupled to the entropy through thermal diffusion.

In general, Stokes' hypothesis is applicable which gives that $\mu' = -2\mu/3$, which means that this factorization applies when $\mathrm{Pr} = 3/4$. Note that for air the Prandtl number ranges from 0.75 to 0.70 over the temperature range $-100$ to 100 Celsius, and is approximately 0.72 for most atmospheric flight conditions. Thus this factorization is very nearly exact for aerodynamic flows.

For convenience, Eq. (15) and its determinant Eq. (16) is written below for the case when the Stokes hypothesis applies and $\mathrm{Pr} = 3/4$:

$$\mathbf{L}_{\mathrm{Pr}=3/4} = \begin{pmatrix} Q_\nu - \frac{1}{3}\nu\partial_i\nabla\cdot & \frac{1}{\rho}\partial_i & 0 \\ \rho c^2\nabla\cdot & Q-\frac{4}{3}(\gamma-1)\nu\Delta & -\frac{4}{3}\frac{\rho c^2}{c_p}\nu\Delta \\ 0 & -\frac{4}{3}\frac{c_p(\gamma-1)}{\rho c^2}\nu\Delta & Q-\frac{4}{3}\nu\Delta \end{pmatrix},$$

$$\det \mathbf{L}_{\mathrm{Pr}=3/4} = Q_\nu^{d-1}\left(Q-\frac{4}{3}\nu\Delta\right)\left[Q\left(Q-\gamma\frac{4}{3}\nu\Delta\right)-c^2\Delta\right].$$

## 5 Modes of the principal linearization

To construct an efficient relaxation scheme for the Navier-Stokes equation that makes effective use of the factors, it is necessary to determine the derived variables upon which the factors are acting. The most natural decomposition of the perturbation variables into modes corresponding to the factors of the operator is

$$\begin{pmatrix} \hat{u}_i \\ \hat{p} \\ \hat{s} \end{pmatrix} = \begin{pmatrix} U_{si} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \partial_i \\ -\rho(Q-\frac{\nu}{\mathrm{Pr}}\Delta) \\ -(\gamma-1)\frac{c_p\nu}{c^2\mathrm{Pr}}\Delta \end{pmatrix}\phi + \begin{pmatrix} \frac{\nu}{c_p\mathrm{Pr}}\partial_i \\ 0 \\ 1 \end{pmatrix}\sigma \tag{17}$$

where $U_{si}$ is a solenoidal velocity field, and $\sigma$ and $\phi$ are scalar potentials. This is related to the classical decomposition of a velocity field into solenoidal and irrotational parts. Applying the operator $\mathbf{L}$ to the above decomposition yields

$$\begin{aligned} \mathbf{L}\begin{pmatrix} \hat{u}_i \\ \hat{p} \\ \hat{s} \end{pmatrix} = Q_\nu \begin{pmatrix} U_{si} \\ 0 \\ 0 \end{pmatrix} &+ \begin{pmatrix} 0 \\ -\rho Q\left(Q-\gamma\frac{\nu}{\mathrm{Pr}}\Delta\right)+\rho c^2\Delta \\ 0 \end{pmatrix}\phi \\ &+ \left(Q-\frac{\nu}{\mathrm{Pr}}\Delta\right)\begin{pmatrix} \frac{\nu}{c_p\mathrm{Pr}}\partial_i \\ 0 \\ 1 \end{pmatrix}\sigma \\ &+ \left(\frac{\nu}{\mathrm{Pr}}-2\nu-\nu'\right)\Delta\begin{pmatrix} \partial_i\left(\frac{\nu}{c_p\mathrm{Pr}}\sigma+\phi\right) \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{18}$$

Note that the last term vanishes for the special case of Subsection 4.3, i.e., $\mathrm{Pr} = 1/(2 + \mu'/\mu)$. In this particular case, three remaining terms correspond exactly to the factors of Eq. (16). Furthermore, in the inviscid limit it is observed that $\phi$ is the velocity potential, and $\sigma$ becomes the entropy. Most significantly, the continuity equation depends on $\phi$ alone, and the entropy equation depends on $\sigma$ alone. The coupling between these modes in the general case appears in the momentum equation, and arises from the effect of the dilatational viscous stresses.

The quantity $U_{si}$ is an incompressible vorticity error transported along the mean flow. The quantity $\sigma$ is an isobaric temperature error transported by the mean flow. The appearance of the velocity component $\nu/(c_p\mathrm{Pr})\partial_i\sigma$ arises from the dilatation of the velocity field induced by heating of the fluid. The quantity $\phi$ is essentially a velocity potential. A change in $\phi$ corresponds to a change in velocity at a constant total enthalpy, or total temperature. The continuity equation is the full potential equation modified by the addition of a term corresponding to the absorption of acoustic energy by thermal diffusion. In the case of $\mathrm{Pr} = 3/4$, the velocity dilatation due to thermal diffusion is balanced exactly by the viscous action of the bulk viscosity. For any $\mathrm{Pr} \sim 1$, the coupling is weak.

In the inviscid limit, these three modes are fully decoupled:

$$\mathbf{L}\begin{pmatrix}\hat{u}_i\\ \hat{p}\\ \hat{s}\end{pmatrix} = Q\begin{pmatrix}U_{si}\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}0\\ -\rho\left(Q^2 - c^2\Delta\right)\\ 0\end{pmatrix}\phi + Q\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\sigma. \tag{19}$$

It is interesting to note that these modes appear in Kovásznay's study of supersonic turbulent flow (Chu & Kovásznay, 1958; Kovásznay, 1953). He developed the perturbation equations for small-amplitude, small-scale fluctuations being transported with a uniform mean velocity. In the present case, the short-wavelength solution errors are analogous to the small-scale turbulent fluctuations, and these errors are transported along the mean-flow approximate solution. Because the equations are linearized about an unconverged state, the residuals act as apparent mass sources, body forces, and heat sources on the right-hand side. Kovásznay's work focuses on disturbances whose Reynolds number is much greater than one, whereas here the viscous terms are principal when $\mathrm{Re}_h = O(1)$. For higher Reynolds numbers, the artificial viscosity of the scheme dominates the physical viscosity.

It is useful to compare the present decomposition to that presented in Thomas et al. (2003). They use internal energy rather than entropy as their thermodynamic variable. Converting their formulation into the present notation, and rescaling some terms, yields

$$\begin{pmatrix}\hat{u}_i\\ \hat{p}\\ \hat{s}\end{pmatrix} = \begin{pmatrix}I\\ \rho(\nu+\nu')\nabla\cdot\\ -\frac{1}{T}(\nu+\nu')\nabla\cdot\end{pmatrix}U_{si} + \begin{pmatrix}\partial_i\\ -\rho(Q-(2\nu+\nu')\Delta)\\ \frac{1}{T}(Q-(2\nu+\nu')\Delta)\end{pmatrix}\phi + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\sigma. \tag{20}$$

Here, the velocity $U_{si}$ is not assumed to be solenoidal. The quantity $\sigma$ is an isobaric temperature mode, but it does not induce any velocity dilatation. Finally, the quantity $\phi$ is not the usual velocity potential, but corresponds to a velocity and pressure change at a constant static temperature, rather than total temperature. The application of $\mathbf{L}$ to this set of variables yields

$$\mathbf{L}\begin{pmatrix}\hat{u}_i\\ \hat{p}\\ \hat{s}\end{pmatrix} = \begin{pmatrix}Q_\nu\\ \mathcal{P}\nabla\cdot\\ -\frac{\nu+\nu'}{T}Q\nabla\cdot\end{pmatrix}U_{si} + \begin{pmatrix}0\\ -\rho Q\left(Q-(2\nu+\nu')\Delta\right)+\rho c^2\Delta\\ \frac{1}{T}Q\left(Q-(2\nu+\nu')\Delta\right)\end{pmatrix}\phi + \begin{pmatrix}0\\ -\frac{\rho\nu c^2}{c_p\mathrm{Pr}}\Delta\\ Q-\frac{\nu}{\mathrm{Pr}}\Delta\end{pmatrix}\sigma \tag{21}$$

where $\mathcal{P} = \rho c^2 + (\mu+\mu')Q$. Note that the energy and continuity equations are coupled for all values of Pr. Also, if the velocity $U_{si}$ is solenoidal, the tranport equation for this mode becomes fully decoupled from the other modes. In the inviscid limit, this system becomes

$$\mathbf{L}\begin{pmatrix}\hat{u}_i\\ \hat{p}\\ \hat{s}\end{pmatrix} = \begin{pmatrix}Q\\ \rho c^2\nabla\cdot\\ 0\end{pmatrix}U_{si} + \begin{pmatrix}0\\ -\rho\left(Q^2-c^2\Delta\right)\\ \frac{1}{T}Q^2\end{pmatrix}\phi + \begin{pmatrix}0\\ 0\\ Q\end{pmatrix}\sigma. \tag{22}$$

# 6 Conclusions

In this paper, the compressible Navier-Stokes equations have been examined with the objective of identifying the factors of the operator. The principle linearization of the systems has been developed by examining those terms of the linearization that are most significant for the short wavelength error components. The factors of the operator are seen to be identical to the modes identified by Kovásznay (1953) in his analysis of compressible turbulent flow.

It must be emphasized that the decomposition of the Navier-Stokes equations into these factors is only the first step in the construction of a multigrid algorithm. A factorizable discretization of the system—one that preserves these factors at the discrete level—must be constructed. Such a discretization must also be $h$-elliptic for an efficient relaxation scheme and multigrid solver to be constructed. Creating a scheme that preserves all these properties is an extremely difficult task (see Sidilkover, 1999). Indeed, there is no *a priori* guarantee that such a scheme exists. Nevertheless, the identification of the factors provides a useful framework for construction of fast converging schemes.

# References


Brandt, A. 1985. Multigrid Techniques: 1984 Guide with Applications to Fluid Dynamics. *GMD-Studie 85*, GMD-FIT

Brandt, A., Yavneh, I. 1992. On multigrid solution of high-Reynolds incompressible entering flow. *J. Comput. Phys.* 101:151–64

Chu, B.-T., Kovásznay, L. S. G. 1958. Non-linear interactions in a viscous heat-conducting compressible gas. *J. Fluid Mech.* 3:494–514

Kovásznay, L. S. G. 1953. Turbulence in supersonic flow. *J. Aeron. Sci.* 20:657–74, 682

Roberts, T. W., Sidilkover, D., Thomas, J. L. 2000. Multigrid relaxation of a factorizable conservative discretization of the compressible Euler equations. *AIAA Paper 2000-2252*

Roberts, T. W., Swanson, R. C., Sidilkover, D. 1999. An algorithm for ideal multigrid convergence for the steady Euler equations. *Comput. Fluids* 28:427–42

Sidilkover, D. 1999. Factorizable scheme for the equation of fluid flow. *ICASE Rep. 99-20*

Ta'asan, S. 1993. Canonical forms of multidimensional steady inviscid flows. *ICASE Rep. 93-34*

Ta'asan, S. 1994. Canonical-variables multigrid for steady-state Euler equations. *ICASE Rep. 94-14*

Thomas, J. L., Diskin, B., Brandt, A. 1999. Distributed relaxation multigrid and defect correction applied to the compressible Navier-Stokes equations. *AIAA Paper 99-3334*

Thomas, J. L., Diskin, B., Brandt, A. 2001. Textbook multigrid efficiency for the incompressible Navier-Stokes equations: high Reynolds number wakes and boundary layers. *Comput. Fluids* 30:853–74

Thomas, J. L., Diskin, B., Brandt, A. 2003. Textbook multigrid efficiency for fluid simulations. *Annu. Rev. Fluid Mech.* 35:317–40

Swanson, R. C. 2001. Towards optimal multigrid efficiency for the Navier-Stokes equations. *AIAA Paper 2001-2574*

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The Navier-Stokes equations for a Newtonian ideal gas are examined to determine the factorizable form of the equations relevant to the construction of a factorizable relaxation scheme. The principal linearization of the equations is found by examining the relative magnitude of the terms for shortwavelength errors. The principal part of the operator is then found. Comparison of the factors of the Navier-Stokes and Euler equations differ qualitatively because of the coupling of entropy and pressure through thermal diffusion. Special cases of the factorization are considered.

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