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CRITICAL EVALUATION OF THE UNSTEADY AERODYNAMICS APPROACH TO  
DYNAMIC STABILITY AT HIGH ANGLES OF ATTACK

Final Report

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During the period of the Grant - March 1, 1984 to February 28, 1985 - theoretical research has been conducted on the topics as proposed and the objectives of the proposal achieved. As a result, the following two conference papers have been completed.

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- (a) "Transient Motion of Hypersonic Vehicles Including Time History Effects", AIAA-85-0201, presented at AIAA 23rd Aerospace Sciences Meeting, January 14-17, 1985, Reno, Nevada; and

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- (b) "Bifurcation Theory Applied to Aircraft Motions", paper to be presented at NATO AGARD Symposium on "Unsteady Aerodynamics - Fundamentals and Applications to Aircraft Dynamics", May 6-9, 1985, Gottingen, West Germany.

~~Copies of these papers are enclosed herewith.~~

BIFURCATION THEORY APPLIED TO AIRCRAFT MOTIONS

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## ABSTRACT

Bifurcation theory is used to analyze the nonlinear dynamic stability characteristics of an aircraft subject to single-degree-of-freedom, e.g., pitching-or rolling-motion perturbations about its trimmed steady flight. The requisite moment of the aerodynamic forces in the equations of motion is shown to be representable in a form equivalent to the response to finite-amplitude oscillations. It is shown how this information can be deduced from the case of infinitesimal-amplitude oscillations. The bifurcation theory analysis reveals that when the bifurcation parameter, e.g., the angle of attack, is increased beyond a critical value at which the aerodynamic damping vanishes, new solutions representing finite-amplitude periodic motions bifurcate from the previously stable steady motion. The sign of a simple criterion, cast in terms of aerodynamic properties, determines whether the bifurcating solutions are stable (supercritical) or unstable (subcritical). For the pitching motion of flat-plate airfoils flying at supersonic/hypersonic speed and for oscillation of flaps at transonic speed, the bifurcation is subcritical, implying either that exchanges of stability between steady and periodic motion are accompanied by hysteresis phenomena, or that potentially large aperiodic departures from steady motion may develop. On the other hand, for the rolling motion of a slender delta wing in subsonic flight (wing rock) the bifurcation is found to be supercritical. This and the predicted amplitude of the bifurcation periodic motion are in good agreement with experiments.

## 1. INTRODUCTION

Problems of aerodynamic stability of aircraft flying at small angles of attack have been studied extensively. With increasing angles of attack the problems become more complicated and typically involve nonlinear phenomena such as coupling between modes, amplitude and frequency effects, and hysteresis. The need for investigating stability characteristics at high angles of attack was clearly demonstrated by Orlik-Ruckemann [1] in his survey paper which largely deals with experiments.

On the theoretical side, the greater part of an extensive body of work is based on the linearized theory, in which the unsteady flow is regarded as a small perturbation of some known steady flow (possibly nonlinear in, e.g., the angle of attack) that prevails under certain flight conditions. The question of the validity and limitations of such a linearized perturbation theory is of fundamental importance and yet has been investigated only rarely. One may argue that in principle, it should be possible to advance to higher and higher angles of attack  $\alpha$  by a series of linear perturbations, since the solution at each step should include a steady-state part which, when added to the previous steady-state solution, would provide the starting point for the next perturbation. This may well be true provided that at each step the steady motion is stable both statically and dynamically, and that the actual disturbances, e.g., the amplitude of oscillation, remain small. However, when the angle of attack exceeds a certain critical value  $\alpha_{cr}$  at which the steady motion is no longer stable, the linear theory predicts an exponential growth of the perturbation with time and, therefore, must itself cease to be valid after finite time. The motion of the aircraft under these conditions can only be studied using a nonlinear theory.

In this paper we investigate the stability characteristics of an aircraft trimmed to a mean angle of attack  $\alpha$  near  $\alpha_{cr}$  at which the steady motion becomes unstable. Padfield [2] studied a similar problem, using the method of multiple scales, which is valid only for weakly non-linear oscillations. We shall study the problem by means of bifurcation theory. This will allow us to draw on recent mathematical developments (e.g., [3]) that are particularly well suited to investigating fundamental questions in linear and nonlinear stability theory. A numerical scheme based on bifurcation theory was proposed earlier [4] for analyzing aircraft dynamic stability in a rather general framework. More recent work by Guichetenu [5] demonstrates the considerable potential of bifurcation theory in flight dynamics studies, particularly toward establishing a method for the design of flight control systems to ensure protection against loss of control. On the other hand, while acknowledging the importance of the aerodynamic model in determining the aircraft stability characteristics, neither of these works contains an adequate assessment of the model requirements. The treatment of unsteady flow effects, in particular, receives no attention. In contrast, we shall focus on just this aspect of the problem at the expense of narrowing the scope of the motion analysis.

We shall restrict ourselves to the simpler case of single-degree-of-freedom motions, e.g., pitching or rolling, of an aircraft about its trimmed flight condition. This will enable us to analyze motions for which complete aerodynamic information is available, for certain shapes, in the form of exact analytical or numerical solutions [6-12] of the unsteady inviscid flow equations, or empirically from experiments [13]. In this way it will be possible to establish a form revealing a precise analytical relationship between the basic aerodynamic coefficients and the characteristics of the motion.

Specifically we shall consider the following three types of oscillatory motions:

- A. Pitching supersonic/hypersonic aerofoils in rectilinear flight (Fig. 1);
- B. Flap oscillations in transonic flight (Fig. 2);
- C. Wing rock of slender delta wings in subsonic flight (Fig. 3).

## §2. MATHEMATICAL FORMULATION

Let the aircraft be in level, steady flight up until time  $t = 0$  when it is perturbed from its trim position. During the subsequent motion the center of gravity continues to follow a rectilinear path at constant velocity  $V_\infty$ . For a single-degree-of-freedom oscillatory motion, the equation of motion is

$$I \frac{d^2 \xi}{dt^2} = G(t; \lambda) \quad (1)$$

where the angular displacement  $\xi(t)$  is the instantaneous pitch angle in example A, flap deflection angle in example B and roll angle in example C,  $I$  is the respective moment of inertia, while  $G(t)$  is the corresponding instantaneous moment of the aerodynamic forces.

In Eq. (1),  $\lambda$  in general represents a set of parameters defining the steady flight at the trim condition, e.g., flight Mach number  $M_\infty$ , ratio of specific heats  $\gamma$ , angle of attack  $\alpha$ , etc. In this paper we shall consider  $\lambda$  to be the angle of attack  $\alpha$  in examples A and C, and the mean flap deflection angle  $\delta_m$  in example B. In other words, all the other defining parameters will be held fixed when considering the consequence of varying  $\lambda$  on the aircraft motion characteristics. We assume that the moment required to trim the aircraft at  $\lambda$  has been accounted for, so that  $G(t; \lambda)$  is a measure of the perturbation moment only.

It is clear that the instantaneous motion state,  $\xi(t)$  and  $\dot{\xi}(t)$ , and the instantaneous moment  $G(t)$  at time  $t$  are a result of the interaction of aircraft motion and the unsteady aerodynamic forces from time zero to time  $t$ .

Consequently, the instantaneous moment  $G(t)$  depends not only on the instantaneous motion state,  $\xi(t)$  and  $\dot{\xi}(t)$ , but also on the past motion history from time zero to  $t$ . This is to say that  $G(t)$  is a functional of  $\xi(t_1)$ , ( $0 \leq t_1 \leq t$ ) as shown by Tobak et. al. [14]. Thus

$$G(t) = G \left[ \begin{array}{c} t \\ \xi(t_1) \\ t_1=0 \end{array} \right] \quad (2)$$

Such a functional, which includes all time-history effects, is however rarely known.

Now for motions for which  $\xi(t_1)$  is analytic, the functional is equivalent to a function of an infinite set of variables, i.e.

$$\begin{aligned} G(t) &= G \left[ \begin{array}{c} t \\ \xi(t_1) \\ t_1=0 \end{array} \right] \\ &= G(\xi(t), \dot{\xi}(t), \ddot{\xi}(t), \dots, \frac{d^n \xi(t)}{dt^n}, \dots) \end{aligned} \quad (3)$$

For most problems encountered in the study of the dynamic stability of an aircraft, the motion is slow although its amplitude may be finite or large. Under these conditions  $\ddot{\xi}(t)$ ,  $\ddot{\ddot{\xi}}(t)$ , ... in Eq. (3) may be neglected and, as a first approximation, we get

$$G(t) = G(\xi(t), \dot{\xi}(t)) \quad (4)$$

We further assume that  $G$  is an analytic function of  $\xi$  and  $\dot{\xi}$ . Expanding (4) as Taylor series and neglecting terms  $O(\xi^2)$  and higher for slow motions we get, after re-introducing the parameter  $\lambda$ ,

$$G(t; \lambda) = G_0(\xi(t); \lambda) + \frac{\dot{\xi}(t)l}{V_\infty} G_1(\xi(t); \lambda) \quad (5)$$

where  $l$  is a characteristic length and

$$G_0(0; \lambda) = 0 \quad (6)$$

as required at the trim condition. We note that the form of the instantaneous moment  $G(t; \lambda)$  in Eq. (5) is consistent with the exact analytic solution for the pitching moment in example A [Ref. 10], the numerical solution for the hinge moment in example B [Ref. 12] and the experimentally derived empirical formula for the rolling moment in example C [Ref. 13].

Summing up, the single-degree-of-freedom motions considered in this paper will be based on the following mathematical problem

$$I \frac{d^2 \xi}{dt^2} = G_0(\xi; \lambda) + \frac{\dot{\xi}l}{V_\infty} G_1(\xi; \lambda) \triangleq G(\xi; \dot{\xi}; \lambda) \quad (7a)$$

$$\xi(0) = \hat{\xi}_0 \quad (7b)$$

$$\dot{\xi}(0) = \hat{\xi}_1 \quad (7c)$$

The functions  $G_0(\xi; \lambda)$  and  $G_1(\xi; \lambda)$  are generally nonlinear in  $\xi$  and have to be determined from the study of unsteady aerodynamics, either theoretically

or experimentally. Evidently  $G_0$  is related to the restoring moment and  $G_1$  to the damping moment.

In many situations, it is known that when the parameter  $\lambda$  reaches some critical value  $\lambda_{cr}$ , the aerodynamic damping  $G_1$  vanishes and the steady flight at the trim condition  $\lambda_{cr}$  loses its stability. The main purpose of this paper is to use Hopf bifurcation theory to determine the motion characteristics of the aircraft whose trim condition is near or beyond  $\lambda_{cr}$ .

### §3. BIFURCATION THEORY

We introduce the dimensionless time  $\tau = V_\infty t / \ell$ , where  $\ell$  is a characteristic length, equal to the chord length of the aerofoil in example A, the chord length of the flap in example B and the chord length of the wing in example C. Hereafter we use  $(\dot{\phantom{x}})$  to denote  $d/d\tau$ . We further let

$$\begin{aligned} F(\xi, \dot{\xi}; \lambda) &= G(\xi, \dot{\xi}; \lambda) / (I V_\infty^2 / \ell^2) \\ &= F_0(\xi; \lambda) + \dot{\xi} F_1(\xi; \lambda) \end{aligned} \quad (8)$$

then Eq. (7a) may be written

$$\frac{d\xi}{d\tau} = \dot{\xi} \quad (9a)$$

$$\frac{d\dot{\xi}}{d\tau} = F(\xi, \dot{\xi}; \lambda) \quad (9b)$$

An expansion of  $F(\xi, \dot{\xi}; \lambda)$  in a Taylor series in  $\xi$  and  $\dot{\xi}$  and a change of notation  $u_1 = \xi$ ,  $u_2 = \dot{\xi}$  yield for Eq. (9)

$$\begin{aligned} \dot{u}_i &= A_{ij}(\lambda) u_j + B_{ijk}(\lambda) u_j u_k + C_{ijkl}(\lambda) u_j u_k u_l \\ &+ O(|\vec{u}|^4), \quad (i = 1, 2) \end{aligned} \quad (10)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -S(\lambda) & -D(\lambda) \end{pmatrix} \quad (11a)$$

$$S(\lambda) = -F_0'(0; \lambda), \quad D(\lambda) = -F_1(0; \lambda) \quad (11b)$$

$$B_{1jk} = 0, \quad B_{2jk} = \frac{1}{2!} \left. \frac{\partial^2 F}{\partial u_j \partial u_k} \right|_{\vec{u}=0} \quad (11c)$$

$$C_{1jkl} = 0, \quad C_{2jkl} = \frac{1}{3!} \left. \frac{\partial^3 \Gamma}{\partial u_j \partial u_k \partial u_l} \right|_{u=0} \quad (11d)$$

(Although Eq. (9) has been derived on the assumption of slow oscillations, our subsequent bifurcation analysis of (9) will hold for general  $F(\xi, \dot{\xi}; \lambda)$ , i.e., as if no restriction had been placed on the magnitude of  $\dot{\xi}$ .)

In Eq. (11) the tensors B and C represent the effects of finite amplitude to the second and third order. We note that the following symmetry properties hold:

$$B_{2jk} = B_{2kj} \quad (12a)$$

$$C_{2jkl} = C_{2jlk} = C_{2kjl} = C_{2ljk} = C_{2lkj} \quad (12b)$$

On the basis of Eq. (10), we shall study now the linear and nonlinear stability of the motion.

### 3.1 LINEAR STABILITY THEORY.

The stability of the steady motion at the trim condition  $\lambda$  to infinitesimal disturbances is determined by the nature of the eigenvalues of the matrix A. They are

$$\eta_{\pm}(\lambda) = \frac{1}{2} [-D(\lambda) \pm \sqrt{D^2(\lambda) - 4S(\lambda)}] \quad (13)$$

Case I:  $S(\lambda) < 0$ . In this case  $\eta_1 > 0$ ,  $\eta_2 < 0$ . The steady motion at this condition  $\lambda$  is always unstable.

Case II:  $S(\lambda) > 0$  .

IIa:  $D(\lambda) < 0$  . In this case  $\text{Re}(\eta_1) > 0$  and the steady motion at  $\lambda$  is unstable.

IIb:  $D(\lambda) > 0$  . In this case  $\text{Re}(\eta_{\frac{1}{2}}) < 0$  and the steady motion at  $\lambda$  is stable.

Thus, only in Case IIb, when both stiffness derivative  $S(\lambda)$  and damping derivative  $D(\lambda)$  are positive, is the steady motion at the trim condition  $\lambda$  stable to infinitesimal disturbances. In fact, stability theory [3] can be used to show that stability of the steady motion in this case is assured if and only if the disturbance is sufficiently small.

In all cases in which the linear theory predicts growth of the disturbance amplitude, the growth predicted is of exponential form and hence the linear theory must cease to be valid after some finite time when the amplitude is no longer small. Thus, what eventually happens to a motion for which linear stability theory predicts a growth of disturbances cannot be determined from the linear theory itself. Instead, the full nonlinear inertial equations of motion, or a suitable approximation of them, such as (10), must be adopted to determine the ultimate state of the motion. Of particular interest is the dynamic stability boundary  $\lambda = \lambda_{cr}$ , where  $S(\lambda_{cr}) > 0$  and,  $D(\lambda_{cr}) = 0$  . The stability characteristics near this boundary will be studied presently.

### 3.2 HOPF BIFURCATION THEORY

At the dynamic stability boundary  $\lambda = \lambda_{cr}$ , we have  $S(\lambda_{cr}) > 0$

hence

$$\eta_{\frac{1}{2}}(\lambda_{cr}) = \pm i\sqrt{S(\lambda_{cr})} \equiv \pm i\omega_0 \quad (14')$$

The existence of purely imaginary eigenvalues of the matrix  $A$  at  $\lambda = \lambda_{cr}$  is the characteristic sign of a Hopf bifurcation [3,15], signaling a changeover from stable steady motion to periodic motion. On crossing  $\lambda_{cr}$ , the steady motion that had been stable for  $\lambda < \lambda_{cr}$  will become unstable to disturbances, resulting (after a transient motion has died away) in the existence of a new motion, which (if it is stable) will be periodic. In the vicinity of  $\lambda = \lambda_{cr}$ , the circular frequency of the periodic motion will be nearly equal to  $\omega_0$ . We call the new solution of the equations of motion a bifurcation solution. In this section we shall determine its character and a criterion for its stability.

For  $\lambda$  slightly larger than  $\lambda_{cr}$ , the eigenvalues of the matrix  $A$  are

$$\eta_{\frac{1}{2}} = -\frac{1}{2}D(\lambda) \pm i\Omega(\lambda) \quad (15)$$

where

$$\Omega(\lambda) = \sqrt{S(\lambda) - D^2(\lambda)/4} \quad (16)$$

We shall assume that

$$D'(\lambda_{cr}) < 0 \quad (17)$$

which is the usual case in applications. (The case  $D'(\lambda_{cr}) > 0$  can be created in exactly the same way.) The normalized eigenvector  $\vec{\zeta}(\lambda)$

associated with the eigenvalue  $\eta(\lambda)$  is

$$\vec{\zeta}(\lambda) = \begin{pmatrix} \zeta_1(\lambda) \\ \zeta_2(\lambda) \end{pmatrix} = \frac{1-i}{2\sqrt{\eta(\lambda)}} \begin{pmatrix} 1 \\ \eta(\lambda) \end{pmatrix} \quad (18)$$

whereas the adjoint eigenvector  $\vec{\zeta}^*(\lambda)$  with eigenvalue  $\bar{\eta}(\lambda)$ , which is the complex conjugate of  $\eta(\lambda)$ , is

$$\vec{\zeta}^*(\lambda) = \begin{pmatrix} \zeta_1^*(\lambda) \\ \zeta_2^*(\lambda) \end{pmatrix} = \frac{1+i}{2\sqrt{\bar{\eta}(\lambda)}} \begin{pmatrix} \bar{\eta}(\lambda) + D(\lambda) \\ 1 \end{pmatrix} \quad (19)$$

A. Hopf Bifurcation. The bifurcation solution  $\vec{u}(\tau, \lambda)$  may be written as

$$\vec{u} = a(\tau)\vec{\zeta} + \bar{a}(\tau)\vec{\zeta}^* \quad (20)$$

Following Iooss and Joseph ([3], p. 125), we get

$$\left. \begin{aligned} a &= \varepsilon b_1(s) + \varepsilon^2 b_2(s) + \varepsilon^3 b_3(s) + O(\varepsilon^4) \\ s &= [\omega_0 + \varepsilon^2 \omega_2 + O(\varepsilon^4)]\tau \\ \lambda &= \lambda_{cr} + \varepsilon^2 \lambda_2 + O(\varepsilon^4) \end{aligned} \right\} \quad (21)$$

where, for brevity, we omit the lengthy solution forms for  $\varepsilon$ ,  $b_n$ ,  $\omega_2$ , and  $\lambda_2$  (cf. [11]). The solution is periodic in  $\tau$  with circular frequency equal to  $\omega_0 + \varepsilon^2 \omega_2 + O(\varepsilon^4)$ .

B. Stability of the Bifurcation Periodic Solution. According to Floquet theory [3], the stability of the bifurcation periodic solution (21) is determined by the sign of an index  $\mu$ . To  $O(\epsilon^4)$ ,  $\mu$  has the form

$$\mu = D'(\lambda_{cr})\lambda_2\epsilon^2 + O(\epsilon^4) \quad (22)$$

and the bifurcation periodic solution is stable if  $\mu < 0$ , unstable if  $\mu > 0$ . Since we have assumed  $D'(\lambda_{cr}) < 0$ , stability thus depends on the sign of  $\lambda_2$ , with  $\lambda_2 > 0$  denoting stability and  $\lambda_2 < 0$  instability. It remains to cast  $\lambda_2$  in more recognizable terms. After considerable manipulation, we get

$$\mu = \frac{\epsilon^2}{4\omega_0^3} \left[ \left( \frac{\partial^2 \Gamma}{\partial u_1^2} + \omega_0^2 \frac{\partial^2 F}{\partial u_2^2} \right) \frac{\partial^2 F}{\partial u_1 \partial u_2} + \omega_0^2 \left( \frac{\partial^3 F}{\partial u_1^2 \partial u_2} + \omega_0^2 \frac{\partial^3 F}{\partial u_2^3} \right) \right]_{\substack{u=0 \\ \lambda=\lambda_{cr}}} \quad (23)$$

in terms of  $F(u_1, u_2; \lambda)$ . From (8) we see that the function  $F$  is directly related to the moment  $C(\xi, \dot{\xi}; \lambda)$  acting on the aircraft which is performing a finite-amplitude oscillation  $\xi$  around the trim condition  $\lambda$ . Equation (23) demonstrates that the stability of the periodic motion near the dynamic stability boundary  $\lambda_{cr}$  is determined by the behavior of the aerodynamic response  $G(\xi, \dot{\xi}; \lambda)$  in that vicinity.

With the assumption of slow oscillations under which the form of (5) was derived (terms of  $O(\dot{\xi}^2, \ddot{\xi})$  neglected), we may substitute (5) with (8) into (23) to get

$$\mu = \frac{\epsilon^2}{4\omega_0^3} (\Gamma_0'' F_1' + \omega_0^2 \Gamma_1''')_{\substack{\xi=0 \\ \lambda=\lambda_{cr}}} \quad (24)$$

Since from (14) and (11b)

$$\omega_0 = \sqrt{S(\lambda_{cr})} = \sqrt{-F_0'(0; \lambda_{cr})} \quad (25)$$

Eq. (24) becomes, after using (8)

$$\mu = -\frac{c^2 \omega_0^2}{4} \left[ \frac{d}{d\xi} \left\{ \frac{F_1'(\xi; \lambda_{cr})}{F_0'(\xi; \lambda_{cr})} \right\} \right]_{\xi=0} \quad (26a)$$

or,

$$\mu = -\frac{c^2 \omega_0^2}{4} \left[ \frac{d}{d\xi} \left\{ \frac{G_1'(\xi; \lambda_{cr})}{G_0'(\xi; \lambda_{cr})} \right\} \right]_{\xi=0} \quad (26b)$$

Using (10) we get

$$\mu = \frac{c^2 \omega_0^2}{2} \frac{C_{2112}(\lambda_{cr})}{S(\lambda_{cr})} \quad (26c)$$

We have thus established the following criterion: the bifurcation periodic motion is stable or unstable according to

$$\mu < 0 \quad \text{or} \quad \mu > 0 \quad (27a)$$

or alternatively, since  $S(\lambda_{cr}) > 0$ , according to

$$C_{2112}(\lambda_{cr}) < 0 \quad \text{or} \quad C_{2112}(\lambda_{cr}) > 0 \quad (27b)$$

The two possibilities are well illustrated in the form of bifurcation diagrams as shown in Figs. 4a and 4b.

In a bifurcation diagram, the abscissa is the bifurcation parameter  $\lambda$ , while the ordinate is a parameter characteristic of the bifurcation solution alone. In our case it is  $\epsilon$ , a measure of the amplitude of the bifurcation periodic solution. Stable solutions are indicated by solid lines, unstable solutions by dashed lines. Thus over the range of the bifurcation parameter  $\lambda < \lambda_{cr}$  where the steady-state motion is stable,  $\epsilon$  is zero, and the stable steady motion is represented along the abscissa by a solid line. The steady motion becomes unstable for all values of  $\lambda > \lambda_{cr}$  as the dashed line along the abscissa indicates. Periodic solutions bifurcate from  $\lambda = \lambda_{cr}$  either supercritically or subcritically.

When  $C_{2112}(\lambda_{cr}) < 0$  hence  $\mu < 0$  (implying  $\lambda_2 > 0$ ), the bifurcation is called supercritical and its characteristic form is shown in Fig. 4a. In this case, stable periodic solutions (solid curves in Fig. 4a) exist for values of  $\lambda > \lambda_{cr}$ . The amplitude of the periodic solution at a given value of  $\lambda - \lambda_{cr}$  is proportional to  $\epsilon$ , hence is vanishingly small when  $\lambda - \lambda_{cr}$  is small, varying essentially as  $(\lambda - \lambda_{cr})^{1/2}$ .

When  $C_{2112}(\lambda_{cr}) > 0$  hence  $\mu > 0$  (implying  $\lambda_2 < 0$ ), the bifurcation is called subcritical and its characteristic form is shown in Fig. 4b. In this case, periodic solutions exist for values of  $\lambda < \lambda_{cr}$ , but they are unstable (dashed curve in Fig. 4b). Whether stable periodic solutions do or do not exist for  $\lambda > \lambda_{cr}$  depends predominantly on the behavior of the damping  $G_1(\xi; \lambda)$  for  $\lambda > \lambda_{cr}$ . If no such stable periodic solutions exist for  $\lambda > \lambda_{cr}$ , then when  $\lambda$  is increased beyond  $\lambda_{cr}$  the aircraft may undergo an aperiodic motion whose departure from the steady

motion at  $\lambda = \lambda_{cr}$  is potentially large.

In the more likely event that stable periodic solutions do exist for  $\lambda > \lambda_{cr}$ , their amplitudes must be finite, and not infinitesimally small, even for small positive values of  $\lambda - \lambda_{cr}$ . It is likely that this branch of stable periodic solutions will join that of the unstable branch in the way illustrated in Fig. 4b. In this event, the form of the bifurcation curve for values of  $\lambda < \lambda_{cr}$  helps explain the situation where the steady-state motion could be stable to sufficiently small disturbances but become unstable to larger disturbances. Thus, Fig. 4b suggests that for  $\lambda < \lambda_{cr}$ , so long as disturbances are of small enough amplitude to lie well below those of the unstable branch of periodic solutions (curve OB in Fig. 4b), they will die out with time and the steady motion will remain stable. However, disturbances with amplitudes sufficiently larger than those of the unstable branch may actually grow up to the ultimate motion as  $t \rightarrow \infty$ , which will be that of the stable branch of periodic solutions (curve BA in Fig. 4b). Finally, we note that if the motion does attain the stable branch of periodic solutions (say, for  $\lambda < \lambda_{cr}$ ) then hysteresis effects will manifest themselves with further changes in  $\lambda$ . When  $\lambda$  is increased beyond  $\lambda_{cr}$ , the motion will continue to be periodic with finite amplitude (point A in Fig. 4b). If  $\lambda$  is now decreased below  $\lambda_{cr}$ , the periodic motion will persist, even at values of  $\lambda$  where previously there had been steady motion when  $\lambda$  was being increased. Not until  $\lambda$  is diminished beyond a certain point (point B in Fig. 1b) will the motion return to the steady-state condition (point C in Fig. 1b) that had been experienced when  $\lambda$  was increasing.

#### §4. APPLICATIONS TO AIRCRAFT MOTIONS

In this section we shall apply the theory developed in §§ 2 and 3 to three different single-degree-of-freedom motions of the aircraft.

##### 4.1. PITCHING MOTION OF SUPERSONIC/HYPERSONIC AIRFOILS

In the case of pitching oscillation (Fig. 1) of a supersonic/hypersonic aerofoil in rectilinear flight,  $\xi(t)$  in Eq. (1) is the instantaneous pitch angle  $\theta(t)$  measured from its trim condition,  $\lambda$  is the mean angle of attack  $\alpha$ , and  $G$  is the pitching moment about the center of gravity. Thus

$$G(t; \lambda) = qSl C_m(t; \alpha, M_\infty, \gamma, h) \quad (28)$$

where  $q$  is the dynamic pressure, and  $S$  and  $l$  are the reference area and length. The instantaneous pitching moment coefficient  $C_m$  is in general also a function of the angle of attack, the flight Mach number  $M_\infty$ , the ratio of specific heats  $\gamma$ , and the pivot axis position  $h$ .

For large amplitude slow oscillations exact analytic solutions of  $C_m$  exist for simple shapes [10,11] and they are in the form consistent with Eq. (2), i.e.

$$C_m(\tau; \alpha) = C_{m_0}(\theta(\tau); \alpha) + \dot{\theta} C_{m_1}(\theta(\tau); \alpha) - C_{m_0}(\theta; \alpha) \quad (29)$$

Moreover, it is shown that [10,11]

$$C_{m_i}(\theta; \alpha) = C_{m_i}(\theta + \alpha), \quad (i=0,1) \quad (30)$$

Accordingly

$$\frac{\partial C_{m_i}}{\partial \theta} = \frac{\partial C_{m_i}}{\partial \alpha} \quad (i=0,1) \quad (31)$$

and Eq. (26b) reduces to

$$\mu = -\frac{\epsilon^2 \omega_0^2}{4} \left[ \frac{d}{d\alpha} \frac{D'(\alpha)}{S(\alpha)} \right]_{\alpha = \alpha_{cr}} \quad (32)$$

where the stiffness derivative  $S(\alpha)$  and the damping-in-pitch derivative  $D(\alpha)$  at angle of attack  $\alpha$  are related to the pitching moment coefficient  $C_m$  by

$$S(\alpha) = -C'_{m_0}(\alpha) \quad , \quad D(\alpha) = -C_{m_1}(\alpha) \quad (33)$$

and

$$D(\alpha_{cr}) = 0 \quad (34)$$

A typical example of the damping derivative  $D(\alpha)$  versus angle of attack  $\alpha$  is shown in Fig. 5 [16].

We therefore conclude that when the angle of attack  $\alpha$  is increased beyond a critical value  $\alpha_{cr}$  at which the aerodynamic damping vanishes, i.e.,  $D(\alpha_{cr}) = 0$ , the steady flight at  $\alpha_{cr}$  loses its stability and, after a transition, results in a finite amplitude periodic motion. Furthermore, the stability of this bifurcation periodic motion depends on whether  $D'(\alpha)/S(\alpha)$  increases or decreases on crossing the stability boundary  $\alpha_{cr}$ .

By utilising an approximate relation [17]

$$D(\alpha) = b[S(\alpha_{cr}) - S(\alpha)] \quad , \quad b > 0$$

in (32), we get

$$\mu = \frac{\epsilon^2 \omega_0^2 b}{4} \left[ \frac{d^2}{d\alpha^2} \ln S(\alpha) \right]_{\alpha = \alpha_{cr}}$$

For the case of a flat plate aerofoil in supersonic/hypersonic flow the stiffness and damping-in-pitch derivative,  $S(\alpha)$  and  $D(\alpha)$ , are known exactly in analytical form [7] for the angle of attack up to the shock detachment angle. It is shown in Table 1 that  $\mu > 0$  for all combinations of  $M_\infty$  and  $h$ , so the bifurcation is subcritical and the bifurcation periodic motion is unstable, implying that exchange of stability between steady and periodic motion are accompanied by hysteresis phenomena, or that potentially large aperiodic departures from steady flight may develop.

#### 4.2 FLAP OSCILLATION IN TRANSONIC FLOW

The case of a flap oscillating about the hinge is conceptually similar to that of an aerofoil pitching about a pivot axis. Thus, the oscillating flap may be regarded as an aerofoil pitching in a non-uniform incoming stream that results from a uniform free stream passing the fore-body fixed aerofoil. (Fig. 2)

In the present case  $\xi(t)$  in Eq. (1) is the instantaneous flap deflection angle  $\delta(t)$ ,  $\lambda$  is the mean deflection angle  $\delta_m$ , and  $G$  is the hinge moment. Thus

$$G(t; \lambda) = q S l C_h(t; \delta_m, M_\infty, \gamma) \quad (35)$$

where  $C_h$  is the instantaneous hinge moment coefficient. In the mathematical modeling of [12], the  $C_h$  used is of the form

$$C_h(\tau; \delta_m) = C_{h_0}(\delta(\tau); \delta_m) + \dot{\delta} C_{h_1}(\delta(\tau); \delta_m) - C_{h_0}(0; \delta_m) \quad (36)$$

Eq. (36) is consistent with (5) and was validated in [12] by comparisons with results of large scale numerical integration of the coupled inertial/flowfield equations.

It should be pointed out that the form (5) or (36) for the moment coefficient enable one to calculate  $C_{h_1}(\delta(\tau); \delta_m)$  by linearising, for small amplitude and frequency, the flowfield equations about the steady flow corresponding to  $\delta_m$  and  $\delta(\tau)$ . Such calculations of  $C_{h_1}(\delta(\tau); \delta_m)$  from the linearised equations require negligible amount of computing time compared with the nonlinear method used in [12], yet, they are consistent with the level of approximation leading to (36) or (5). With this improvement on the method

of calculating  $C_{h_1}(\delta(\tau); \delta_m)$ , the main conclusion of [12] is further strengthened in that it costs very little, in all cases, to calculate the motion of the flap using the mathematical modelling approach of Tobak et al. [14] than to solve the coupled inertial/flowfield equations.

It was further established [11] that

$$C_{h_i}(\delta; \delta_m) = C_{h_i}(\delta + \delta_m) \quad (i=0,1) \quad (37)$$

which in turn reduces Eq. (26b) to

$$\mu = -\frac{\varepsilon^2 \omega_0^2}{4} \left[ \frac{d}{d\delta_m} \frac{D'(\delta_m)}{S(\delta_m)} \right]_{\delta_m = \delta_{cr}} \quad (38)$$

where

$$S(\delta_m) = -C_{h_0}'(\delta_m), \quad D(\delta_m) = -C_{h_1}(\delta_m), \quad D(\delta_{cr}) = 0 \quad (39)$$

We therefore reach a similar conclusion regarding the motion characteristics of the flap oscillation in the neighborhood of  $\delta_{cr}$  as that in 3.4.1 regarding the motion characteristics of the aerofoil pitching motion in the neighborhood of  $\alpha_{cr}$ .

Numerical results of Ref. 12 for  $S(\delta_m)$  and  $D(\delta_m)$  are reproduced, using spline fitting, in Fig. 6 for NACA 64A010 aerofoil at zero mean angle of attack in a transonic stream with  $M_\infty = 0.8$  and  $\gamma = 1.4$ . It is found that

$$\left[ \frac{d}{d\delta_m} \frac{D'(\delta_m)}{S(\delta_m)} \right]_{\delta_m = \delta_{cr}} = -59.20 \quad \text{and hence the bifurcation is subcritical and the}$$

bifurcation periodic solution is therefore unstable. This implies that, similar

to the flat plate at supersonic/hypersonic flow, exchange of stability between steady and periodic motion of the flap of the transonic aerofoil are accompanied by hysteresis phenomena, or that potentially large aperiodic departures from the steady motion may develop.

#### 4.2. WING ROCK OF SLENDER DELTA WINGS IN SUBSONIC FLIGHT

The phenomenon known as "wing rock" of a slender delta wing in subsonic flight has been a subject for intensive investigations by many researchers [13, 18-21] and is now well documented. It is known that the steady flow past a slender delta wing at small enough angle of attack  $\alpha$  is symmetric and remains steady. However, as the angle of attack  $\alpha$  is increased past a critical value  $\alpha_{cr}$  the symmetric configuration of the leading edge vortices becomes asymmetric, causing a loss of roll damping at small angles of roll [21]. Consequently, at  $\alpha \geq \alpha_{cr}$  small disturbances introduced into the flowfield cause the rolling motion of the wing to develop, resulting in wing rock.

For rolling motion of the wing,  $\xi(t)$  in Eq. (1) is the instantaneous roll angle  $\phi(t)$ ,  $\lambda$  the steady angle of attack  $\alpha$ , and  $G$  the rolling moment. Thus

$$G(t; \lambda) = qS\ell C_{\xi}(t; \alpha, \Lambda) \quad (40)$$

where, for incompressible flow, the instantaneous rolling moment coefficient  $C_{\xi}$  depends on the angle of attack  $\alpha$  and the sweep back angle  $\Lambda$  of the delta wing. It is found from experiments [13, 18] and numerical calculations [19] that for a given  $\Lambda$  a good approximation to the instantaneous rolling

moment  $C_{\ell}$  is of the form

$$C_{\ell} = C_{\ell_0}(\phi(\tau); \alpha) + \dot{\phi} C_{\ell_1}(\phi(\tau); \alpha) \quad (41)$$

which is consistent with Eq. (5).

An example, for the case of an  $80^\circ$  sweep-back flat delta wing used in Ref. 18,  $C_{\ell}$  is well approximated by a power series [19] which leads to the following equation of the rolling motion.

$$\begin{aligned} \ddot{\phi} &= F(\phi, \dot{\phi}; \alpha) \\ &= [b_1(\alpha)\phi + b_3(\alpha)\phi^3] + \dot{\phi}[b_0 + b_2(\alpha) + b_4(\alpha)\phi^2] \\ &= F_0(\phi; \alpha) + \dot{\phi} F_1(\phi; \alpha) \end{aligned} \quad (42)$$

where, with different scalings in [19] accounted for,

$$b_i(\alpha) = \kappa^2 C_1 a_i(\alpha) \quad (i=1,3) \quad (43a)$$

$$b_j(\alpha) = \kappa C_1 a_j(\alpha) \quad (j=2,4) \quad (43b)$$

In (43),  $\kappa$  is a factor arising from the different scalings used for the time variable in Ref. 19 and in the present paper. Thus, from [19]

$\kappa = 2c/L = 2 \times 0.429/0.107 = 8$ ,  $C_1 = 0.038$  and  $a_i(\alpha)$  are tabulated in

Table 5 of Ref. 19 which yield the following table for  $b_i(\alpha)$

$\alpha$	10°	15°	20°	25°
$b_1$	-0.0265	-0.0721	-0.1977	-0.3320
$b_2$	-0.0101	0.0090	0.0596	0.0959
$b_3$	-0.1222	-0.2714	-0.0501	0.2894
$b_4$	0.1491	0.1159	-0.1799	-0.9977

From Eq. (11b), the linear damping in roll is

$$D(\alpha) = -F_1(0; \alpha) = -b_0 - b_2(\alpha) \quad (44)$$

where  $-b_0 > 0$  is proportional to the damping coefficient for the bearings in sting in the experiments of [18]. In order to compare our theoretical prediction with the experiments of [18] we choose the value of  $-b_0$  such that Eq. (44) yields zero damping in roll at the same angle  $\alpha_{cr} \doteq 18.6^\circ$  as that in [18]. The function  $D(\alpha)$  is then plotted in figure 7 using spline fitting.

From Eq. (26c) the index for stability of the bifurcation periodic rolling motion is

$$\mu = -\frac{c^2 \omega_0^2}{2} \frac{b_4(\alpha_{cr})}{b_1(\alpha_{cr})} \quad (45)$$

At  $\alpha_{cr} = 18.6^\circ$ , we know from the above table that  $b_1(\alpha_{cr}) = -0.1591 < 0$ , we also find from the spline-fit curve for  $b_4(\alpha)$  (Fig. 8) that  $b_4(\alpha_{cr}) = -0.05473 < 0$  hence  $\mu < 0$ , and the bifurcation is supercritical, implying that the

bifurcation periodic rolling motion is stable, in agreement with [18].

We now further compare the supercritical bifurcation diagram predicted by the present theory with experimental results of Ref. 18. Combining (21c) with (22) to eliminate  $\lambda_2$  and then use Eq. (45) to get

$$\begin{aligned} \alpha - \alpha_{cr} &= -\epsilon^2 \left[ \frac{\omega_0}{2} \frac{b_4}{D' b_1} \right]_{\alpha = \alpha_{cr}} \\ &= \left[ \frac{b_4}{2D' \sqrt{-b_1}} \right]_{\alpha = \alpha_{cr}} \epsilon^2 \end{aligned} \quad (46)$$

The last equation was obtained by using Eq. (25) and (42). From Fig. 7 and noting that  $1^\circ = \pi/180$ , we get  $D'(\alpha_{cr}) = -0.6131$ . Hence Eq. (46) reduces to

$$\alpha - \alpha_{cr} = 0.1120 \epsilon^2 \quad (47)$$

where  $\epsilon$  is the amplitude of the bifurcation periodic solution [11].

Eq. (47) is plotted in Fig. 9 for the  $80^\circ$  sweep back delta wing and compared with experimental results of Levin and Katz [18]. The agreement is seen to be excellent in the neighborhood of  $\alpha_{cr}$  where the bifurcation theory applies.

It should be noted that although  $\mu < 0$  and hence the bifurcation is supercritical, the value of  $|\mu|$  is small in the example. This implies that the bifurcation is close to the boundary between subcritical and supercritical. Consequently, on increasing the angle of attack  $\alpha_{cr}$  by a small amount the amplitude of the resulting periodic motion will be quite large.

This theoretical prediction is confirmed by experiments as seen from Fig. 9; for instance, increasing the angle of attack by  $1^\circ$  past  $\alpha_{cr}$  ( $\approx 18.6^\circ$ ) results in a stable periodic motion whose amplitude is  $22.6^\circ$ .

## §5. CONCLUSIONS AND DISCUSSIONS

We have shown how bifurcation theory can be used to study the nonlinear dynamic stability characteristics of an aircraft subject to single-degree-of freedom motion about its trimmed steady flight near the stability boundary. In most cases of slow motion, the required moment of the aerodynamic forces is in the form of Eq. (5). The theory shows that when the bifurcation parameter  $\lambda$ , e.g., the angle of attack, is increased past the stability boundary  $\lambda_{cr}$  where the aerodynamic damping vanishes, the steady flight loses its stability, resulting (after the transient motion has died out) in a finite-amplitude periodic motion. We have also established a simple criterion (Eq. (27)), for the stability of the bifurcation periodic motion in terms of the aerodynamic coefficients. The theory predicts that the bifurcation solutions are unstable (subcritical) in examples A and B and stable (supercritical) in example C, the latter prediction being also in good agreement with available experimental data.

In the case the theory predicts subcritical bifurcation, the abrupt change resulting from increasing  $\lambda$  past  $\lambda_{cr}$  could cause an abrupt structural change of the flow field which may in turn render invalid the form of the moment of the aerodynamic forces, Eq. (5). Under these conditions aerodynamic information in a different form may be needed. However, the theory as developed in this paper is valid up to  $\lambda_{cr}$  and can be used to predict the onset of subcritical bifurcation.

In the case the theory predicts supercritical bifurcation, the bifurcation periodic solution is stable to small enough disturbances for  $\lambda$  in the neighborhood of  $\lambda_{cr}$ . However, with further increase in  $\lambda$ , the periodic motion might lose its stability, causing another bifurcation at

$\lambda = \lambda_2 > \lambda_{cr}$ , say, which can be either subcritical or supercritical. The resulting bifurcation solution may also be almost periodic and this sequence of bifurcation may continue.

Finally, the theory in principle may be generalized to the motions of the aircraft involving more than one-degree-of-freedom by the method of projection [3]. In that case one should be aware of the possibility of chaotic motion (strange attractor) occurring after a finite number of successive bifurcations.

Table 1 Values of stability criterion  $\mu(M_\infty, h)$  for flat-plate airfoil:

$c = 1, \gamma = 1.4; \mu > 0$  subcritical bifurcation,  $\mu < 0$  supercritical

bifurcation

$M_\infty$	h					$M_\infty$	h				
	0	0.1	0.2	0.3	0.4		0	0.1	0.2	0.3	0.4
1.5	36.5	19.0	--	--	--	2	39.5	25.8	17.2	13.0	14.7
1.6	39.6	23.4	13.6	8.8	10.7	3	57.2	38.6	26.8	20.9	23.3
1.7	40.0	24.8	15.5	11.0	12.8	4	84.7	58.4	41.4	32.8	36.4
1.8	39.7	25.3	16.3	12.0	13.7	5	107.3	74.8	53.7	42.9	47.5
1.9	39.4	25.5	16.8	12.6	14.3	6	123.6	86.7	62.6	50.3	55.6
2.0	39.5	25.8	17.2	13.0	14.7						

$M_\infty$	h										
	0.25	0.26	0.27	0.28	0.29	0.30	0.31	0.32	0.33	0.34	0.35
1.6	10.6	10.1	9.7	9.3	9.0	8.8	8.6	8.5	8.4	8.4	8.5
1.7	12.6	12.2	11.8	11.5	11.2	11.0	10.8	10.7	10.6	10.6	10.7
1.8	13.6	13.2	12.8	12.5	12.2	12.0	11.8	11.7	11.6	11.7	11.7
1.9	14.1	13.7	13.4	13.0	12.8	12.5	12.4	12.3	12.2	12.2	12.3
2.0	14.5	14.1	13.8	13.5	13.2	13.0	12.8	12.7	12.6	12.7	12.7

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## FIGURE CAPTIONS

- Fig. 1. Pitching aerofoil in supersonic/hypersonic flight
- Fig. 2. Flap oscillation in transonic flight
- Fig. 3. Wing rock of slender delta wing in subsonic flight.
- Fig. 4. Typical forms of bifurcation diagrams near the dynamic stability boundary  $\lambda_{cr}$  where  $D(\lambda_{cr}) = 0$
- (a) supercritical,  $\mu < 0$ .
- (b) subcritical,  $\mu > 0$ .
- Fig. 5. Damping-in-pitch derivative  $D$  versus angle of attack  $\alpha$  for a bi-convex circular arc aerofoil. (Ref. 16)  $h = 0.5$ ,  $\gamma = 1.4$ ,  $\tau = 0.075$ .
- Fig. 6a. Stiffness derivative  $S$  versus mean deflection angle  $\delta_m$  of a transonic flap on NASA 64A010 aerofoil (Ref. 12).  $M_\infty = 0.8$ ,  $\gamma = 1.4$ ,  $\alpha = 0$ .
- Fig. 6b. Damping derivative  $D$  versus mean deflection angle  $\delta_m$  of a transonic flap on NASA 64A010 aerofoil (Ref. 12).  $M_\infty = 0.8$ ,  $\gamma = 1.4$ ,  $\alpha = 0$ ,  $\delta_{cr} = 2.7^\circ$ .
- Fig. 7. Damping in roll  $D$  of a  $80^\circ$  sweep back flat delta wing versus angle of attack  $\alpha$  in incompressible flow (Refs. 18,19), Eq. (44).  $\alpha_{cr} = 18.6^\circ$ .
- Fig. 8. Rolling aerodynamic coefficient  $b_4(\alpha)$  of a  $80^\circ$  sweep back flat delta wing versus angle of attack  $\alpha$  in incompressible flow (Refs. 18,19). Observe  $b_4(\alpha_{cr}) = -0.05473 < 0$ .
- Fig. 9. Amplitude of bifurcation periodic motion of a  $80^\circ$  sweep back delta wing versus angle of attack  $\alpha$ . ——— Eq. (47);  $\diamond$  experiments (Ref. 18).  $\alpha_{cr} = 18.6^\circ$ .

Fig. 1

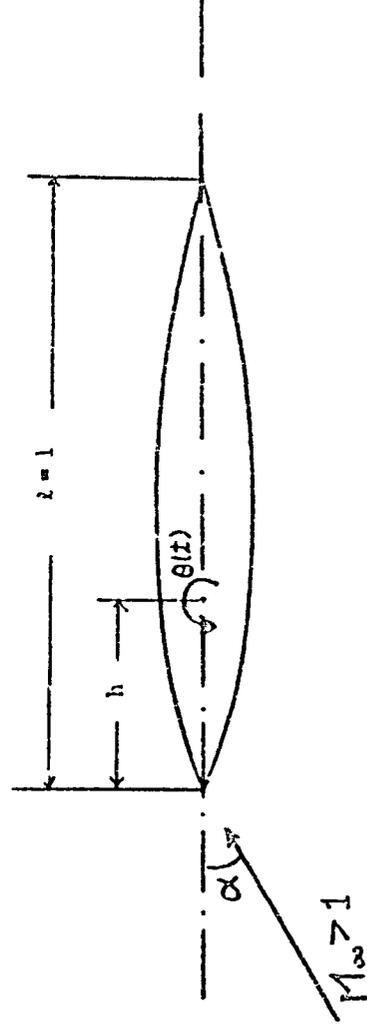


Fig. 2

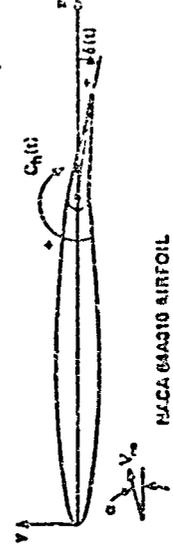
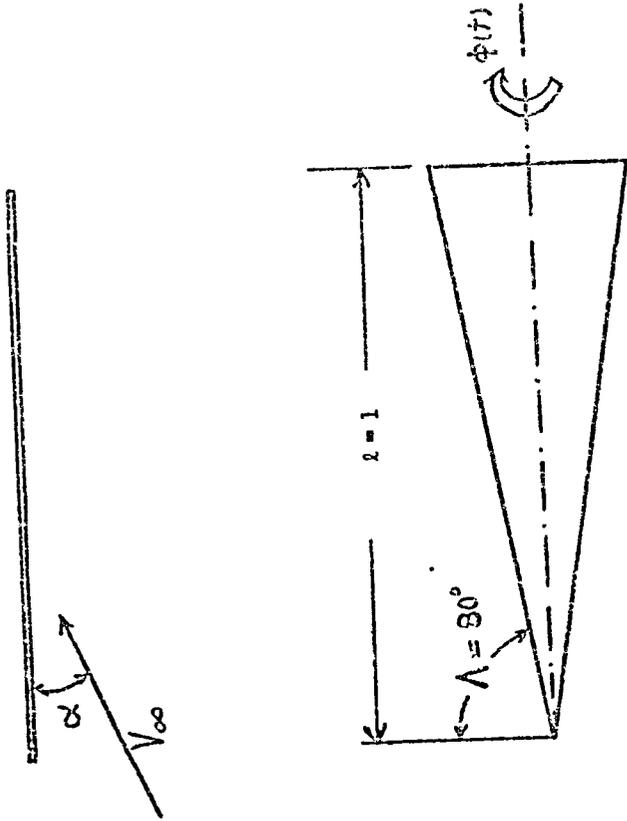


Fig 3



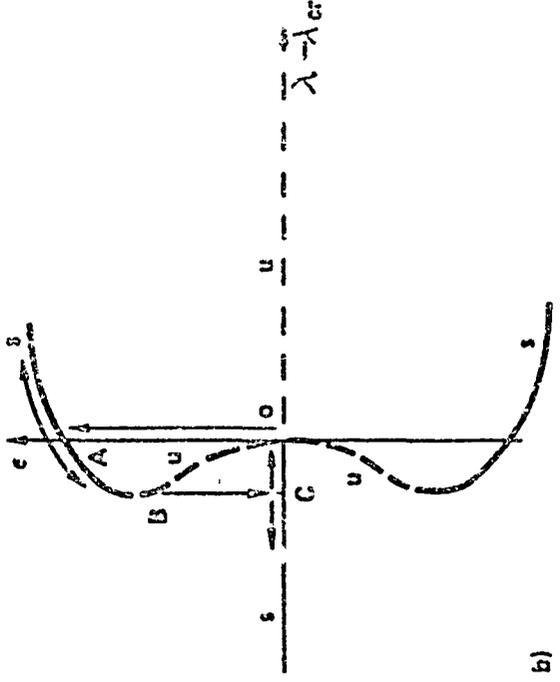
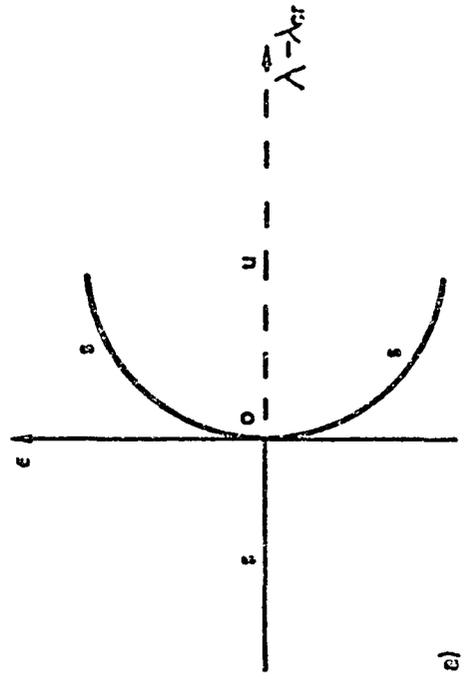


Fig. 4

Fig. 5. Bi-circular arc aerofoil,  
 $\tau=0.075$ ,  $\gamma=1.4$ ,  $h=0.5$

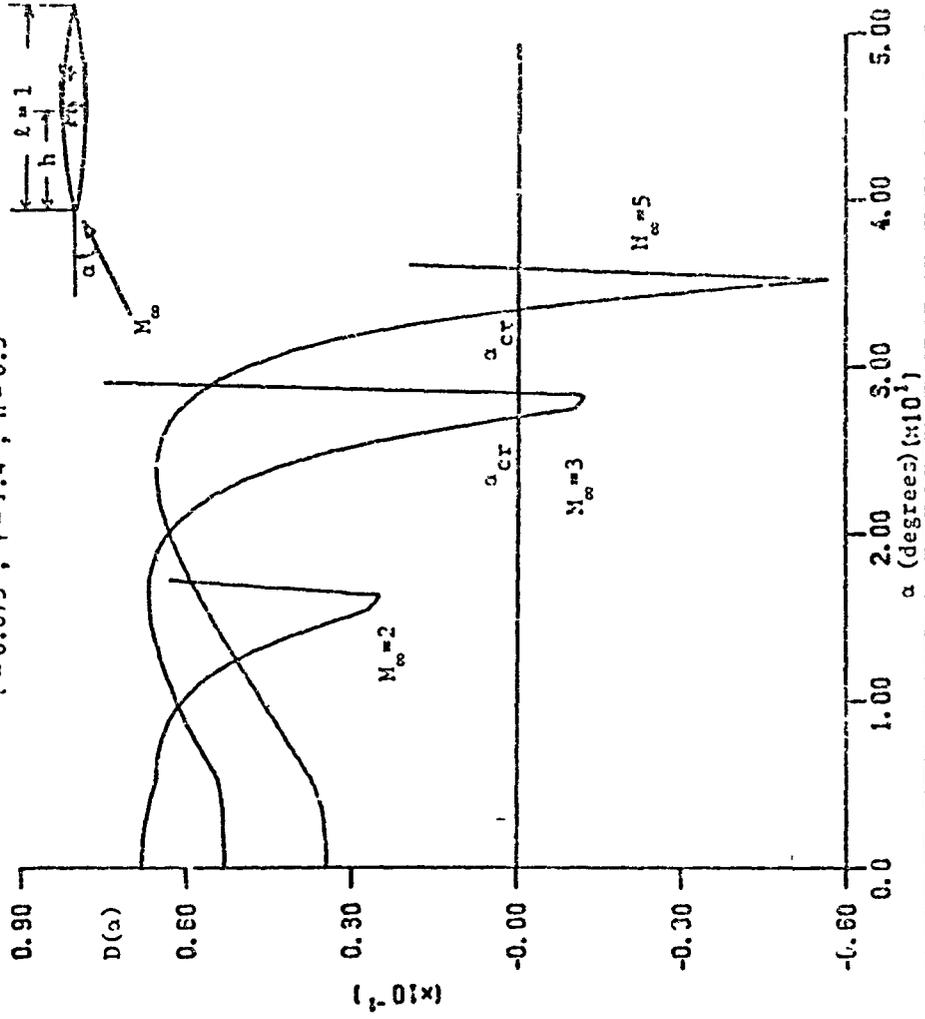


FIG. 6a  $M_\infty=0.8$ ,  $\gamma=1.4$

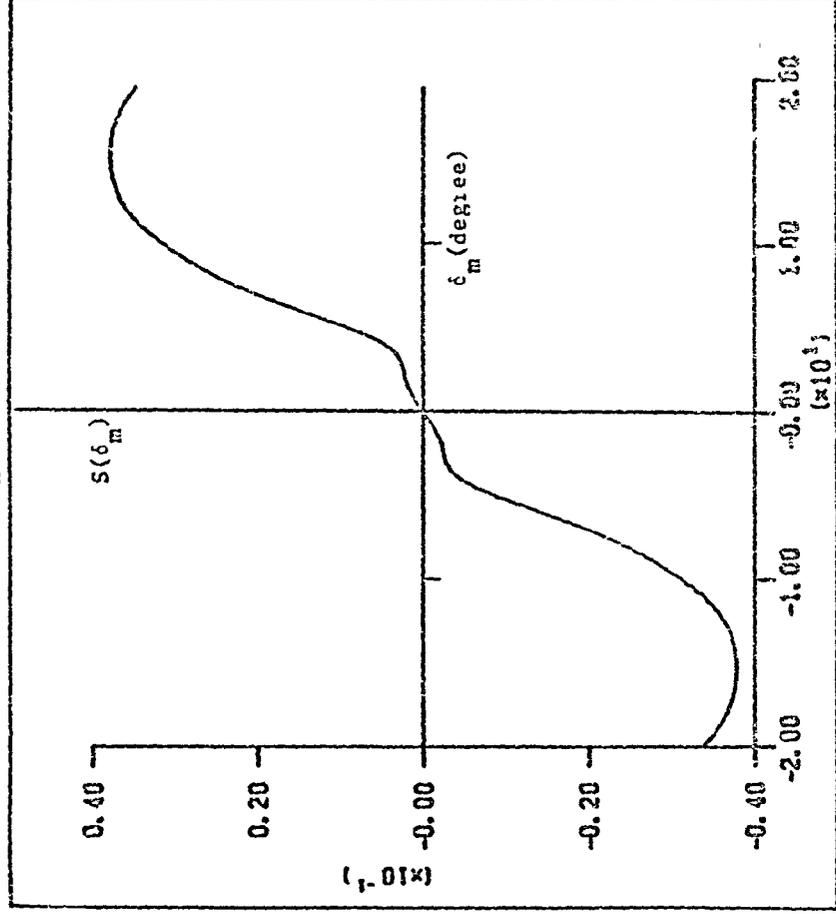


Fig. 6b  $M_\alpha=0.8$ ,  $\tau=1.4$ ,  $\xi_{cr}=2.7^\circ$

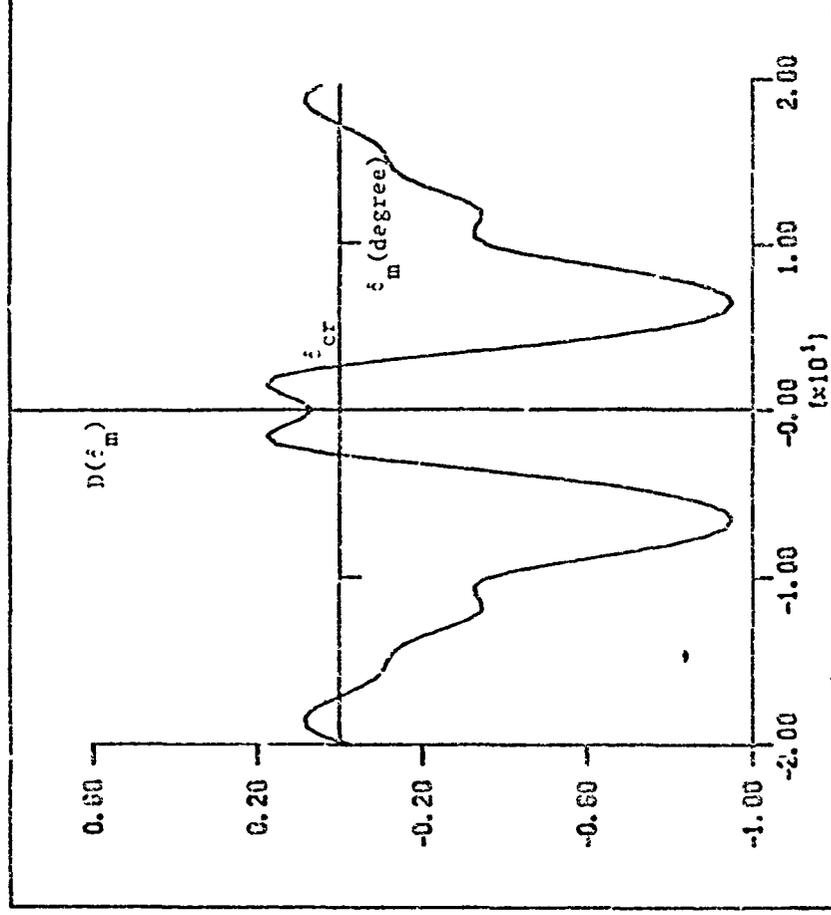


Fig. 7

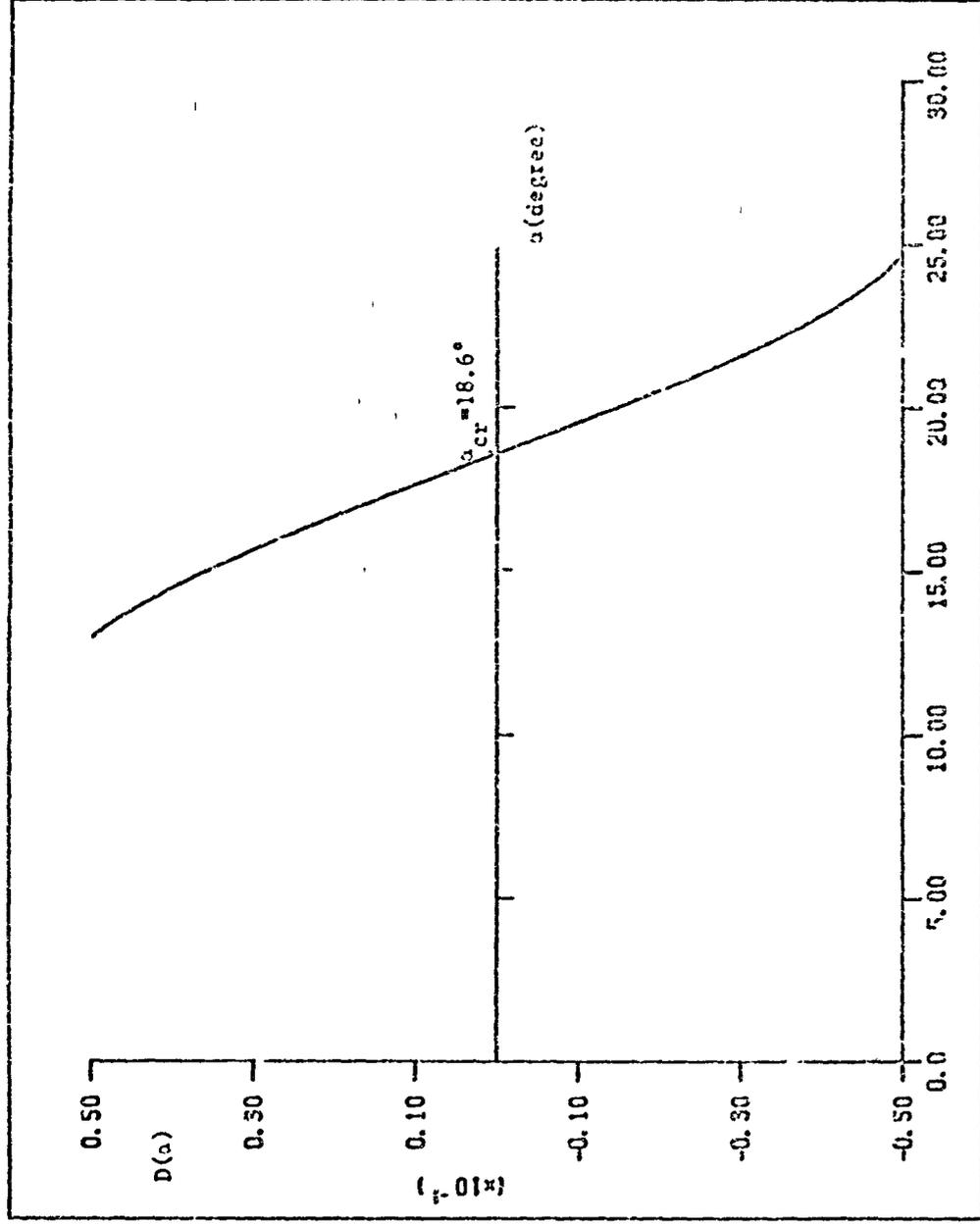


Fig. 8

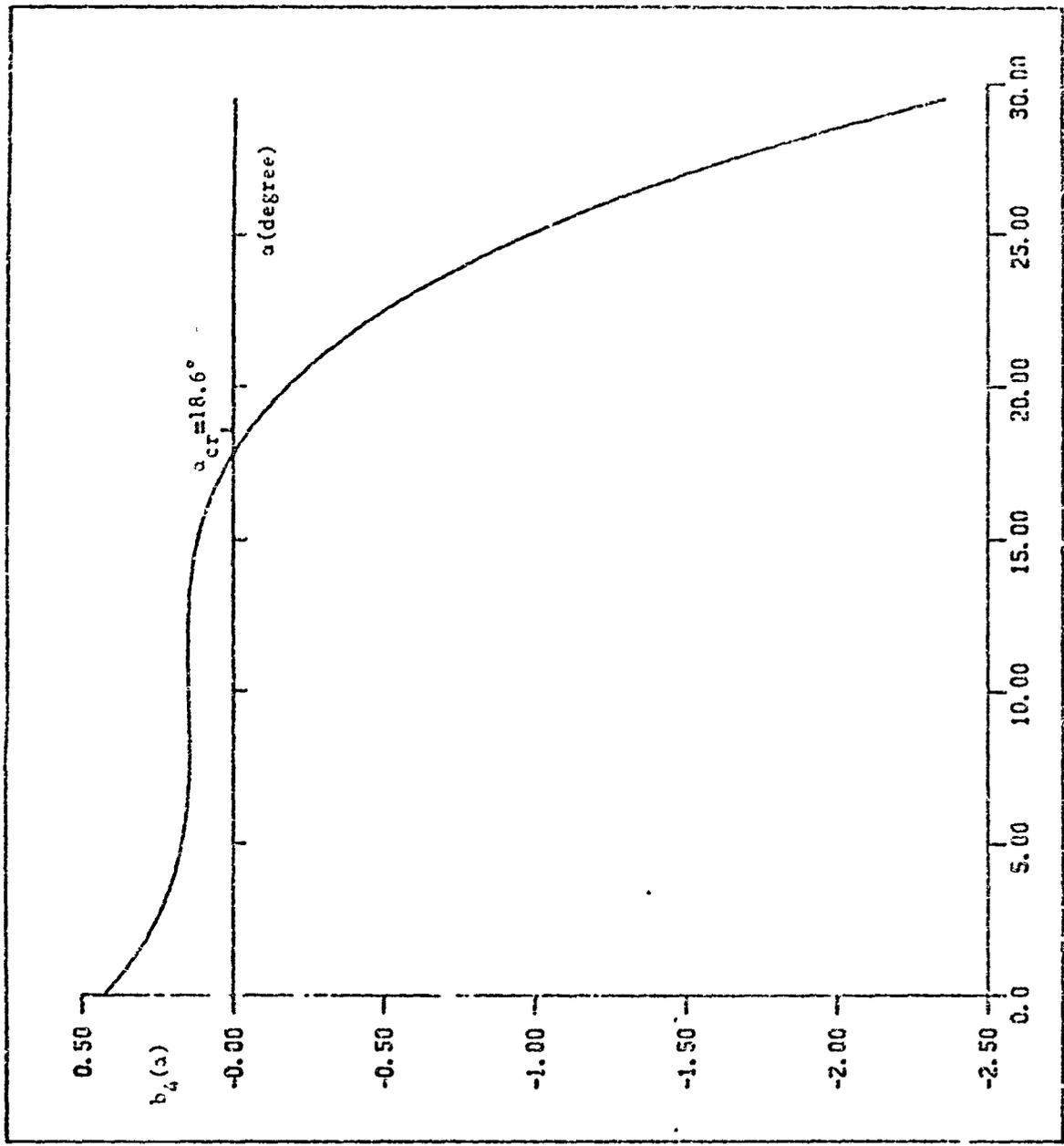
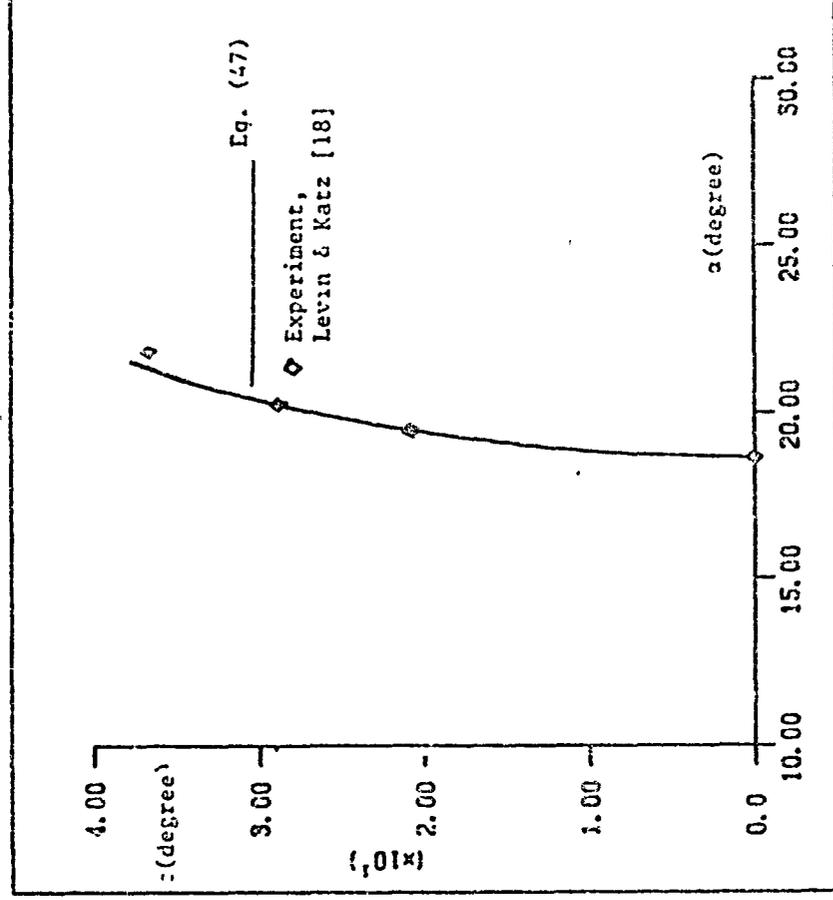


Fig. 9



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