Multiphase Optimal and Non-Singular Power Flow by Successive Linear Approximations

Preprint

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Presented at the IEEE Power Systems Computation Conference (PSCC)
Dublin, Ireland
June 11-15, 2018

Suggested Citation
Bernsetin, Andrey, Cong Wang, and Jean-Yves Le Boudec. 2018. “Multiphase Optimal and Non-Singular Power Flow by Successive Linear Approximations: Preprint.” Golden, CO: National Renewable Energy Laboratory. NREL/CP-5D00-70368
https://www.nrel.gov/docs/fy18osti/70368.pdf

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Conference Paper
NREL/CP-5D00-70368
August 2018

Contract No. DE-AC36-08GO28308

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Multiphase Optimal and Non-Singular Power Flow by Successive Linear Approximations

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***Abstract*—We propose an optimal power flow method for multiphase power systems. The method works for both radial and meshed networks, and is able to accommodate different types of load/source connections such as wye, delta, and combination thereof. Moreover, the method generates a sequence of non-singular points (i.e., associated with non-singular power-flow Jacobian) in the electrical state space and a corresponding sequence of power injections, which can be used in real-time control settings. Under certain conditions, the sequences attain the local minimum. The method has been tested using several typical networks, including the IEEE 37-bus and 123-bus test feeders.**

***Index Terms*—Multiphase power networks; AC optimal power flow; successive linear approximation; non-singularity; existence; uniqueness; feasibility.**

I. Introduction

The AC optimal power flow (OPF) problem is one of the fundamental problems in power system operation and analysis. Mainly due to the nonlinearity of the power flow equations, the OPF problem is nonconvex and NP-hard in general (see, e.g., [1]). In transmission systems, OPF methods are typically based on DC power-flow models and on the assumption that the system is balanced. In distribution systems, however, these two simplifying assumptions are no longer valid, since distribution systems generally have high R/X ratio and are unbalanced with a variety of different types of connections. Therefore a full multiphase power-flow model has to be used, and AC OPF methodology has to be developed and applied.

There is a wide range of literature that addressed the AC OPF problem by using relaxation techniques, such as semidefinite relaxation (SDR) [2], [3] and generalizations thereof (e.g., [4]); due to the scope of this paper, we will not present an exhaustive list here. Several papers addressed the AC OPF problem for *multiphase distribution networks* (see [5] for an extensive review). A majority of them (see, e.g., [6]–[8] and pertinent references therein) utilized generic solvers to identify a solution of this challenging nonconvex (and NP-hard) optimization problem; nonetheless, no theoretical convergence guarantees are offered. In [9], an SDR approach was proposed, and a feasible (and globally optimal) solution can be identified under certain conditions. Similarly, in [10], an SDR approach was applied for multiphase radial distribution networks. Finally, [11] developed a successive convex approximation methodology, which is proven to converge to a KKT point of the original non-convex AC OPF. Overall, with the exception of [7], the network models utilized in the existing literature can support only wye-connected sources, and they cannot be straightforwardly extended to delta connections.

In this paper, we propose to solve the AC OPF for multiphase unbalanced systems using a new solution methodology based on successive linear approximations. The distinctive characteristics of the proposed method are highlighted as following: 1) We leverage the general multiphase power-flow model recently proposed in [12]. This allows us to explicitly consider different types of load/source connections, such as wye, ungrounded delta, and a combination thereof; 2) We propose the following iterative method. At each iteration:

(1) We convexify the original non-convex problem by replacing the exact AC power-flow constraints with an appropriate linear approximation, leveraging the model in [12]. The linearization is performed around the power-flow solution obtained in the previous iteration.
(2) In this convex optimization problem, we explicitly impose constraints on the power injections to guarantee the existence of the exact AC power-flow solution that is *non-singular* (namely, having a non-singular power-flow Jacobian) and *unique* in an analytically specified domain.
(3) After obtaining the optimal power injections in the convexified problem, we compute its exact AC power-flow solution using the algorithm in [12].
   a) If the exact AC power-flow solution is feasible in the sense that it truly satisfies the security constraints, then we set this solution as the next linearization point and move on to the next iteration.
   b) Otherwise, we shrink the constraints imposed by The-

The author from NREL (1st) and the authors from EPFL (2nd and 3rd) have equal contribution in the paper. The work of A. Bernstein, was supported by the U.S. Department of Energy under Contract No. DE-AC36-08GO28308 with the National Renewable Energy Laboratory. The U.S. Government retains and the publisher, by accepting the article for publication, acknowledges that the U.S. Government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this work, or allow others to do so, for U.S. Government purposes. The work of C. Wang and J.-Y. Le Boudec was supported by the Swiss National Foundation SNSF under grant NRP70 - "Energy Turnaround".

orem 2 in [12], recompute the solution and repeat this step until the conditions in (a) are satisfied.

We prove that the algorithm generates a non-singular sequence of power-flow solutions that, under certain conditions, approaches a local minimum in the original multiphase AC OPF problem. Moreover, all the members in this sequence belong to the same non-singular path-connected component in the voltage space, which means that we can move from one member to another without passing the steady-state stability limit.

Overall, the main advantages of the proposed algorithm are:

(i) It generates a sequence of non-singular points in the state space and a corresponding sequence of power injections; the latter can be used in real-time control settings to dispatch setpoints.

(ii) It is applicable to the general multiphase model and accommodates different types of connections, including wye, delta, and a combination thereof.

(iii) It does not require radiality of the network topology.

It is worth mentioning that feature (i) above is important in practice as it ensures that the *entire trajectory* from the current operating point to the optimal one satisfies the operational constraints and does not pass through the stability limits.

Recently, there were several works on successive linear or convex approximations for solving OPF problems (see, e.g., [11], [13], [14]). In particular, in [13], [14], a sequential linearization with consecutive power-flow verification (of the similar type proposed in our paper) was used to solve a certain planning problem for transmission networks; however, the approach was tailored to balanced networks (hence, no wye/delta connections were considered), and no convergence results were provided.

The paper is structured as follows. In Section II, we present the multiphase distribution network model. In Section III, we formulate the AC OPF problem and its convex relaxation using linearized power-flow model. In Section IV, we present our algorithm and main results. In Section V, we evaluate the performance of the algorithm on several typical test feeders. Finally, we present some concluding remarks in Section VI.

Notation. Upper-case (resp. lower-case) boldface letters are used for matrices (resp. column vectors); $(\cdot)^\intercal$ for transposition; and $|\cdot|$ for the absolute value of a number or the component-wise absolute value of a vector or a matrix. For a complex number $c \in \mathbb{C}$, $\Re\{c\}$ and $\Im\{c\}$ denote its real and imaginary part, respectively; and $\bar{c}$ denotes the conjugate of c. For an $N \times 1$ vector $\mathbf{x} \in \mathbb{C}^N$, $\|\mathbf{x}\|_\infty := \max(|x_1|...|x_n|)$, $\|\mathbf{x}\|_1 := \sum_{i=1}^N |x_i|$, and $\mathsf{diag}(\mathbf{x})$ returns an $N \times N$ matrix with the entries of $\mathbf{x}$ in its diagonal. For an $M \times N$ matrix $\mathbf{A} \in \mathbb{C}^{M \times N}$, the ℓ_∞-induced norm is defined as $||\mathbf{A}||_\infty = \max_{i=1,...,M} \sum_{j=1}^N |(\mathbf{A})_{ij}|$.

II. System Model

In this section, we briefly overview the general multiphase power-flow model recently proposed and analyzed in [12]. A simpler version, without delta loads and with only fully three-phase scenarios, can be found in the two-page letter [15].

A. Power-Flow Model

Consider a generic three-phase distribution network with one slack bus and N three-phase PQ buses. Each bus in the network can have different types of load/source connections, such as wye, ungrounded delta, and a combination thereof. This model is compactly described by the following set of non-linear equations:

$$\mathbf{s}^Y = \mathsf{diag}(\mathbf{v}) \left(\overline{\mathbf{Y}}_{L0}\overline{\mathbf{v}}_0 + \overline{\mathbf{Y}}_{LL}\overline{\mathbf{v}}\right) - \mathsf{diag}\left(\mathbf{H}^\intercal \overline{\mathbf{i}^\Delta}\right)\mathbf{v}, \tag{1a}$$

$$\mathbf{s}^\Delta = \mathsf{diag}\left(\mathbf{H}\mathbf{v}\right)\overline{\mathbf{i}^\Delta}. \tag{1b}$$

In (1), $\mathbf{s}^Y, \mathbf{s}^\Delta \in \mathbb{C}^{3N}$ are the complex power injection vectors for wye and delta connections, respectively; $\mathbf{v} \in \mathbb{C}^{3N}$ collects the complex voltages at every bus and phase of the PQ buses; $\mathbf{v}_0 \in \mathbb{C}^3$ collects the complex voltages at the slack bus; $\mathbf{i}^\Delta \in \mathbb{C}^{3N}$ collects the phase-to-phase currents at every bus; $\mathbf{Y}_{00} \in \mathbb{C}^{3\times3}, \mathbf{Y}_{L0} \in \mathbb{C}^{3N\times3}, \mathbf{Y}_{0L} \in \mathbb{C}^{3\times3N}$, and $\mathbf{Y}_{LL} \in \mathbb{C}^{3N\times3N}$ are the submatrices of the three-phase admittance matrix

$$\mathbf{Y} := \begin{bmatrix} \mathbf{Y}_{00} & \mathbf{Y}_{0L} \\ \mathbf{Y}_{L0} & \mathbf{Y}_{LL} \end{bmatrix} \in \mathbb{C}^{3(N+1)\times3(N+1)}, \tag{2}$$

which can be formed from the topology of the network, the π-model of the transmission lines, and other passive network devices, as shown in, e.g., [16]; and $\mathbf{H}$ is a $3N \times 3N$ block-diagonal matrix defined by

$$\mathbf{H} := \begin{bmatrix} \mathbf{\Gamma} & & \\ & \ddots & \\ & & \mathbf{\Gamma} \end{bmatrix}, \quad \mathbf{\Gamma} := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}. \tag{3}$$

Note that (1) defines an explicit mapping from the state vector $(\mathbf{v}^\intercal, (\mathbf{i}^\Delta)^\intercal)^\intercal \in \mathbb{C}^{6N}$ to the vector of power injections $\mathbf{s} := ((\mathbf{s}^Y)^\intercal, (\mathbf{s}^\Delta)^\intercal)^\intercal \in \mathbb{C}^{6N}$. For further development, we sometimes find it useful to represent the power-flow equations in real (rather than complex) space. To this end, let $\mathbf{x} := \left(\Re\{\mathbf{s}^Y\}^\intercal, \Im\{\mathbf{s}^Y\}^\intercal, \Re\{\mathbf{s}^\Delta\}^\intercal, \Im\{\mathbf{s}^\Delta\}^\intercal\right)^\intercal$ collect the active and reactive power injections of wye and delta sources. Also, let $\mathbf{y} := \left(\Re\{\mathbf{v}\}^\intercal, \Im\{\mathbf{v}\}^\intercal, \Re\{\mathbf{i}^\Delta\}^\intercal, \Im\{\mathbf{i}^\Delta\}^\intercal\right)^\intercal$ denote the vector of the state variables. Then, the power-flow equations can be written as

$$\mathbf{x} = \mathbf{h}(\mathbf{y}), \tag{4}$$

where $\mathbf{h} : \mathbb{R}^{12N} \to \mathbb{R}^{12N}$ is the mapping defined explicitly by (1). When need to revert the real-valued expressions back to the complex-valued counterparts, we define the operator $\mathrm{comp}()$ such that $\mathrm{comp}_s(\mathbf{x})$ and $\mathrm{comp}_v(\mathbf{y})$ represent the $\mathbf{s}$ and $\mathbf{v}$ that correspond to $\mathbf{x}$ and $\mathbf{y}$, respectively.

B. Conditions for Existence, Uniqueness, and Non-Singularity

In [12], explicit conditions for existence, uniqueness, and non-singularity of the solution to the multiphase power-flow equations were derived. We next summarize these conditions, as they will be used explicitly in our proposed AC OPF algorithm in Section IV; see [12] for a thorough review of other similar conditions recently proposed in the literature.

Let $\mathbf{w} \in \mathbb{C}^{3N}$ denote the zero-load voltage, which is given explicitly by $\mathbf{w} := -\mathbf{Y}_{LL}^{-1}\mathbf{Y}_{L0}\mathbf{v}_0$. Also, let $\mathbf{W} := \mathsf{diag}(\mathbf{w})$, and $\mathbf{L} := |\mathbf{H}|$ be the component-wise absolute value of the matrix $\mathbf{H}$ defined in (3). For $\mathbf{s} := ((\mathbf{s}^Y)^\intercal, (\mathbf{s}^\Delta)^\intercal)^\intercal \in \mathbb{C}^{6N}$ define the following norm $\xi(\cdot)$ on $\mathbb{C}^{6N}$ (see Lemma 1 in [12]):

$$\xi^Y(\mathbf{s}) := \left\| \mathbf{W}^{-1}\mathbf{Y}_{LL}^{-1}\mathbf{W}^{-1}\mathsf{diag}(\mathbf{s}^Y) \right\|_\infty, \tag{5a}$$

$$\xi^\Delta(\mathbf{s}) := \left\| \mathbf{W}^{-1}\mathbf{Y}_{LL}^{-1}\mathbf{H}^\intercal\mathsf{diag}(\mathbf{L}|\mathbf{w}|)^{-1}\mathsf{diag}(\mathbf{s}^\Delta) \right\|_\infty, \tag{5b}$$

$$\xi(\mathbf{s}) := \xi^Y(\mathbf{s}) + \xi^\Delta(\mathbf{s}), \tag{5c}$$

where $|\mathbf{w}|$ is the component-wise absolute value of the vector $\mathbf{w}$, and $\|\mathbf{A}\|_\infty$ is the induced ℓ_∞-norm of a complex matrix $\mathbf{A}$.

Next, for $\mathbf{v} \in \mathbb{C}^{3N}$, let

$$\alpha(\mathbf{v}) := \min_j \frac{|(\mathbf{v})_j|}{|(\mathbf{w})_j|} \tag{6a}$$

$$\beta(\mathbf{v}) := \min_j \frac{|(\mathbf{H}\mathbf{v})_j|}{(\mathbf{L}|\mathbf{w}|)_j} \tag{6b}$$

$$\gamma(\mathbf{v}) := \min\{\alpha(\mathbf{v}),\, \beta(\mathbf{v})\}\,. \tag{6c}$$

Finally, for any $\rho \geq 0$, define

$$\mathcal{D}_\rho(\mathbf{v}) := \{\mathbf{v}' : |(\mathbf{v}')_j - (\mathbf{v})_j| \leq \rho|(\mathbf{w})_j|,\, j = 1 \ldots 3N\}. \tag{7}$$

We next wrap up the main results from [12] for completeness.

Theorem 1 (Theorems 2 and 3 in [12])**.** *Let $\widehat{\mathbf{v}}$ be a given solution to the power-flow equations with power injection $\widehat{\mathbf{s}}$ satisfying:*

$$\xi(\widehat{\mathbf{s}}) < (\gamma(\widehat{\mathbf{v}}))^2, \tag{8}$$

where $\xi(\cdot)$ and $\gamma(\cdot)$ are given in (5) *and* (6)*, respectively. Consider some other candidate power injections vector $\mathbf{s}$, and assume that*

$$\xi(\mathbf{s} - \widehat{\mathbf{s}}) < \frac{1}{4}\left(\frac{(\gamma(\widehat{\mathbf{v}}))^2 - \xi(\widehat{\mathbf{s}})}{\gamma(\widehat{\mathbf{v}})}\right)^2. \tag{9}$$

Let

$$\rho^\ddagger(\widehat{\mathbf{v}}, \widehat{\mathbf{s}}) := \frac{1}{2}\left(\frac{(\gamma(\widehat{\mathbf{v}}))^2 - \xi(\widehat{\mathbf{s}})}{\gamma(\widehat{\mathbf{v}})}\right) \tag{10a}$$

$$\rho^\dagger(\widehat{\mathbf{v}}, \widehat{\mathbf{s}}, \mathbf{s}) := \rho^\ddagger(\widehat{\mathbf{v}}, \widehat{\mathbf{s}}) - \sqrt{(\rho^\ddagger(\widehat{\mathbf{v}}, \widehat{\mathbf{s}}))^2 - \xi(\mathbf{s} - \widehat{\mathbf{s}})} \tag{10b}$$

Then:

(i) The power-flow solution $(\widehat{\mathbf{v}}, \widehat{\mathbf{s}})$ is non-singular, in the sense that the Jacobian matrix of the mapping $\mathbf{h}$ defined in (4) *evaluated at $\left(\widehat{\mathbf{v}}, \widehat{\mathbf{i}}^\Delta := \mathsf{diag}^{-1}\left(\mathbf{H}\overline{\widehat{\mathbf{v}}}\right)\overline{\widehat{\mathbf{s}}}^\Delta\right)$ is invertible;*

(ii) There exists a unique power-flow solution $\mathbf{v}$ in $\mathcal{D}_\rho(\widehat{\mathbf{v}})$ defined in (7) *with $\rho = \rho^\ddagger(\widehat{\mathbf{v}}, \widehat{\mathbf{s}})$;*

(iii) The solution is located in $\mathcal{D}_\rho(\widehat{\mathbf{v}})$ with $\rho = \rho^\dagger(\widehat{\mathbf{v}}, \widehat{\mathbf{s}}, \mathbf{s})$;

(iv) The solution $\mathbf{v}$ satisfies $\xi(\mathbf{s}) < (\gamma(\mathbf{v}))^2$, hence is non-singular as well;

(v) The solution $\mathbf{v}$ can be found by iteration (11) *starting from anywhere in $\mathcal{D}_\rho(\widehat{\mathbf{v}})$ with $\rho = \rho^\ddagger(\widehat{\mathbf{v}}, \widehat{\mathbf{s}})$:*

$$\mathbf{v} \leftarrow \mathbf{w} + \mathbf{Y}_{LL}^{-1}\left(\mathsf{diag}(\overline{\mathbf{v}})^{-1}\overline{\mathbf{s}}^Y + \mathbf{H}^T\mathsf{diag}\left(\mathbf{H}\overline{\mathbf{v}}\right)^{-1}\overline{\mathbf{s}}^\Delta\right). \tag{11}$$

Once $\mathbf{v}$ is obtained, $\mathbf{i}^\Delta$ can be uniquely recovered using (1).

Remark 1. For general multiphase networks, the vectors $\mathbf{v}$, $\mathbf{s}^Y$, and $\mathbf{w}$ collect their corresponding electrical quantities for the existent phases; the vectors $\mathbf{i}^\Delta$ and $\mathbf{s}^\Delta$ collect the electrical quantities for the existent phase-to-phase connections; $\mathbf{H}$ contains rows that correspond to the existent phase-to-phase connections. More precisely, $\mathbf{H}$ is an $N^\Delta \times N^{phase}$ matrix, where N^Δ is the total number of phase-to-phase connections and N^{phase} is the total number of phases. When there are no phase-to-phase connections in the network, everything still holds after removing the terms that involve $\mathbf{H}$, $\mathbf{L}$, $\mathbf{i}^\Delta$ and $\mathbf{s}^\Delta$.

III. OPF Problem Formulation

We consider the following prototypical multiphase AC OPF problem

$$\text{(P0)} \quad \min_{\mathbf{x}\in\mathbb{R}^{12N}, \mathbf{y}\in\mathbb{C}^{12N}} \quad f(\mathbf{x}) \tag{12a}$$

$$\text{subject to}: \quad \mathbf{x} \in \mathcal{X} \tag{12b}$$

$$\mathbf{x} = \mathbf{h}(\mathbf{y}) \tag{12c}$$

$$V^{\min} \leq |\mathrm{comp}_v(\mathbf{y})| \leq V^{\max}, \tag{12d}$$

where $f(\cdot)$ is a convex function, $\mathcal{X} \subseteq \mathbb{R}^{12N}$ is a convex compact set, $\mathbf{h}(\cdot)$ is the multiphase power-flow mapping, and $|\mathrm{comp}_v(\mathbf{y})| \in \mathbb{R}^{3N}$ is the vector of voltage magnitudes that is decided by $\mathbf{y}$. (Recall the definition of $\mathrm{comp}()$ at the end of Section II-A.)

As it is well-known, (P0) is a non-convex problem due to the non-linear equality constraints (12c) and the lower bound constraint in (12d). Also, from the practical point of view, there might be several solutions to the power-flow equations (12c), some of which might be singular[1].

To tackle these challenges, we propose a successive linear approximation of the constraints (12c) and (12d). To this end, let $\widehat{\mathbf{x}}$ be a feasible power injection vector with a voltage $\widehat{\mathbf{v}}$ that satisfy condition (8). Consider now a linear approximation of the voltage magnitudes as a function of power injections:

$$|\mathbf{v}| \approx \mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\mathbf{x} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} \tag{13}$$

with the property that $\mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\widehat{\mathbf{x}} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} = |\widehat{\mathbf{v}}|$. This linear approximation can be efficiently computed, for example, by computing the inverse of the load-flow Jacobian, or any other method; see, e.g., [12], [18], [19]. In this paper, we assume that the matrix $\mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} \in \mathbb{R}^{12N\times 3N}$ and vector $\mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} \in \mathbb{R}^{3N}$ are *continuous* in $(\widehat{\mathbf{v}}, \widehat{\mathbf{x}})$. These assumptions hold for the approximations proposed in, e.g., [12]. Let

$$\epsilon^{\max}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} := \frac{1}{4}\left(\frac{(\gamma(\widehat{\mathbf{v}}))^2 - \xi(\widehat{\mathbf{s}})}{\gamma(\widehat{\mathbf{v}})}\right)^2, \tag{14}$$

and for any $\epsilon \in (0, \epsilon^{\max}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})})$, consider the following *local* optimization problem:

$$(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon)) \quad \min_{\mathbf{x}\in\mathbb{R}^{12N}} \quad f(\mathbf{x}) \tag{15a}$$

$$\text{subject to}: \quad \mathbf{x} \in \mathcal{X} \tag{15b}$$

$$V^{\min} \leq \mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\mathbf{x} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} \leq V^{\max} \tag{15c}$$

$$\xi\left(\mathrm{comp}_s(\mathbf{x} - \widehat{\mathbf{x}})\right) \leq \epsilon, \tag{15d}$$

where $\xi(\cdot)$ is defined in (5)[2]. Observe that $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ is a convex optimization problem, and constraint (15d) guarantees existence, uniqueness, and non-singularity of the exact power-flow solution as prescribed by Theorem 1. An iterative algorithm based on $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ is proposed in the next section.

Remark 2. Other constraints can be added to (P0) and (P1). For example, if we plan to consider the ampacity of branch currents, we could approximate the branch current by affine functions of $\mathbf{x}$ that are similar to (13). Indeed, this can be efficiently conducted, since the branch currents are linear functions of the nodal voltages in generic two-port device modeling. However, for the ease of exposition, we prefer to analyze here a simpler problem formulation.

IV. Algorithm and Main Result

In this section, we propose a successive convex approximation algorithm that is based on solving a sequence of $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ until convergence. In order to initialize the algorithm, a *strictly feasible* solution $(\mathbf{v}^{(0)}, \mathbf{x}^{(0)})$ to the power-flow problem is required, in the sense that

$$\mathbf{x}^{(0)} \in \mathcal{X},\, V^{\min} < |\mathbf{v}^{(0)}| < V^{\max}. \tag{16}$$

In addition, it is required that this solution satisfies (8). Practically, such initial values can be obtained from state estimation procedures. However, in the cases where state estimation is not available, one can either simply choose $\mathbf{v}^{(0)} = \mathbf{w}$ and $\mathbf{x}^{(0)} = \mathbf{0}$ as $\mathbf{w}$ is almost always strictly feasible, or utilize techniques as in [11] to find a feasible point.

The algorithm is given in Algorithm 1.

[1] See [17] for a practical example where singular state can be feasible.

[2] Note that (5) defines a norm on $\mathbb{C}^{6N}$ which is equivalent to the norm on $\mathbb{R}^{12N}$.

Algorithm 1 Multiphase AC OPF

Input: Strictly feasible power-flow solution $(\mathbf{v}^{(0)}, \mathbf{x}^{(0)})$, satisfying (16) and (8).
Input: A parameter $\beta \in (0, 1)$.
Output: $(\mathbf{v}^*, \mathbf{x}^*)$, e, $\{\mathbf{v}^{(k)}, \mathbf{x}^{(k)}, \epsilon_{\text{seq}}^{(k)}\}$

1: $k \leftarrow 0$, flagA$\leftarrow 1$
2: **while** flagA **do**
3:   Compute the linear model $(\mathbf{K}_{(\mathbf{v}^{(k)}, \mathbf{x}^{(k)})}, \mathbf{a}_{(\mathbf{v}^{(k)}, \mathbf{x}^{(k)})})$ using $(\mathbf{v}^{(k)}, \mathbf{x}^{(k)})$.
4:   $n \leftarrow 0$, flagB$\leftarrow 1$
5:   **while** flagB **do**
6:     Set $\epsilon^{(n)} := \beta^{n+1} \epsilon^{\max}_{(\mathbf{v}^{(k)}, \mathbf{x}^{(k)})}$, where $\epsilon^{\max}_{(\mathbf{v}^{(k)}, \mathbf{x}^{(k)})}$ is given in (14).
7:     {SOLVE THE LOCAL OPTIMIZATION PROBLEM}
8:     Solve $(\text{P1}(\mathbf{v}^{(k)}, \mathbf{x}^{(k)}, \epsilon^{(n)}))$, obtain the optimal solution $\widetilde{\mathbf{x}}$.
9:     {SOLVE THE POWER-FLOW EQUATIONS FOR THE EXACT SOLUTION}
10:     Solve the power-flow equations for $\widetilde{\mathbf{x}}$ using the method in [12], let $\widetilde{\mathbf{v}}$ denote the solution,
11:     {CHECK FEASIBILITY}
12:     **if** $\widetilde{\mathbf{v}}$ is strictly feasible **then**
13:       $\mathbf{v}^{(k+1)} \leftarrow \widetilde{\mathbf{v}}$, $\mathbf{x}^{(k+1)} \leftarrow \widetilde{\mathbf{x}}$, $\epsilon_{\text{seq}}^{(k)} \leftarrow \epsilon^{(n)}$, flagB$\leftarrow 0$
14:     **else**
15:       $n \leftarrow n + 1$
16:     **end if**
17:   **end while**
18:   **if** $\left| f(\mathbf{x}^{(k+1)}) - f(\mathbf{x}^{(k)}) \right| <$ ErrBound **then**
19:     flagA$\leftarrow 0$, $e \leftarrow \left\| \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)} \right\|_2$
20:   **end if**
21:   $k \leftarrow k + 1$
22: **end while**
23: $\mathbf{v}^* \leftarrow \mathbf{v}^{(k)}$, $\mathbf{x}^* \leftarrow \mathbf{x}^{(k)}$
24: **return** $(\mathbf{v}^*, \mathbf{x}^*)$, e, $\{\mathbf{v}^{(k)}, \mathbf{x}^{(k)}, \epsilon_{\text{seq}}^{(k)}\}$

Remark 3. In fact, each obtained $(\mathbf{v}^{(k)}, \mathbf{x}^{(k)})$ has an equivalent $(\mathbf{x}^{(k)}, \mathbf{y}^{(k)})$. The latter will be used in the statement of the main theorem.

Remark 4. Some of the practical scenarios may require that the objective function explicitly involves the electrical state $\mathbf{y}$. If that is the case, then we want to minimize $f(\mathbf{x}, \mathbf{y})$. To extend the proposed method to these scenarios, we have to make following modifications:

- In $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$, an extra constraint $\mathbf{y} = \mathbf{K}'_{(\widehat{\mathbf{v}}, \widehat{\mathbf{x}})}\mathbf{x} + \mathbf{a}'_{(\widehat{\mathbf{v}}, \widehat{\mathbf{x}})}$ should be added, where $\mathbf{K}'_{(\widehat{\mathbf{v}}, \widehat{\mathbf{x}})}$ and $\mathbf{a}'_{(\widehat{\mathbf{v}}, \widehat{\mathbf{x}})}$ can be obtained in the way given by [12];
- In line 12 of the algorithm, the condition should be replaced by "If $\widetilde{\mathbf{v}}$ is strictly feasible and $f(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \leq f(\mathbf{x}^{(k)}, \mathbf{y}^{(k)})$".

For brevity, in this paper, we focus on the cases where $f(\cdot)$ does not explicitly involve $\mathbf{y}$.

Below is our convergence result regarding Algorithm 1. The proof is deferred to the Appendix.

Theorem 2. *The following statements hold for Algorithm 1:*

(i) The sequence of solutions $(\mathbf{x}^{(k)}, \mathbf{y}^{(k)})$ *is*
- *feasible for problem (P0);*
- *non-singular, in the sense that the Jacobian matrix of* $\mathbf{h}(\cdot)$ *evaluated at* $\mathbf{y}^{(k)}$ *is invertible;*
- *connected to each other via non-singular paths.*

(ii) $f(\mathbf{x}^{(k)}) \leq f(\mathbf{x}^{(k-1)})$ *for all* k.

(iii) When ErrBound > 0, *the algorithm converges (stops) in a finite number of steps. Upon convergence, if* $e = 0$, *then* $(\mathbf{x}^*, \mathbf{y}^*)$ *is a local minimum of (P0).*

(iv) In addition, when ErrBound *is set to 0, we obtain an infinite sequences such that:*

(a) If

$$\epsilon_{\min} := \liminf_{k \to \infty} \epsilon_{\text{seq}}^{(k)} > 0, \tag{17}$$

we have that any *limit point of the sequence of solutions* $\{\mathbf{x}^{(k)}, \mathbf{y}^{(k)}\}$ *produced by Algorithm 1 is a* non-singular local minimum *of (P0).*

(b) If

$$\epsilon_{\max} := \limsup_{k \to \infty} \epsilon_{\text{seq}}^{(k)} > 0, \tag{18}$$

then there exists a limit point of the sequence $\{\mathbf{x}^{(k)}, \mathbf{y}^{(k)}\}$ *that is a* non-singular local minimum *of (P0).*

Remark 5. We note that Theorem 2 does not establish convergence of the sequence of solutions to a local minimum in general. Rather, under condition (17), it is guaranteed that any limit point is a local minimum, while under a milder condition (18), we know that there exists such a limit point. In case the sequence indeed converges, both part (iv) (a) and (b) state that it converges to a local minimum. Conditions under which convergence in solutions is guaranteed is left for future work.

Remark 6. Conditions (17) and (18) are *a-posteriori* conditions that cannot be verified before the algorithm's run. As we show in the numerical examples in Section V, these conditions are typically satisfied in practice.

V. Performance Evaluation

In this section, we first illustrate the performance of the algorithm using a simple example, which involves a single-phase two-bus network[3] and a quadratic objective function. Then, we adopt the IEEE 37-bus and 123-bus test feeders to evaluate performance of the proposed method. In particular, (i) the regulators in the two feeders are set to their nominal values; (ii) the objective function is linear with randomly generated coefficients. In addition, note that (15d) is essentially a collection of linear constraints imposed on $|\mathbf{s} - \widehat{\mathbf{s}}|$. Therefore, we introduce auxiliary variables and formulate $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ as a second-order conic programming (SOCP) problem, which can be efficiently solved by the interior-point method.

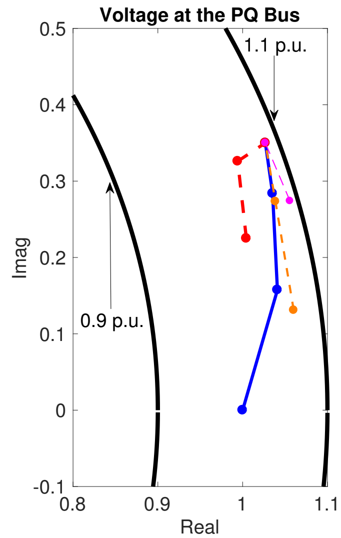
Finally, we assume that (i) the slack bus voltages are known and set to nominal values; (ii) the voltage bounds are 0.9 and 1.1 p.u.; (iii) ErrBound is 0.0001 in all numerical analyses unless otherwise specified.

A. Illustrative Example

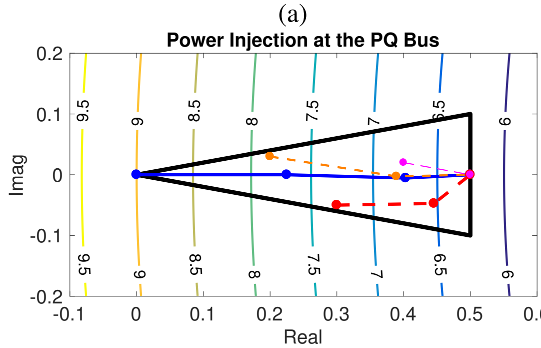
Here, we consider a single-phase two-bus network, where the PQ bus is directly attached to the slack bus via a line with series impedance $0.3 + 0.7\jmath$ p.u. Suppose that (i) there is no shunt element; (ii) the PQ-bus power injection belongs to the triangular region shown on Figure 1(b), which defines $\mathcal{X}$; (iii) the objective function is $\|\mathbf{x} - (3, 0)^T\|_2^2$. Note that, with this objective function, we try to encourage the local (active) power generation.

In this network, it is easy to check that $\mathbf{v}^{(0)} = 1$ and $\mathbf{x}^{(0)} = (0, 0)^T$ form a valid initial point. With this initial point and $\beta = 0.9$, the algorithm converges in 3 steps (i.e., 4 instances of $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ are solved). We plot the trajectory of voltage $\mathbf{v}^{(k)}$ in Figure 1(a) and

[3] In fact, it is equivalent to a balanced three-phase network.



(a)



(b)

Figure 1: Trajectories in (a) voltage space and (b) power injection space. For each trajectory of (a), the lowest point represents $\mathbf{v}^{(0)}$; for each trajectory of (b), the leftmost point stands for $\mathbf{x}^{(0)}$. Note that in voltage space, a point is nonsingular if its real part is not 0.5 p.u.

its corresponding trajectory $\mathbf{x}^{(k)}$ in Figure 1(b) via thick solid lines. Clearly, the obtained $\mathbf{v}^{(k)}$ are all strictly feasible and $\mathbf{x}^* = (0.5, 0)^T$ is the unique minimum in $\mathcal{X}$. Moreover, the obtained points are nonsingular and connected to each other via nonsingular paths. Further consider that this network can be solved analytically, therefore we are able to generate other valid initial values for $\mathbf{v}^{(0)}$ and $\mathbf{x}^{(0)}$. Similar to the case where $\mathbf{v}^{(0)} = 1$ and $\mathbf{x}^{(0)} = (0,0)^T$, we randomly generate some $\mathbf{v}^{(0)}$, $\mathbf{x}^{(0)}$ and plot their corresponding trajectories in Figure 1 via dashed lines of different thickness. As can be seen, all of them are feasible and nonsingular in voltage space and converge to the same optimal value.

It is also worth noticing that, when set $\mathrm{ErrBound}$ to 0, we obtain numerically $\liminf_{k\to\infty} \epsilon_{\text{seq}}^{(k)} = \limsup_{k\to\infty} \epsilon_{\text{seq}}^{(k)} = 0.1211$. This means that the conditions in the fourth item of Theorem 2 hold.

B. IEEE 37-Bus Feeder

In the IEEE 37-bus feeder, all the loads are delta-connected. We choose $\mathbf{v}^{(0)} = \mathbf{w}$, $\mathbf{x}^{(0)} = \mathbf{0}$, which is feasible for this network.

Now, for a positive real number κ, construct set $\mathcal{X}$ to be: $\mathcal{X} = \{\mathbf{x} : \kappa x_j^{\text{benchmark}} \leq x_j \leq 0, j \in \{1, ..., 12N\}\}$, where x_j is the j-th entry of $\mathbf{x}$ and $x_j^{\text{benchmark}}$ is the benchmark value of x_j (coming from the feeder data) that is negative due to consumption.

First, we analyze how the required number of local problem $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ (i.e., the number of inner loops) depends on κ. In other words, we want to test whether a larger set $\mathcal{X}$ results in a much greater number of local problem $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ to solve. Indeed, this is important for the algorithm performance as the total complexity includes the complexity of $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ and the number of $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ instances solved. For a good algorithm, the number of required $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ should not change dramatically with the size of $\mathcal{X}$.

We fix β at 0.8 and gradually increase κ from 1 to 6. The numerical results show that the algorithm finishes after solving (i) 1 local problem $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ when $\kappa = 1$; (ii) 5 to 7 local problem $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ when $\kappa = 6$. Thus, the algorithm scales well with the size of $\mathcal{X}$.

Next, we fix the value of κ at 6, and check the impact of β on the required number of local problem $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ to solve. By letting β be from $\{0.2, 0.4, 0.6, 0.8\}$, we observe that the number of local problem $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ decreases as β increases. In fact, this is partially attributed to the accuracy of the linear approximation (13). In particular, the fact that the linear model is accurate means that the true values of the state variables are accurately represented by their approximations. Therefore, if the approximations are feasible, then the true values are also feasible in most of the cases. By such observation, any small β becomes conservative and only leads to more instances of $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ to solve.

C. IEEE 123-Bus Feeder

In the IEEE 123-bus feeder, we add an additional delta-connected power injection $(-0.03 - 0.01\jmath, -0.03 - 0.01\jmath, -0.03 - 0.01\jmath)$ p.u. to bus 1, and hence create a bus with mixed wye-delta load. Denote the modified benchmark power vector as $\mathbf{x}^{\text{benchmark}}$ and let (i) $\mathbf{v}^{(0)}$ be the solution to $\mathbf{x}^{\text{benchmark}}$ that is guaranteed in $\mathcal{D}_{\rho^\ddagger(\mathbf{w},\mathbf{0})}(\mathbf{w})$ by Theorem 1; (ii) $\mathbf{x}^{(0)} = \mathbf{x}^{\text{benchmark}}$. Then, after repetition of the analysis in the IEEE 37-bus feeder, we obtain similar results. In particular, for $\kappa = 6$, the number of required $(\text{P1}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon))$ is between 8 and 11. This proves that the algorithm is able to work on networks that have sizes comparable to the IEEE 123-bus feeder.

However, we also notice a limitation of the performance. For large networks, the number of constraints coming from (15d) is very large. In order to reduce the complexity, we have to manually remove some of the constraints that are redundant (i.e., the ones that are implied by some other constraints). For radial networks with negligible shunt elements, this can be done through the way given in [20]. Nonetheless, how to reduce the complexity in general networks still requires further study.

VI. Conclusion

In the paper, we propose an optimal power flow method for multiphase power systems. The method does not require any specific network topology and is able to accommodate different types of load/source connections. Moreover, the method generates a sequence of non-singular points in the state space and a corresponding sequence of power injections, which can be used in real-time control settings. Under certain conditions, the sequences attain the local minimum. Despite the advantages of this method, there are several limitations as we mentioned in the paper. The improvement is left as a future work.

Appendix

Auxiliary Sets and Lemmas

A. Auxiliary Sets

For any $\epsilon > 0$, define the open set

$$\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}}, \widehat{\mathbf{x}}, \epsilon) := \Big\{\mathbf{x} \in \mathcal{X} : V^{\min} < \mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\mathbf{x} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} < V^{\max}, \\ \xi(\text{comp}_s(\mathbf{x} - \widehat{\mathbf{x}})) < \epsilon\Big\} \quad (19)$$

and set

$$\mathcal{Y}(\widehat{\mathbf{x}},\epsilon) := \Big\{\mathbf{x}\in\mathcal{X} :\ \exists\mathbf{y}\in\mathbb{C}^{12N} \text{ s.t. } \mathbf{x}=\mathbf{h}(\mathbf{y}),\\ V^{\min} < |\mathrm{comp}_v(\mathbf{y})| < V^{\max},\ \xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}})) < \epsilon\Big\}. \tag{20}$$

Moreover, let $\mathcal{B}(\widehat{\mathbf{x}},\epsilon) := \{\mathbf{x}\in\mathcal{X} :\ \xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}})) < \epsilon\}$. These auxiliary sets form the core of following lemmas.

B. Lemmas

Lemma 1. *Assume that $\widehat{\mathbf{x}}\in\mathcal{X}$ and the power flow solution $\widehat{\mathbf{v}}$ to $\widehat{\mathbf{x}}$ satisfy (i) $V^{\min} < \mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\widehat{\mathbf{x}} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} < V^{\max}$ and (ii) $\xi(\mathrm{comp}_s(\widehat{\mathbf{x}})) < (\gamma(\widehat{\mathbf{v}}))^2$. Then, there exists $\epsilon_0(\widehat{\mathbf{v}},\widehat{\mathbf{x}}) > 0$ such that for all $\epsilon\in(0,\epsilon_0(\widehat{\mathbf{v}},\widehat{\mathbf{x}})]$, we have that*

$$\mathcal{B}(\widehat{\mathbf{x}},\epsilon) = \widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon) = \mathcal{Y}(\widehat{\mathbf{x}},\epsilon). \tag{21}$$

Proof. First, consider that:

1) $\mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\mathbf{x} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}$ is an affine function from $\mathbb{R}^{12N}$ to $\mathbb{R}^{3N}$;
2) $V^{\min} < \mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\widehat{\mathbf{x}} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} < V^{\max}$;
3) $\xi(\cdot)$ is a norm and defines a metric that is Lipschitz equivalent to the Euclidean distance.

Therefore, there exists $\epsilon_1(\widehat{\mathbf{v}},\widehat{\mathbf{x}}) > 0$ such that

$$\begin{aligned} &\xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}})) < \epsilon_1(\widehat{\mathbf{v}},\widehat{\mathbf{x}}) \\ \Rightarrow& V^{\min} < \mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\mathbf{x} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} < V^{\max} \end{aligned} \tag{22}$$

Namely, we have

$$\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon) = \mathcal{B}(\widehat{\mathbf{x}},\epsilon), \forall\epsilon\in(0,\epsilon_1(\widehat{\mathbf{v}},\widehat{\mathbf{x}})]. \tag{23}$$

Next, consider:

1) $|\widehat{\mathbf{v}}| = \mathbf{K}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}\widehat{\mathbf{x}} + \mathbf{a}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})} \Rightarrow V^{\min} < |\widehat{\mathbf{v}}| < V^{\max}$;
2) $\xi(\mathrm{comp}_s(\widehat{\mathbf{x}})) < (\gamma(\widehat{\mathbf{v}}))^2$;
3) Based on item 2), if $\xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}})) < \epsilon^{\max}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}$, then there is a solution in $\mathcal{D}_{\rho^\dagger}(\widehat{\mathbf{v}})$;
4) The value of $\rho^\dagger(\widehat{\mathbf{v}},\mathrm{comp}_s(\widehat{\mathbf{x}}),\mathrm{comp}_s(\mathbf{x}))$ is controlled by $\xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}}))$ as

$$\begin{aligned} &\rho^\dagger(\widehat{\mathbf{v}},\mathrm{comp}_s(\widehat{\mathbf{x}}),\mathrm{comp}_s(\mathbf{x})) \\ =&\rho^\ddagger(\widehat{\mathbf{v}},\mathrm{comp}_s(\widehat{\mathbf{x}})) - \sqrt{(\rho^\ddagger(\widehat{\mathbf{v}},\mathrm{comp}_s(\widehat{\mathbf{x}})))^2 - \xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}}))} \\ =&\rho^\ddagger(\widehat{\mathbf{v}},\mathrm{comp}_s(\widehat{\mathbf{x}}))\left(1-\sqrt{1-\frac{\xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}}))}{(\rho^\ddagger(\widehat{\mathbf{v}},\mathrm{comp}_s(\widehat{\mathbf{x}})))^2}}\right) \\ \le&\frac{\xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}}))}{\rho^\ddagger(\widehat{\mathbf{v}},\mathrm{comp}_s(\widehat{\mathbf{x}}))}. \end{aligned} \tag{24}$$

Therefore, there exists $\epsilon_2(\widehat{\mathbf{v}},\widehat{\mathbf{x}}) \in (0,\epsilon^{\max}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})}]$ such that

$$\begin{aligned} &\xi(\mathrm{comp}_s(\mathbf{x}-\widehat{\mathbf{x}})) < \epsilon_2(\widehat{\mathbf{v}},\widehat{\mathbf{x}}) \\ \Rightarrow&\exists\mathbf{y} \text{ s.t. } \mathbf{x}=\mathbf{h}(\mathbf{y}), V^{\min} < |\mathrm{comp}_v(\mathbf{y})| < V^{\max} \end{aligned} \tag{25}$$

Namely, we have

$$\mathcal{Y}(\widehat{\mathbf{x}},\epsilon) = \mathcal{B}(\widehat{\mathbf{x}},\epsilon), \forall\epsilon\in(0,\epsilon_2(\widehat{\mathbf{v}},\widehat{\mathbf{x}})]. \tag{26}$$

Let $\epsilon_0(\widehat{\mathbf{v}},\widehat{\mathbf{x}}) = \min\{\epsilon_1(\widehat{\mathbf{v}},\widehat{\mathbf{x}}),\epsilon_2(\widehat{\mathbf{v}},\widehat{\mathbf{x}})\}$, we complete the proof. □

Using Lemma 1, we are able to prove that at each iteration, we either output a feasible solution with a smaller objective value, or converge to a local minimum.

Lemma 2. *Assume that the conditions of Lemma 1 hold. Let $\widetilde{\mathbf{x}} = \arg\min_{\mathbf{x}\in\mathsf{cl}(\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon))} f(\mathbf{x})$ and suppose that $\widetilde{\mathbf{x}}\in\mathsf{cl}(\mathcal{Y}(\widehat{\mathbf{x}},\epsilon))$, for some $\epsilon\in(0,\epsilon^{\max}_{(\widehat{\mathbf{v}},\widehat{\mathbf{x}})})$. Then $\widetilde{\mathbf{x}} = \arg\min_{\mathbf{x}\in\mathsf{cl}(\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon))\cap\mathsf{cl}(\mathcal{Y}(\widehat{\mathbf{x}},\epsilon))} f(\mathbf{x})$. Moreover, if $\widetilde{\mathbf{x}} = \widehat{\mathbf{x}}$, then $\widehat{\mathbf{x}}$ is a local minimum of $f(\cdot)$ over $\mathsf{cl}(\mathcal{Y}(\widehat{\mathbf{x}},\infty))$.*

Proof. First, observe that under the conditions of the lemma, we have that

$$\begin{aligned} &f(\widetilde{\mathbf{x}}) \le f(\mathbf{x}),\ \forall\mathbf{x}\in\mathsf{cl}(\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon))\cap\mathsf{cl}(\mathcal{Y}(\widehat{\mathbf{x}},\epsilon)) \subseteq \mathsf{cl}(\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon)) \\ &\widetilde{\mathbf{x}}\in\mathsf{cl}(\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon))\cap\mathsf{cl}(\mathcal{Y}(\widehat{\mathbf{x}},\epsilon)). \end{aligned}$$

Therefore, the first part of the lemma trivially follows.

Second, notice that by Lemma 1, we have that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathcal{B}(\widehat{\mathbf{x}},\delta) \subseteq \widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon)\cap\mathcal{Y}(\widehat{\mathbf{x}},\epsilon) \subseteq \mathsf{cl}(\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon))\cap\mathsf{cl}(\mathcal{Y}(\widehat{\mathbf{x}},\epsilon)). \tag{27}$$

Indeed, this is true for $\delta=\epsilon$ if $\epsilon\le\epsilon_0(\widehat{\mathbf{v}},\widehat{\mathbf{x}})$, and $\delta=\epsilon_0(\widehat{\mathbf{v}},\widehat{\mathbf{x}})$ otherwise. Therefore, by the definition of a local minimum, if $\arg\min_{\mathbf{x}\in\mathsf{cl}(\widetilde{\mathcal{Y}}(\widehat{\mathbf{v}},\widehat{\mathbf{x}},\epsilon))\cap\mathsf{cl}(\mathcal{Y}(\widehat{\mathbf{x}},\epsilon))} f(\mathbf{x}) = \widehat{\mathbf{x}}$, then $\widehat{\mathbf{x}}$ is a local minimum.

□

PROOF OF THEOREM 2

C. Properties of $\mathbf{v}^{(k)}$, $\mathbf{x}^{(k)}$ and $\epsilon^{(k)}_{\mathrm{seq}}$

It is easy to verify that the sequences $\mathbf{v}^{(k)}$, $\mathbf{x}^{(k)}$ and $\epsilon^{(k)}_{\mathrm{seq}}$ have following properties:

- $\mathbf{x}^{(k+1)}\in\mathcal{Y}(\mathbf{x}^{(k)},\epsilon^{(k)}_{\mathrm{seq}})$;
- $\mathbf{x}^{(k+1)}$ minimizes $f()$ over $\mathsf{cl}(\widetilde{\mathcal{Y}}(\mathbf{v}^{(k)},\mathbf{x}^{(k)},\epsilon^{(k)}_{\mathrm{seq}}))$;
- $\epsilon^{(k)}_{\mathrm{seq}} < \epsilon^{\max}_{(\mathbf{v}^{(k)},\mathbf{x}^{(k)})}$;
- $\forall k$, $\mathbf{x}^{(k)}$ and $\mathbf{v}^{(k)}$ satisfies the conditions of Lemma 1.

These properties will be used in the subsequent proof of Theorem 2.

D. Proof of Theorem 2

First note that by Theorem 1, Lemma 1 and Lemma 2, we have that

- the inner loop on n in Algorithm 1 (lines 5-16) completes within finite number of steps with a strictly feasible solution for the constraints in (P0);
- $f(\mathbf{x}^{(k)})$ is an infinite monotonically non-increasing sequence if without the lines 18-20 in the algorithm;
- the Jacobian matrix of $\mathbf{h}(\cdot)$ evaluated at $\mathbf{y}^{(k)}$ is invertible;
- $\forall k$, $\mathbf{x}^{(k)}$ and $\mathbf{x}^{(k+1)}$ can be connected by a continuous path in $\mathcal{B}(\mathbf{x}^{(k)},\epsilon^{\max}_{(\mathbf{v}^{(k)},\mathbf{x}^{(k)})})$, which must have a non-singular (hence continuous) pre-image in the state space as guaranteed by Theorem 1.

This proves items (i) and (ii) of the theorem.

Now, consider that $f(\cdot)$ is continuous and $\mathcal{X}$ is compact, we have that the sequence $f(\mathbf{x}^{(k)})$ is bounded. By Monotone Convergence Theorem, $f(\mathbf{x}^{(k)})$ converges. In other words, given $\mathrm{ErrBound} > 0$, $\exists M\in\mathbb{N}$ such that $\left|f(\mathbf{x}^{(k+1)}) - f(\mathbf{x}^{(k)})\right| < \mathrm{ErrBound},\ \forall k\ge M$. Furthermore, by Lemma 2, $\mathbf{x}^*$ is a local minimum of $f(\cdot)$ over the feasible set (12b)–(12d) whenever $e=0$. This proves item (iii).

To prove item (iv) (a), note that the sequence $\{\mathbf{x}^{(k)}\}$ stays in a compact set, hence there exists a convergent subsequence. Let $\{\sigma(k)\}$ denote the indexes of a subsequence of $\{\mathbf{x}^{(k)}\}$ with a limit point $\mathbf{x}^*$, namely $\mathbf{x}^{\sigma(k)}\to\mathbf{x}^*$ as $k\to\infty$. Moreover, observe that, in the framework of the proposed Algorithm 1, the feasibility constraints can be wrapped up as $\boldsymbol{\phi}(\widehat{\mathbf{x}},\mathbf{x})\le 0$. In details, the continuous function $\boldsymbol{\phi}$ is defined as

$$\boldsymbol{\phi}(\widehat{\mathbf{x}},\mathbf{x}) := \begin{bmatrix}\mathbf{K}_{(\mathbf{g}(\widehat{\mathbf{x}}),\widehat{\mathbf{x}})}\mathbf{x} + \mathbf{a}_{(\mathbf{g}(\widehat{\mathbf{x}}),\widehat{\mathbf{x}})} - V^{\max} \\ V^{\min} - \mathbf{K}_{(\mathbf{g}(\widehat{\mathbf{x}}),\widehat{\mathbf{x}})}\mathbf{x} - \mathbf{a}_{(\mathbf{g}(\widehat{\mathbf{x}}),\widehat{\mathbf{x}})}\end{bmatrix},$$

where $\mathbf{g}(\cdot)$ is a continuous function defined by means of Theorem 1. Now we proceed by contradiction. Suppose that $\mathbf{x}^*$ is not a local minimum of (P0). Therefore, it is not a minimizer of (P1$(\mathbf{v}^*,\mathbf{x}^*,\epsilon)$) for any $\epsilon > 0$. In particular, there exists $\widetilde{\mathbf{x}}$ such that $\boldsymbol{\phi}(\mathbf{x}^*,\widetilde{\mathbf{x}}) < 0$, $\xi(\mathrm{comp}_s(\mathbf{x}^*-\widetilde{\mathbf{x}})) < \epsilon_{\min}$, and

$$f(\widetilde{\mathbf{x}}) < f(\mathbf{x}^*). \tag{28}$$

Now, from the fact that

(i) $\mathbf{x}^{\sigma(k)} \to \mathbf{x}^*$;
(ii) $\boldsymbol{\phi}(\cdot, \mathbf{x})$ is continuous for all $\mathbf{x}$; and
(iii) $\xi(\mathrm{comp}_x(\cdot))$ is continuous;
(iv) hypothesis (17) holds;

there exists k_0 such that for all $k \geq k_0$, we have that $\boldsymbol{\phi}(\mathbf{x}^{\sigma(k)}, \widetilde{\mathbf{x}}) \leq 0$ and $\xi(\mathrm{comp}_s(\mathbf{x}^{\sigma(k)} - \widetilde{\mathbf{x}})) \leq \epsilon_{\min} \leq \epsilon_{\mathrm{seq}}^{\sigma(k)}$. Therefore, $\widetilde{\mathbf{x}} \in \mathsf{cl}(\widetilde{\mathcal{Y}}(\mathbf{g}(\mathbf{x}^{\sigma(k)}), \mathbf{x}^{\sigma(k)}, \epsilon_{\mathrm{seq}}^{\sigma(k)}))$. Since $\mathbf{x}^{\sigma(k)+1}$ is an optimum in $\mathsf{cl}(\widetilde{\mathcal{Y}}(\mathbf{g}(\mathbf{x}^{\sigma(k)}), \mathbf{x}^{\sigma(k)}, \epsilon_{\mathrm{seq}}^{\sigma(k)}))$ by its definition, it follows that

$$f(\widetilde{\mathbf{x}}) \geq f(\mathbf{x}^{\sigma(k)+1}). \tag{29}$$

However, since $\{f(\mathbf{x}^{(k)})\}$ is a monotonically non-increasing sequence, we have that $f(\mathbf{x}^{\sigma(k)+1}) \geq f(\mathbf{x}^*)$. Combining this last inequality with (28) and (29), we obtain that

$$f(\mathbf{x}^{\sigma(k)+1}) \geq f(\mathbf{x}^*) > f(\widetilde{\mathbf{x}}) \geq f(\mathbf{x}^{\sigma(k)+1}),$$

a contradiction. Therefore, $\mathbf{x}^*$ is a local minimum of (P0).

For item (iv) (b), let $\{\sigma(k)\}$ be a sequence such that $\epsilon_{\mathrm{seq}}^{\sigma(k)} \to \epsilon_{\max}$ as $k \to \infty$. Let $\epsilon_0 \in (0, \epsilon_{\max})$. There exists k_0 such that for all $k \geq k_0$, we have that $\epsilon_{\mathrm{seq}}^{\sigma(k)} > \epsilon_0$. Now let $\{\sigma'(k)\}$ be a subsequence of $\{\sigma(k)\}$ such that $\mathbf{x}^{\sigma'(k)} \to \mathbf{x}^*$ for some $\mathbf{x}^*$. Then, the arguments in proof for item (iv) (a) apply to $\mathbf{x}^*$, by replacing $\{\sigma(k)\}$ with $\{\sigma'(k)\}$ and $\epsilon_{\min}$ with ϵ_0. This completes the proof of theorem.

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