TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

No. 109

THE TWISTED WING WITH ELLIPTIC PLAN FORM.

By Max M. Munk.

August, 1922.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

TECHNICAL NOTE NO. 109.

THE TWISTED WING WITH ELLIPTIC PLAN FORM.

By Max M. Munk.

SUMMARY

A method for computing the aerodynamic induction of wings with elliptic plan form if arbitrarily twisted.

REFERENCES.

(1) Munk. Beitrag zur Aerodynamik Technische Berichte, II.
(2) Munk. General Biplane Theory, N.A.C.A. Report No. 151.
(3) Betz. Beitrag zur Tragflugel Theorie.
(4) Munk. The Minimum Induced Drag of Aerofoils, N.A.C.A. Report No. 121.
(5) Prandtl. Applications of Modern Hydrodynamics to Aeronautics, N.A.C.A. Report No. 116.

The chief numerical results referring to the aerodynamic induction of single wings are derived from the investigation of wings with elliptic plan views with the angle of attack constant all over the span. It was found from wind tunnel tests (reference (1)) and later confirmed by means of a very laborious computation by A. Betz (reference (3)) that a wing with rectangular plan form has practically the same average induction as the elliptic wing. The plan form of actual wings is in general neither rectangular nor elliptic but something between these two, and it is often better described by comparing it with an ellipse than by comparing it with a rectangle. The results obtained for the elliptic wing are thus even more useful for actual wings than for rectangular wings. It seems therefore instructive and interesting to explore the elliptic wing farther and to investigate more general elliptic wings, no longer subjected to the condition of constant angle of attack. I will assume in this paper the plan view to be elliptic but the angle of attack to be variable and to be different at different distances from the center. I consider first cases where the angle of attack is some special function of the span b and proceed afterwards to the most general case where any twist or any distribution of the lift is given. The results can be applied beyond the original conception of a twisted elliptic wing. Some other problems can be treated by the consideration of wings with equivalent twist. These applications however are still under study and will be presented later separately.

The elliptic wing with the twist zero, that is, with constant geometric angle of attack is distinguished by an extremely simple relation between the effective angle of attack and the induced angle of attack. Under the usual assumptions (reference (3)) these two can be described by the same function of the span; the ratio of the two angles is constant all over the span. Hence this is also true for their sum, the geometric angle of attack, and all three angles are constant. The relation between the three angles is linear and independent of the distance from the center and this characteristic rather than the fact that the three angles themselves are constant makes the results so simple and useful. Therefore the question suggests itself whether there cannot be found other variations of the angle of attack for which the same relation holds true, viz., that the effective angle of attack and the induced angle of attack are expressed by the same function with a different constant factor, which function than also expresses the geometric angle of attack. It is not easy to arrive in a systematic way at the solution of this problem. It must be enough to describe in the next paragraph the solutions and to demonstrate that they really conform to the condition.

Let the span of the wing have the length 2, and the chord be $c = C\sqrt{1 - x^2}$ where C is a constant and x denotes the distance of the chord from the center of the wing. V may denote the velocity of flight and ρ the density of the air. As abbreviation write $\sin \varphi$ for x, that is, $\cos \varphi = \sqrt{1 - x^2}$.

Then the distribution of the lift is a solution of the problem if the density of the lift per unit of the span is proportional to $\sin n \left(\varphi + \frac{\pi}{2}\right)$.

To prove this, consider the two-dimensional flow in the transverse plane through the wing, characteristic for all quantities connected with the present problem, (references (4) and (5)). This flow is formed by the components of the actual flow at right angles to the direction of flight, provided that the wing is imagined to be concentrated into a straight line. The flow is a potential flow and hence can be represented by a complex function F which in the present case has the form

$$F = BV\left(\sqrt{1 - x^2} - i\,x\right)^n$$

x denotes here the complex independent variable and B a constant; n is an integer, and by substituting all integers for n infinitely many solutions of the problem are obtained. The wing is represented by the straight line between the two points $x = \pm 1$. The density of lift corresponding to this function F is the difference of the values the function assumes on opposite sides of the wing, multiplied by $2 V$, and hence is

$$L' = 8\, BV^2 \frac{\rho}{2} \cos n\,\varphi, \quad n \text{ odd}$$

resp.

$$= 8\, BV^2 \frac{\rho}{2} \sin n\,\varphi, \quad n \text{ even}$$

This lift requires for its production the effective angle of attack

$$\alpha_e = \frac{L'}{2\,\pi c q} = \frac{4B}{\pi C}\,\frac{\cos n\,\varphi}{\cos\,\varphi} \text{ resp. } \frac{4B}{\pi C}\,\frac{\sin n\,\varphi}{\cos\,\varphi}\,.$$

The induced angle of attack, if n is odd, is the imaginary part of dF/dx divided by V, otherwise it is the real part.

$$\alpha_i = n\,B\,\frac{\cos n\,\varphi}{\cos\,\varphi} \text{ resp. } n\,B\,\frac{\sin n\,\varphi}{\cos\,\varphi}$$

It is thus proven that the induced angle of attack and the effective angle of attack are represented by the same function and have always the same ratio.

This function can be described in a little simpler way by introducing the angle $(\varphi + \frac{\pi}{2})$ instead of φ, that is, by choosing the angle <u>zero</u> at one wing tip instead of at the center of one wing. If n is even, $\sin n\,\varphi = \pm \sin n\,(\varphi + \frac{\pi}{2})$ but if n is odd, $\cos n\,\varphi = \sin n\,(\varphi + \frac{\pi}{2})$. Hence, the following relations are valid for n being any integer.

$$L' = 8\,BV^2 \frac{\rho}{2} \sin n \left(\varphi + \frac{\pi}{2}\right)$$

$$\alpha_i = n\,B \frac{\sin n \left(\varphi + \frac{\pi}{2}\right)}{\sin \left(\varphi + \frac{\pi}{2}\right)}$$

From all these relations follows, as was to be expected, that for each of these distributions of lift, less favorable than the elliptic distribution $n = 1$, the induced angle of attack is comparatively greater than for the most favorable distribution.

The entire lift is always zero with the exception of $n = 1$. Then the lift is $4\pi B\ V^2\ \rho/2$.

II.

The distributions of lift discussed in the first part are distinguished by a constant ratio of the effective angle of attack to the induced angle of attack. For other distributions these two have by no means a constant ratio and that makes it more difficult to determine the effective angle of attack and hence the density of lift, if the geometric angle of attack is given at each point. The solution can be accomplished however by the use of these very particular distributions discussed before, and this it is which makes them useful.

Within the assumptions of this investigation the three angles of attack and the density of lift are connected with each other by linear relations and hence new solutions can be derived by means of the superposition of known ones. Two or more known solutions give a new solution by simply superposing the distributions of the given quantity, then the distribution of the unknown quantities are the superpositions of the corresponding original distributions. The natural way of attacking the general problem is therefore to attempt to express the given distribution of, say, the geometric angle of attack as the sum of several or infinite many of the particular distributions discussed right now. The solutions for each of the single terms thus obtained are then only to be added. Now this expansion of the given angle of attack is always possible and easy. The function expressing the geometric angle of attack is to be expanded into the series

$$\alpha_g = a_1 \frac{\sin \beta}{\sin \beta} + a_2 \frac{\sin 2\beta}{\sin \beta} + \ldots + a_n \frac{\sin n \beta}{\sin \beta} + \ldots$$

where β denotes $(\varphi + \frac{\pi}{2})$.

That is to say $(\alpha_g \sin \beta)$ is to be expanded into the Fourier series

$$\alpha_g \sin \beta = a_1 \sin \beta + a_2 \sin 2\beta + \ldots + a_n \sin n \beta + \ldots$$

which, as is well known, is done by determining the factors a by means of

$$a_n = \frac{2}{\pi} \int_0^\pi \alpha_g \sin \beta \quad \sin n \beta \, d\beta .$$

Then the effective angle of attack results

$$\alpha_e = \frac{1}{\sin \beta} \left(\frac{a_1 \sin \beta}{1 + 2\, S/b^2} + \frac{a_2 \sin 2\beta}{1 + 4\, S/b^2} + \ldots + \frac{a_n \sin n \beta}{1 + 2n\, S/b^2} + \ldots \right)$$

and the density of lift is accordingly

$$L' + 8q \frac{S}{b} \left(\frac{a_1 \sin \beta}{1 + 2\, S/b^2} + \frac{a_2 \sin 2\beta}{1 + 4\, S/b^2} + \ldots + \frac{a_n \sin n \beta}{1 + 2n\, S/b^2} + \ldots \right)$$

where S denotes the area of the wing, q the dynamic pressure and b the span.

The case that the distribution of the density of lift is given can be treated in a similar way by expanding the density of lift into a Fourier's series. Care has to be taken however to choose the distribution of lift so as to be physically possible. The derived series are now less rapidly convergent than the original series and if the lift is not realizable they are not convergent at all.