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by Ernest S. Armstrong and Athena T. Markos

Langley Research Center
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SUMMARY

A method is presented for obtaining space-rendezvous trajectories between a one-stage rocket vehicle and a target in a circular Keplerian orbit. The trajectories satisfy the necessary conditions of the Pontryagin maximum principle for time-optimal rendezvous in which no terminal mass constraint is placed on the rocket. The use of Pontryagin's theory leads to a two-point boundary-value problem. A digital program is given for the iterative solution of this problem. The method is successfully applied for the determination of time-optimal rendezvous trajectories between a vehicle launched from the surface of the moon and a target in an 80-nautical-mile circular orbit.

INTRODUCTION

The study of trajectories for rendezvous in space has been of interest for many years. An early summary of some aspects of the problem is given in reference 1. The application of optimization theory to trajectory computation has also received attention (refs. 2 and 3). Paiewonsky and Woodrow (ref. 4) have considered the problem of a three-dimensional time-optimal space rendezvous between a single-stage rocket vehicle, with constrained terminal mass, and a target in a circular Keplerian orbit. Linearized dynamic equations were used to describe the motion of the vehicle, and the mass limitation was of the form of a terminal inequality constraint. Linearization of the equations imposes the condition that the rocket be in proximity to the target. Reference 5 has considered the three-dimensional time-optimal rendezvous problem with unsimplified dynamic equations and fixed terminal mass through the dual problem of three-dimensional fuel-optimal rendezvous with specified final time. The present paper continues the extention of the problem of reference 4 to unsimplified dynamics by considering three-dimensional time-optimal rendezvous with unspecified terminal mass.

The mathematical model of reference 5, which treats the rocket as a point mass and takes into account rotation of the attracting center, is employed. The Pontryagin maximum principle (ref. 6) is applied to find the correct thrust magnitude and direction for time optimality. This operation leads to a two-point boundary-value problem in which certain initial conditions on a set of differential equations introduced by the maximum principle have to be found such that certain terminal conditions are met. Following a method developed in reference 5, a digital program is written to solve the boundary-value problem by iteration. The procedure is illustrated numerically by solving the problem of finding time-optimal trajectories of a rocket vehicle launched from the moon to rendezvous with a target in an 80-nautical-mile circular orbit.

In addition to extending the work of reference 4, the digital program and accompanying analysis provide a useful method for obtaining time-optimal rendezvous trajectories and control laws. Therefore, the digital program is discussed and a listing included.

SYMBOLS

A, M, N, K, L, T_1, T_2 constant matrices

$$A_i = \beta\left[\frac{1}{x_7\sqrt{\psi}} - \frac{\psi_i^2}{x_7\left(\sqrt{\psi}\right)^3}\right]$$

B seven-dimensional diagonal matrix with elements b_i

$$B_{ik} = -\frac{\beta\psi_i\psi_k}{x_7\left(\sqrt{\psi}\right)^3}$$

b_i positive weighting elements ($i = 1, 2, \ldots 7$)

$$C_{ik} = \frac{3\Omega^2 R_s{}^3 v_i v_k}{\left(\sqrt{x}\right)^5}$$

c effective exhaust velocity

$$D_{ik} = -\frac{3\Omega^2 R_s{}^3\left(2\psi_i v_k + d\right)}{\left(\sqrt{x}\right)^5}$$

$$d = \psi_2\left(x_1 + R_{s_x}\right) + \psi_4\left(x_3 + R_{s_y}\right) + \psi_6\left(x_5 + R_{s_z}\right)$$

$$E_{ik} = \frac{15\Omega^2 R_s{}^3 d v_i v_k}{\left(\sqrt{x}\right)^7}$$

$E\left[\bar{e}(\bar{\alpha})\right]$ scalar measure of terminal error, $\sum_{i=1}^{7}\frac{b_i\left[e_i(\bar{\alpha})\right]^2}{2}$

$\bar{e}$ seven-dimensional vector with elements $e_i(\bar{\alpha})$

e_i error criteria (i = 1, 2, . . . 7)

$$F_{em}^{ik} = -\frac{3\Omega^2 R_s{}^3\left(\psi_e v_i + \psi_m v_k\right)}{\left(\sqrt{x}\right)^5}$$

$\underset{\sim}{H}$ pseudo-Hamiltonian function of the maximum principle

I identity matrix

i,j,n integers (i = 0, 1, . . . n; j = 1, 2, . . . 6)

$\hat{i},\hat{j},\hat{k}$ unit vectors

M constant matrix defined in appendix A

$\underset{\sim}{M}$ maximum value of $\underset{\sim}{H}$ with respect to $\bar{u}$

m mass of launch vehicle

m_o initial mass of launch vehicle

R_s magnitude of $\bar{R}_s$

$\bar{R}_s$ vector from center of attracting body to target

R_{s_x},R_{s_y},R_{s_z} elements of $\bar{R}_s$

$\dot{\bar{R}}_s$ first derivative of $\bar{R}_s$

$\dot{R}_{s_x},\dot{R}_{s_y},\dot{R}_{s_z}$ first derivatives of R_{s_x}, R_{s_y}, and R_{s_z}

R_v magnitude of $\bar{R}_v$

$\bar{R}_v$ vector from center of attracting body to vehicle

R_{v_x},R_{v_y},R_{v_z} elements of $\bar{R}_v$

$\dot{\overline{R}}_V$	first derivative of $\overline{R}_V$
$\dot{R}_{V_x}, \dot{R}_{V_y}, \dot{R}_{V_z}$	first derivatives of R_{V_x}, R_{V_y}, and R_{V_z}
$\bar{r} = \overline{R}_V - \overline{R}_S$	
r_x, r_y, r_z	elements of $\bar{r}$
$\dot{r}_x, \dot{r}_y, \dot{r}_z$	first derivatives of r_x, r_y, and r_z
s	dummy integration variable
sgn ρ	signum function defined by $\begin{cases} 1 \text{ if } & \rho > 0 \\ -1 \text{ if } & \rho < 0 \\ \text{Unspecified if } & \rho = 0 \end{cases}$
T	magnitude of thrust vector
$\overline{T}$	thrust vector
t	time
t_o	initial time
t_f	final time
$[t_2, t_3]$	nonzero subinterval of $[t_o, t_f]$
t'	isolated point at which $M'\overline{\psi}(t) = \overline{0}$
u_i	elements of $\hat{u}$ $(i = 1, 2, 3)$
u_4	magnitude of $\overline{T}$
$\bar{u} = u_4 \hat{u}$	
$\hat{u}$	unit vector in direction of $\bar{u}$
$u_4{}^*$	optimal form of u_4

$\bar{u}^* = u_4{}^* \hat{u}^*$

$\hat{u}^*$ optimal form of $\hat{u}$

v_i, v_k variable v_1, v_3, or v_5

$v_1 = x_1 + R_{s_x}$

$v_3 = x_3 + R_{s_y}$

$v_5 = x_5 + R_{s_z}$

x,y,z rotating axis system defined by figure A-1

$\left. \begin{matrix} x',y',z' \\ X,Y,Z \end{matrix} \right\}$ inertial axis system defined by figure A-2

$\sqrt{x} = \sqrt{\left(x_1 + R_{s_x}\right)^2 + \left(x_3 + R_{s_y}\right)^2 + \left(x_5 + R_{s_z}\right)^2}$

x_i state variables (i = 0, 1, . . . 8)

$\dot{x}_i$ first derivative of x_i

x_{i_o} initial values of x_i

$\bar{x} = \mathrm{col}\left(x_1, \ . \ . \ . \ x_6\right)$

$\dot{\bar{x}}$ first derivative of $\bar{x}$

$\bar{x}_o$ initial value of $\bar{x}$

$\overline{Y}\left(\bar{x}, x_o\right)$ vector defined in equation (A9)

α_i, α_j unknown parameters

$\overline{\alpha}$ seven-dimensional vector with elements α_i

β bound on thrust magnitude

$\delta\bar{\alpha}$ correction to $\bar{\alpha}$

$\delta\alpha_7$ variation in α_7

θ_c, φ_c control angles (see fig. A-3)

θ_v^o, φ_v^o initial vehicle angles (see fig. A-4)

$\theta_o, \iota_o, \varphi_o$ angles determining target orbital plane

λ parameter governing step size of $\delta\bar{\alpha}$

μ universal gravitational constant multiplied by mass of attracting body

ρ switching function

$\psi_e, \psi_i, \psi_j, \psi_k, \psi_m$ variables introduced by the maximum principle, with the subscripts equal to 0, 1, . . . 8

$\dot{\psi}_i$ first derivative of ψ_i

$\bar{\psi} = \mathrm{col}(\psi_1, \ldots \psi_6)$

$\dot{\bar{\psi}}$ first derivative of $\bar{\psi}$

$\sqrt{\Psi} = \sqrt{\psi_2^{\,2} + \psi_4^{\,2} + \psi_6^{\,2}}$

Ω constant angular velocity of the target in its orbital plane

ω angular velocity of body about axis of rotation

Mathematical notation (with arbitrary symbols used as examples):

$\dot{a}(t)$ first derivative of $a(t)$ with respect to t

$\ddot{a}(t)$ second derivative of $a(t)$ with respect to t

$\bar{a} \cdot \bar{b} = \sum_{i=1}^{n} a_i b_i$ where $\bar{a} = \mathrm{col}(a_1, \ldots a_n)$ and $\bar{b} = \mathrm{col}(b_1, \ldots b_n)$

$\|\bar{a}\| = (\bar{a} \cdot \bar{a})^{1/2}$

$[a,b]$ closed interval

$[a,b)$ interval closed at a and open at b

$\frac{\partial \bar{b}(\bar{a})}{\partial \bar{a}}$ Jacobian matrix with elements $c_{ij} = \frac{\partial b_i(\bar{a})}{\partial a_j}$

$\frac{\partial b_i(\bar{a})}{\partial a_j}$ first partial derivative of $b_i(\bar{a})$ with respect to a_j evaluated at $\bar{a} = \mathrm{col}(a_1, \ldots a_n)$

ϵ belongs to a set

$'$ denotes matrix transpose

$\bar{0}$ null vector

-1 as superscript to matrix, denotes inverse

DEVELOPMENT OF THE BOUNDARY-VALUE PROBLEM

In appendix A the dynamic equations are developed for a space vehicle which is required to rendezvous with a target in a circular Keplerian orbit about a rotating body. The vehicle is a one-stage rocket, treated as a point mass, with bounded thrust magnitude. The controls are the magnitude and direction of the thrust vector.

From equation (A8), the dynamic equations when written as first-order differential equations in relative coordinates take the form

$$\left.\begin{aligned}
\dot{x}_0 &= 1 && \left(x_0(t_o) = 0\right) \\
\dot{\bar{x}} &= \frac{u_4 M\hat{u}}{x_7} + \bar{Y}\left(\bar{x}, x_8\right) && \left(\bar{x}(t_o) = \bar{x}_o;\ \ \bar{x}(t_f) = \bar{0}\right) \\
\dot{x}_7 &= -\frac{u_4}{c} && \left(x_7(t_o) = m_o\right) \\
\dot{x}_8 &= 1 && \left(x_8(t_o) = t_o\right)
\end{aligned}\right\} \tag{1}$$

The vector $\bar{x}$ is a six-dimensional vector whose elements $x_1, x_2, \ldots x_6$ are the relative position (x_1, x_3, x_5) and velocity (x_2, x_4, x_6) components of the vehicle and target. The variables x_7 and x_8 are the instantaneous vehicle mass and the time, respectively. The vector $\bar{Y}(\bar{x}, x_8)$, given by equation (A9), includes the kinematic parts of the equations. The scalar u_4 and unit vector $\hat{u}$ are, respectively, the magnitude and direction of the thrust vector $\bar{T}$. The matrix M is a constant matrix defined in appendix A.

Pontryagin's maximum principle (ref. 6) is now applied to find controls which minimize $\int_{t_0}^{t_f} dt$ while satisfying equation (1). In using the maximum principle, a new variable x_0 is introduced such that $\dot{x}_0 = 1\left(x_0(t_0) = 0\right)$ and the problem becomes one of minimizing $x_0(t_f)$. From reference 6 the following conditions must be satisfied:

(1) A function $u_4 \leqq \beta$ and a function $\hat{u}$ $\left(\text{with } \|\hat{u}\| = 1\right)$ must be chosen to maximize

$$\underset{\sim}{H}\left(\psi_0, \ldots \psi_8;\ x_1, \ldots x_8;\ u_1, \ldots u_4\right) = \psi_0 \dot{x}_0(t) + \bar{\psi}(t) \cdot \dot{\bar{x}}(t) + \psi_7(t)\dot{x}_7(t) + \psi_8(t)\dot{x}_8(t) \tag{2}$$

for fixed ψ_i and x_i $(i = 0, 1, \ldots 8)$. The vector $\bar{\psi}$ is equal to $\text{col}\left(\psi_1, \ldots \psi_6\right)$. The terms ψ_i $(i = 0, 1, \ldots 8)$ are nine additional variables introduced by the Pontryagin maximum principle and defined by

$$\dot{\psi}_i = -\frac{\partial \underset{\sim}{H}}{\partial x_i} \qquad (i = 0, 1, \ldots 8) \tag{3}$$

(2) For any $t \epsilon \left[t_0, t_f\right]$, $\psi_0(t) = \text{Constant} \leqq 0$ and the maximum value of $\underset{\sim}{H}\left(\psi_0, \ldots \psi_8;\ x_1, \ldots x_8;\ u_1, \ldots u_4\right)$ with respect to $u_1, \ldots u_4$, given by $\underset{\sim}{M}\left(\psi_0, \ldots \psi_8;\ x_1, \ldots x_8\right)$, must be identically zero over $\left[t_0, t_f\right]$.

(3) The transversality condition must be satisfied.

Since $\bar{x}(t_f)$ is specified and $x_7(t_f)$ and $x_8(t_f)$ are unspecified, the transversality condition discussed in reference 6 yields

$$\psi_7(t_f) = \psi_8(t_f) = 0$$

From a consideration of condition (1),

$$\underset{\sim}{H} = \psi_0 + \bar{\psi} \cdot \left[\frac{u_4 M \hat{u}}{x_7} + \bar{Y}\left(\bar{x}, x_8\right)\right] - \frac{\psi_7 u_4}{c} + \psi_8$$

whereby, from equation (3),

$$\left.\begin{aligned}
\dot{\psi}_0 &= 0 && (\psi_0 \leqq 0) \\
\dot{\bar{\psi}} &= -\frac{\partial \bar{Y}'(\bar{x}, x_8)}{\partial \bar{x}} \bar{\psi} && (\bar{\psi}(t_o) \text{ undetermined}) \\
\dot{\psi}_7 &= -\frac{u_4 \bar{\psi} \cdot M\hat{u}}{x_7^2} && (\psi_7(t_f) = 0) \\
\dot{\psi}_8 &= -\frac{\partial \bar{Y}(\bar{x}, x_8)}{\partial x_8} \cdot \bar{\psi} && (\psi_8(t_f) = 0)
\end{aligned}\right\} \tag{4}$$

If $M'\bar{\psi} \neq 0$, the $\hat{u}$ which maximizes $\underset{\sim}{H}$ and satisfies $\|\hat{u}\| = 1$ is

$$\hat{u}^* = \frac{M'\psi}{\|M'\psi\|} \tag{5}$$

since $\bar{\psi} \cdot M\hat{u}$ can be written as $M'\bar{\psi} \cdot \hat{u}$. Then $\underset{\sim}{H}$ becomes

$$\underset{\sim}{H} = \psi_0 + u_4\left(\frac{\|M'\bar{\psi}\|}{x_7} - \frac{\psi_7}{c}\right) + \bar{\psi} \cdot \bar{Y}(\bar{x}, x_8) + \psi_8$$

and the function $u_4 \leqq \beta$ which maximizes $\underset{\sim}{H}$, if $\frac{\|M'\bar{\psi}\|}{x_7} - \frac{\psi_7}{c}$ does not vanish identically over a nonzero interval in $[t_o, t_f]$, is

$$u_4^* = \frac{\beta}{2}(1 + \operatorname{sgn} \rho) = \left\{\begin{matrix} \beta & & (\rho(t) > 0) \\ 0 & & (\rho(t) < 0) \end{matrix}\right\} \tag{6}$$

with

$$\rho = \frac{\|M'\bar{\psi}\|}{x_7} - \frac{\psi_7}{c}$$

The complete control now takes the form

$$\bar{u}^* = u_4^* \hat{u}^* = \frac{\beta}{2}(1 + \operatorname{sgn} \rho)\frac{M'\bar{\psi}}{\|M'\bar{\psi}\|} \tag{7}$$

The function $\rho(t)$ is the switching function for the system. The function $\rho(t)$ has no zeros on $[t_o, t_f]$ except possibly at t_f, which simply means the thrust is always at its maximum value. Since $\psi_7(t)$ is such that

$$\dot{\psi}_7 = \frac{\beta}{2}(1 + \operatorname{sgn} \rho)\frac{\|M'\bar{\psi}\|}{x_7^2} \geqq 0$$

and $\psi_7(t_f) = 0$, it follows that $\psi_7(t) \leqq 0$ for all $t\epsilon\left[t_o,t_f\right]$. If $\psi_7(t') = 0$ at some $t'\epsilon\left[t_o,t_f\right)$, the condition $\rho(t) < 0$ must be satisfied for $t > t'$ in order to meet the condition $\psi_7(t_f) = 0$. The implication is that from t' until t_f, the vehicle coasts; that is, $T = 0$ from t' to t_f. However, to rendezvous at t_f requires that the vehicle and target have the same position and velocity. Therefore, $\psi_7(t)$ can vanish only at t_f, since it is not possible for the vehicle to coast into the same position and velocity as the target; that is, $\psi_7(t) < 0$ and $\rho(t) > 0$ for $t\epsilon\left[t_o,t_f\right)$. Equation (7) then takes the form

$$\bar{u}^* = \beta \frac{M'\bar{\psi}}{\left\|M'\bar{\psi}\right\|} \tag{8}$$

and $\underset{\sim}{M}$ becomes

$$\underset{\sim}{M} = \psi_0 + \beta\rho + \bar{\psi} \cdot \bar{Y}\left(\bar{x},x_8\right) + \psi_8$$

In an optimal system, $M'\bar{\psi} = \left(\psi_2,\psi_4,\psi_6\right)' = \bar{0}$ over a nonzero interval (for example, over $\left[t_2,t_3\right]$ of $\left[t_o,t_f\right]$) cannot be allowed. For, if it is, the differential equations for ψ_2, ψ_4, and ψ_6 imply $\left(\psi_1,\psi_3,\psi_5\right) \equiv \bar{0}$ over $\left[t_2,t_3\right]$. In fact, $\bar{\psi}(t) \equiv 0$ would be the solution of the $\bar{\psi}$ system over $\left[t_2,t_f\right]$. Since $\psi_7(t_f)$ and $\psi_8(t_f)$ both vanish, $\psi_7(t) \equiv 0$ and $\psi_8(t) \equiv 0$ over $\left[t_2,t_f\right]$. Also $\underset{\sim}{M} \equiv 0$ implies that $\psi_0 = 0$ over $\left[t_2,t_f\right]$, giving $\left(\psi_0,\bar{\psi}(t),\psi_7(t),\psi_8(t)\right) \equiv 0$ over $\left[t_2,t_f\right]$, which is a contradiction to the maximum principle. The maximum principle states that for each $t\epsilon\left[t_o,t_f\right]$, the vector $\left(\psi_0(t),\psi_1(t), \ldots \psi_8(t)\right)$ is nonzero. Isolated points at which $M'\bar{\psi} = 0$ have no effect on the solution, since $\bar{u}(t)$ is bounded. For definiteness,

$$\hat{u}(t') = \lim_{t \to t'} \frac{M'\bar{\psi}(t)}{\left\|M'\bar{\psi}(t)\right\|}$$

at any isolated points t' where $M'\bar{\psi}(t') = 0$.

Equations (1) and (4) now take the form

$$\left.\begin{aligned}
&\dot{x}_0 = 1 && \left(x_0(t_o) = 0\right) \\
&\dot{\bar{x}} = \frac{\beta MM'\bar{\psi}}{\left\|M'\bar{\psi}\right\| x_7} - \frac{\Omega^2 R_s{}^3}{\left\|A\bar{x} + \bar{R}_s\left(x_8\right)\right\|^3}\left[N\bar{x} + M\bar{R}_s\left(x_8\right)\right] + \Omega^2 M\bar{R}_s\left(x_8\right) && \\
&\qquad + \left(N' + 2\omega K + \omega^2 L\right)\bar{x} && \left(\bar{x}(t_o) = \bar{x}_o;\ \ \bar{x}(t_f) = \bar{0}\right) \\
&\dot{x}_7 = -\frac{\beta}{c} && \left(x_7(t_o) = m_o\right) \\
&\dot{x}_8 = 1 && \left(x_8(t_o) = t_o\right)
\end{aligned}\right\} \tag{9a}$$

and

$$
\left.
\begin{aligned}
&\dot{\psi}_0 = 0 && \left(\psi_0 \leqq 0\right) \\
&\dot{\overline{\psi}} = \frac{\Omega^2 R_s{}^3 N'\overline{\psi}}{\left\|A\bar{x} + \overline{R}_s(x_8)\right\|^3} - \frac{3\Omega^2 R_s{}^3}{\left\|A\bar{x} + \overline{R}_s(x_8)\right\|^5}\left\{\left[N\bar{x} + M\overline{R}_s(x_8)\right] \cdot \overline{\psi}\right\} A'\left[A\bar{x} + \overline{R}_s(x_8)\right] \\
&\qquad - \left(N + 2\omega K' + \omega^2 L'\right)\overline{\psi} && \left(\overline{\psi}(t_o) \quad \text{undetermined}\right) \\
&\dot{\psi}_7 = \frac{\beta\left\|M'\overline{\psi}\right\|}{x_7{}^2} && \left(\psi_7(t_f) = 0\right) \\
&\dot{\psi}_8 = \frac{\Omega^2 R_s{}^3 \overline{\psi} \cdot M\overline{R}_s(x_8)}{\left\|A\bar{x} + \overline{R}_s(x_8)\right\|^3} - \Omega^2 \overline{\psi} \cdot M\dot{\overline{R}}_s(x_8) \\
&\qquad - \frac{3\Omega^2 R_s{}^3}{\left\|A\bar{x} + \overline{R}_s(x_8)\right\|^5}\left\{\dot{\overline{R}}_s(x_8) \cdot \left[A\bar{x} + \overline{R}_s(x_8)\right]\right\}\left\{\overline{\psi} \cdot \left[N\bar{x} + M\overline{R}_s(x_8)\right]\right\} && \left(\psi_8(t_f) = 0\right)
\end{aligned}
\right\} \quad (9b)
$$

By comparing equations (9a) and (9b), a two-point boundary-value problem is recognized; ψ_0, $\overline{\psi}(t_o)$, $\psi_7(t_o)$, $\psi_8(t_o)$, and t_f need to be found such that $\bar{x}$, $\underset{\sim}{M}$, ψ_7, and ψ_8 are zero at t_f. This boundary-value problem can be placed in the simpler form

$$
\underset{\sim}{M}\left(\psi_0, \ . \ . \ . \ \psi_8; \ x_1, \ . \ . \ . \ x_8\right) = 0
$$

which yields

$$
\psi_8(t_o) = -\left[\psi_0 + \beta\rho(t_o) + \overline{\psi}(t_o) \cdot \overline{Y}\left(\bar{x}_o, t_o\right)\right] \tag{10}
$$

Since $\underset{\sim}{M} \equiv 0$ over $\left[t_o, t_f\right]$, since $\psi_8(t_f) = 0$, since $\bar{x}(t_f) = \bar{0}$, and since $\overline{Y}\left(\bar{0}, t_f\right) = \bar{0}$, it follows that

$$
\underset{\sim}{M} = \psi_0 + \beta\rho(t_f) = 0
$$

or

$$
\psi_0 = -\beta\rho(t_f) \tag{11}
$$

The boundary-value problem then reduces to satisfying equations (9a), (9b), (10), and (11) and finding parameters $\psi_1(t_o)$, . . . $\psi_7(t_o)$ and t_f such that $\bar{x}(t_f)$ and $\psi_7(t_f)$ are zero. There appear to be eight parameters and seven boundary conditions. However, because of the homogeneous form of their differential equations, one of the parameters $\psi_i(t_o)$ (i = 1, 2, . . . 7) can be removed by normalization – that is, by fixing its value.

Also, by observing the differential equation for ψ_7, it can be noted that the parameter $\psi_7(t_o)$ could be determined after the other parameters are found by setting

$$\psi_7(t_o) = -\beta \int_{t_o}^{t_f} \frac{\|M'\bar{\psi}(s)\|}{x_7^2(s)} ds$$

The minimal combination is then six parameters $\left(\bar{\psi}(t_o)\right.$ and t_f, with one of the elements of $\bar{\psi}(t_o)$ normalized$\left.\right)$ and six boundary conditions $\left(\bar{x}(t_f) = 0\right)$. However, since the correct algebraic sign of an element of $\bar{\psi}(t_o)$ may not be known a priori, the problem is best solved by finding seven parameters $\left(\bar{\psi}(t_o)\right.$ and t_f, with $\psi_7(t_o)$ normalized$\left.\right)$ such that the seven boundary conditions $\left(\bar{x}(t_f) = 0\right.$ and $\left.\psi_7(t_f) = 0\right)$ are met.

SOLUTION OF THE BOUNDARY-VALUE PROBLEM

Development of Iterative Logic

The approach taken to obtain a solution of the foregoing boundary-value problem is that discussed in reference 5. In reference 5, given a similar boundary-value problem, a vector $\bar{e}(\bar{\alpha})$ is defined such that when $\bar{e}(\bar{\alpha}) = 0$ the boundary conditions are satisfied and the unknown parameters for the problem are $\bar{\alpha}$.

The vectors $\bar{\alpha}$ and $\bar{e}(\bar{\alpha})$ correspond to $\mathrm{col}(\alpha_i)$ and $\mathrm{col}\left(e_i(\bar{\alpha})\right)$ $(i = 1, 2, \ldots 7)$, respectively, with elements

$$\left.\begin{array}{ll} \alpha_1 = \psi_1(t_o) & e_1(\bar{\alpha}) = x_1(\bar{\alpha}) \\ \alpha_2 = \psi_2(t_o) & e_2(\bar{\alpha}) = x_2(\bar{\alpha}) \\ \vdots & \vdots \\ \alpha_6 = \psi_6(t_o) & e_6(\bar{\alpha}) = x_6(\bar{\alpha}) \\ \alpha_7 = t_f & e_7(\bar{\alpha}) = \psi_7(\bar{\alpha}) \end{array}\right\} \tag{12}$$

The quantities x_i $(i = 1, 2, \ldots 6)$ and ψ_7 are written $x_i(\bar{\alpha})$ and $\psi_7(\bar{\alpha})$ to indicate their implicit dependence on $\bar{\alpha}$.

The magnitude of $\bar{e}(\bar{\alpha})$ is measured by a scalar quantity

$$E\left[\bar{e}(\bar{\alpha})\right] = \frac{\bar{e}(\bar{\alpha}) \cdot B\bar{e}(\bar{\alpha})}{2}$$

where B is a positive definite diagonal matrix of weighting elements. Here

$$E\left[\bar{e}(\bar{\alpha})\right] = \frac{1}{2}\left[b_1 x_1^2(\bar{\alpha}) + b_2 x_2^2(\bar{\alpha}) + \ldots + b_6 x_6^2(\bar{\alpha}) + b_7 \psi_7^2(\bar{\alpha})\right] \tag{13}$$

where $b_i > 0$ $(i = 1, 2, \ldots 7)$.

A value of $\bar{\alpha}$ is assumed, the differential equations are integrated forward in time until $t = t_f = \alpha_7$, and $E\left[\bar{e}(\bar{\alpha})\right]$ is evaluated. If $\bar{e}(\bar{\alpha})$ vanishes or, for practical purposes, is sufficiently small, the boundary-value problem is considered solved. Otherwise, the assumed $\bar{\alpha}$ is corrected by

$$\delta\bar{\alpha} = -\left[\frac{\partial\bar{e}(\bar{\alpha})'}{\partial\bar{\alpha}} B \frac{\partial\bar{e}(\bar{\alpha})}{\partial\bar{\alpha}} + \lambda I\right]^{-1} \frac{\partial\bar{e}(\bar{\alpha})'}{\partial\bar{\alpha}} B\bar{e}(\bar{\alpha}) \tag{14}$$

Where $\lambda > 0$ is adjusted such that

$$E\left[\bar{e}(\bar{\alpha} + \delta\bar{\alpha})\right] < E\left[\bar{e}(\bar{\alpha})\right] \tag{15}$$

For this problem $\frac{\partial\bar{e}(\bar{\alpha})}{\partial\bar{\alpha}}$ is given as the partitioned matrix

$$\frac{\partial\bar{e}(\bar{\alpha})}{\partial\bar{\alpha}} = \left[\begin{array}{c|c} \frac{\partial\bar{x}(\bar{\alpha})}{\partial\bar{\alpha}} & \dot{\bar{x}}(\bar{\alpha}) \\ \hline \frac{\partial\psi_7(\bar{\alpha})}{\partial\bar{\alpha}} & \dot{\psi}_7(\bar{\alpha}) \end{array}\right] \tag{16}$$

where $\frac{\partial\bar{x}(\bar{\alpha})}{\partial\bar{\alpha}}$ is a Jacobian matrix with elements $\frac{\partial x_i(\bar{\alpha})}{\partial\alpha_j}$ $(i = 1, 2, \ldots 6;$ $j = 1, 2, \ldots 6)$ and $\frac{\partial\psi_7(\bar{\alpha})}{\partial\bar{\alpha}}$ is a row vector with elements $\frac{\partial\psi_7(\bar{\alpha})}{\partial\alpha_j}$ $(j = 1, 2, \ldots 6)$.

The first six elements of $\delta\bar{\alpha}$ are the corrections on $\psi_i(t_o)$ $(i = 1, 2, \ldots 6)$. The seventh element $\delta\alpha_7$ is the correction on the last value of t_f. The next final time is given by $\alpha_7 + \delta\alpha_7$.

The procedure is designed to be applied iteratively and generate a monotone decreasing sequence of $E\left[\bar{e}(\bar{\alpha})\right]$ converging to the smallest $E\left[\bar{e}(\bar{\alpha})\right]$ available relative to the initial choice of $\bar{\alpha}$. Success of the method is dependent on the user's ability to find starting values of $\bar{\alpha}$ and at each stage find values of λ such that equation (15) is satisfied. At each stage a one-dimensional search may be performed to find the values of λ. However, for this problem it was found that a value of λ of 10 or 1 would suffice throughout. In general, the method does not guarantee a solution for an arbitrary boundary-value problem. It has, however, been highly successful in yielding solutions to the boundary-value problem under consideration herein and to others (see ref. 5).

Expanded versions of equations (9a) and (9b), with x_8 replaced by t, are

$$\left.\begin{aligned}
&\dot{x}_i = x_{i+1} && \left(i = 1, 3, 5;\ \ x_i(t_o) = x_{i_o};\ \ x_i(t_f) = 0\right)\\
&\dot{x}_2 = \frac{\beta\psi_2}{x_7\sqrt{\psi}} - \Omega^2 R_s{}^3 \frac{x_1 + R_{s_x}}{(\sqrt{x})^3} + \Omega^2 R_{s_x} + \omega^2 x_1 + 2\omega x_4 && \left(x_2(t_o) = x_{2_o};\ \ x_2(t_f) = 0\right)\\
&\dot{x}_4 = \frac{\beta\psi_4}{x_7\sqrt{\psi}} - \Omega^2 R_s{}^3 \frac{x_3 + R_{s_y}}{(\sqrt{x})^3} + \Omega^2 R_{s_y} - 2\omega x_2 + \omega^2 x_3 && \left(x_4(t_o) = x_{4_o};\ \ x_4(t_f) = 0\right)\\
&\dot{x}_6 = \frac{\beta\psi_6}{x_7\sqrt{\psi}} - \Omega^2 R_s{}^3 \frac{x_5 + R_{s_z}}{(\sqrt{x})^3} + \Omega^2 R_{s_z} && \left(x_6(t_o) = x_{6_o};\ \ x_6(t_f) = 0\right)\\
&\dot{x}_7 = -\frac{\beta}{c} && \left(x_7(t_o) = m_o\right)
\end{aligned}\right\} \tag{17a}$$

and

$$\left.\begin{aligned}
&\dot{\psi}_1 = \frac{\Omega^2 R_s{}^3 \psi_2}{(\sqrt{x})^3} - 3\Omega^2 R_s{}^3 d \frac{x_1 + R_{s_x}}{(\sqrt{x})^5} - \omega^2 \psi_2 && \left(\psi_1(t_o) = \alpha_1\right)\\
&\dot{\psi}_2 = -\psi_1 + 2\omega\psi_4 && \left(\psi_2(t_o) = \alpha_2\right)\\
&\dot{\psi}_3 = \frac{\Omega^2 R_s{}^3 \psi_4}{(\sqrt{x})^3} - 3\Omega^2 R_s{}^3 d \frac{x_3 + R_{s_y}}{(\sqrt{x})^5} - \omega^2 \psi_4 && \left(\psi_3(t_o) = \alpha_3\right)\\
&\dot{\psi}_4 = -2\omega\psi_2 - \psi_3 && \left(\psi_4(t_o) = \alpha_4\right)\\
&\dot{\psi}_5 = \frac{\Omega^2 R_s{}^3 \psi_6}{(\sqrt{x})^3} - 3\Omega^2 R_s{}^3 d \frac{x_5 + R_{s_z}}{(\sqrt{x})^5} && \left(\psi_5(t_o) = \alpha_5\right)\\
&\dot{\psi}_6 = -\psi_5 && \left(\psi_6(t_o) = \alpha_6\right)\\
&\dot{\psi}_7(t) = \frac{\beta\sqrt{\psi}}{x_7{}^2} && \left(\psi_7(t_o)\ \ \text{normalized};\ \ \psi_7(t_f) = 0\right)
\end{aligned}\right\} \tag{17b}$$

where

$$\sqrt{\psi} = \sqrt{\psi_2^{\,2} + \psi_4^{\,2} + \psi_6^{\,2}}$$

$$\sqrt{x} = \sqrt{\left(x_1 + R_{s_x}\right)^2 + \left(x_3 + R_{s_y}\right)^2 + \left(x_5 + R_{s_z}\right)^2}$$

and

$$d = \psi_2\left(x_1 + R_{s_x}\right) + \psi_4\left(x_3 + R_{s_y}\right) + \psi_6\left(x_5 + R_{s_z}\right)$$

The derivatives needed to form equation (15) can be obtained (ref. 7) by solving the following system in conjunction with equations (17a) and (17b) from t_o to t_f, with $j = 1, 2, \ldots 6$:

$$\left.\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial x_i}{\partial \alpha_j}\right) = \frac{\partial x_{i+1}}{\partial \alpha_j} && \left(\frac{\partial x_i(t_o)}{\partial \alpha_j} = 0;\quad i = 1, 3, 5\right) \\
&\frac{d}{dt}\left(\frac{\partial x_2}{\partial \alpha_j}\right) = A_2 \frac{\partial \psi_2}{\partial \alpha_j} + B_{24} \frac{\partial \psi_4}{\partial \alpha_j} + B_{26} \frac{\partial \psi_6}{\partial \alpha_j} + \left[C_{11} - \frac{\Omega^2 R_s^{\,3}}{(\sqrt{x})^3} + \omega^2\right]\frac{\partial x_i}{\partial \alpha_j} && \\
&\qquad + C_{13} \frac{\partial x_3}{\partial \alpha_j} + 2\omega \frac{\partial x_4}{\partial \alpha_j} + C_{15} \frac{\partial x_5}{\partial \alpha_j} && \left(\frac{\partial x_2(t_o)}{\partial \alpha_j} = 0\right) \\
&\frac{d}{dt}\left(\frac{\partial x_4}{\partial \alpha_j}\right) = B_{24} \frac{\partial \psi_2}{\partial \alpha_j} + A_4 \frac{\partial \psi_4}{\partial \alpha_j} + B_{46} \frac{\partial \psi_6}{\partial \alpha_j} + C_{13} \frac{\partial x_1}{\partial \alpha_j} - 2\omega \frac{\partial x_2}{\partial \alpha_j} && \\
&\qquad + \left[C_{33} - \frac{\Omega^2 R_s^{\,3}}{(\sqrt{x})^3} + \omega^2\right]\frac{\partial x_3}{\partial \alpha_j} + C_{35} \frac{\partial x_5}{\partial \alpha_j} && \left(\frac{\partial x_4(t_o)}{\partial \alpha_j} = 0\right) \\
&\frac{d}{dt}\left(\frac{\partial x_6}{\partial \alpha_j}\right) = B_{26} \frac{\partial \psi_2}{\partial \alpha_j} + B_{46} \frac{\partial \psi_4}{\partial \alpha_j} + A_6 \frac{\partial \psi_6}{\partial \alpha_j} + C_{15} \frac{\partial x_1}{\partial \alpha_j} + C_{35} \frac{\partial x_3}{\partial \alpha_j} && \\
&\qquad + \left[C_{55} - \frac{\Omega^2 R_s^{\,3}}{(\sqrt{x})^3}\right]\frac{\partial x_5}{\partial \alpha_j} && \left(\frac{\partial x_6(t_o)}{\partial \alpha_j} = 0\right)
\end{aligned}\right\} \tag{18a}$$

$$
\left.
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \psi_1}{\partial \alpha_j}\right) = \left[\frac{\Omega^2 R_s{}^3}{\left(\sqrt{x}\right)^3} - C_{11} - \omega^2\right]\frac{\partial \psi_2}{\partial \alpha_j} - C_{13}\frac{\partial \psi_4}{\partial \alpha_j} - C_{15}\frac{\partial \psi_6}{\partial \alpha_j} + \left(D_{21} + E_{11}\right)\frac{\partial x_1}{\partial \alpha_j} \\
&\qquad + \left(F_{24}^{31} + E_{13}\right)\frac{\partial x_3}{\partial \alpha_j} + \left(F_{26}^{51} + E_{15}\right)\frac{\partial x_5}{\partial \alpha_j} \qquad \left(\frac{\partial \psi_1(t_o)}{\partial \alpha_j} = \begin{cases} 1 & (j = 1) \\ 0 & (j \neq 1) \end{cases}\right) \\
&\frac{d}{dt}\left(\frac{\partial \psi_2}{\partial \alpha_j}\right) = 2\omega\frac{\partial \psi_4}{\partial \alpha_j} - \frac{\partial \psi_1}{\partial \alpha_j} \qquad \left(\frac{\partial \psi_2(t_o)}{\partial \alpha_j} = \begin{cases} 1 & (j = 2) \\ 0 & (j \neq 2) \end{cases}\right) \\
&\frac{d}{dt}\left(\frac{\partial \psi_3}{\partial \alpha_j}\right) = -C_{13}\frac{\partial \psi_2}{\partial \alpha_j} + \left[\frac{\Omega^2 R_s{}^3}{\left(\sqrt{x}\right)^3} - C_{33} - \omega^2\right]\frac{\partial \psi_4}{\partial \alpha_j} - C_{35}\frac{\partial \psi_6}{\partial \alpha_j} + \left(F_{42}^{13} + E_{13}\right)\frac{\partial x_1}{\partial \alpha_j} \\
&\qquad + \left(D_{43} + E_{33}\right)\frac{\partial x_3}{\partial \alpha_j} + \left(F_{46}^{53} + E_{35}\right)\frac{\partial x_5}{\partial \alpha_j} \qquad \left(\frac{\partial \psi_3(t_o)}{\partial \alpha_j} = \begin{cases} 1 & (j = 3) \\ 0 & (j \neq 3) \end{cases}\right) \\
&\frac{d}{dt}\left(\frac{\partial \psi_4}{\partial \alpha_j}\right) = -2\omega\frac{\partial \psi_2}{\partial \alpha_j} - \frac{\partial \psi_3}{\partial \alpha_j} \qquad \left(\frac{\partial \psi_4(t_o)}{\partial \alpha_j} = \begin{cases} 1 & (j = 4) \\ 0 & (j \neq 4) \end{cases}\right) \\
&\frac{d}{dt}\left(\frac{\partial \psi_5}{\partial \alpha_j}\right) = -C_{15}\frac{\partial \psi_2}{\partial \alpha_j} - C_{35}\frac{\partial \psi_4}{\partial \alpha_j} + \left[\frac{\Omega^2 R_s{}^3}{\left(\sqrt{x}\right)^3} - C_{55}\right]\frac{\partial \psi_6}{\partial \alpha_j} + \left(F_{62}^{15} + E_{15}\right)\frac{\partial x_1}{\partial \alpha_j} \\
&\qquad + \left(F_{64}^{35} + E_{35}\right)\frac{\partial x_3}{\partial \alpha_j} + \left(D_{65} + E_{55}\right)\frac{\partial x_5}{\partial \alpha_j} \qquad \left(\frac{\partial \psi_5(t_o)}{\partial \alpha_j} = \begin{cases} 1 & (j = 5) \\ 0 & (j \neq 5) \end{cases}\right) \\
&\frac{d}{dt}\left(\frac{\partial \psi_6}{\partial \alpha_j}\right) = -\frac{\partial \psi_5}{\partial \alpha_j} \qquad \left(\frac{\partial \psi_6(t_o)}{\partial \alpha_j} = \begin{cases} 1 & (j = 6) \\ 0 & (j \neq 6) \end{cases}\right) \\
&\frac{d}{dt}\left(\frac{\partial \psi_7}{\partial \alpha_j}\right) = \frac{\beta\left(\psi_2\frac{\partial \psi_2}{\partial \alpha_j} + \psi_4\frac{\partial \psi_4}{\partial \alpha_j} + \psi_6\frac{\partial \psi_6}{\partial \alpha_j}\right)}{x_7{}^2\sqrt{\psi}} \qquad \left(\frac{\partial \psi_7(t_o)}{\partial \alpha_j} = 0\right)
\end{aligned}
\right\} \quad (18b)
$$

In this system

$$A_i = \beta\left[\frac{1}{x_7\sqrt{\psi}} - \frac{\psi_i{}^2}{x_7\left(\sqrt{\psi}\right)^3}\right]$$

$$B_{ik} = -\frac{\beta\psi_i\psi_k}{x_7\left(\sqrt{\psi}\right)^3}$$

$$C_{ik} = \frac{3\Omega^2 R_s{}^3 v_i v_k}{\left(\sqrt{x}\right)^5}$$

$$D_{ik} = -\frac{3\Omega^2 R_s{}^3\left(2\psi_i v_k + d\right)}{\left(\sqrt{x}\right)^5}$$

$$E_{ik} = \frac{15\Omega^2 R_s{}^3 d v_i v_k}{\left(\sqrt{x}\right)^7}$$

and

$$F_{em}^{ik} = -\frac{3\Omega^2 R_s{}^3\left(\psi_e v_i + \psi_m v_k\right)}{\left(\sqrt{x}\right)^5}$$

with

$$v_1 = x_1 + R_{s_x}$$

$$v_3 = x_3 + R_{s_y}$$

and

$$v_5 = x_5 + R_{s_z}$$

Iteration Sequence

In summary, the procedure used to solve the boundary-value problem presented in the preceding section is as follows:

(1) Assume a value of $\overline{\alpha}$ given by equation (12).

(2) Solve equations (17a) and (17b) to $t_f = \alpha_7$ and evaluate $\bar{e}(\overline{\alpha})$ given by equation (12).

(3) If $\bar{e}(\overline{\alpha})$ is sufficiently small, then the problem is solved; $\overline{\alpha}$ gives the unknown initial conditions and final time, and the corresponding solution of equation (13) gives the optimal trajectory.

(4) Otherwise, with the use of the solution given by equation (13), for the assumed $\overline{\alpha}$, solve equations (18a) and (18b) and evaluate equation (16).

(5) If previous value is not satisfactory, find λ such that equation (15) follows.

(6) Replace $\overline{\alpha}$ by $\overline{\alpha} + \delta\overline{\alpha}$ and return to step (2).

A digital computer program for this procedure is discussed in the next section.

DESCRIPTION OF PROGRAM

General

The program was written in FORTRAN IV language for the Control Data series 6000 computer systems at the Langley Research Center. A complete listing of the program is found in appendix B.

Upon acceptance of an assumed value of $\overline{\alpha}$ and appropriate system constants characterizing the particular rendezvous problem to be solved, the program proceeds according to the steps listed in the preceding section. The program does not contain a search routine for the determination of λ in step (5). It is expected that in each instance a value of λ which will work for the complete iteration process can be found.

The mathematical symbols used in the theory and their FORTRAN equivalents are given in table 1.

TABLE 1.- FORTRAN EQUIVALENTS OF MATHEMATICAL SYMBOLS

Mathematical symbol		FORTRAN equivalent	
b_i	$(i = 1, 2, \ldots 7)$	B(I)	(I = 1, 2, . . . 7)
c		C	
$e_i(\overline{\alpha})$	$(i = 1, 2, \ldots 7)$	E(J)	(J = 1, 2, . . . 7)
$E\left[\bar{e}(\overline{\alpha})\right]$		EDP	
ι_o		IO	
R_s		RS	
R_{s_x},R_{s_y},R_{s_z}		RSV(I)	(I = 1, 2, 3)
$\dot{R}_{s_x},\dot{R}_{s_y},\dot{R}_{s_z}$		DRSV(I)	(I = 1, 2, 3)
R_v		RV	
R_{v_x},R_{v_y},R_{v_z}		RVX,RVY,RVZ	
$\dot{R}_{v_x},\dot{R}_{v_y},\dot{R}_{v_z}$		DRVX,DRVY,DRVZ	
t_o		VAR(1)	
t_f		TF	
T_1		XT1(I,J)	(I = 1, 2, 3; J = 1, 2, 3)
T_2		XT2(I,J)	(I = 1, 2, 3; J = 1, 2, 3)
u_1,u_2,u_3		UX,UY,UZ	
x_i	$(i = 1, 2, \ldots 7)$	VAR(I)	(I = 2, 3, . . . 8)
$\frac{\partial\bar{e}'}{\partial\overline{\alpha}} B \frac{\partial\bar{e}}{\partial\overline{\alpha}} + \lambda I$		PEMAT(I,J)	(I = 1, 2, . . . 7; J = 1, 2, . . . 7)
$\frac{\partial x_i(\overline{\alpha})}{\partial\alpha_j}$	$(i = 1, 2, \ldots 6;\ j = 1, 2, \ldots 6)$	PX(I,J)	(I = 1, 2, . . . 6; J = 1, 2, . . . 6)
α_i	$(i = 1, 2, \ldots 7)$	PEVEC(I,1)	(I = 1, 2, . . . 7)
β		BETA	
θ_o		THETAO	
$\theta_v{}^o$		THETVO	
λ		LAMBDA	
μ		MU	
φ_o		PHIO	
$\varphi_v{}^o$		PHIVO	
ψ_i	$(i = 1, 2, \ldots 7)$	VAR(I)	(I = 9, 10, . . . 15)
ω		SØMEG	
Ω		CØMEG	
$\frac{\partial\psi_i(\overline{\alpha})}{\partial\alpha_j}$	$(i = 1, 2, \ldots 6;\ j = 1, 2, \ldots 6)$	PPSI(I,J)	(I = 1, 2, . . . 6; J = 1, 2, . . . 6)
$\frac{\partial\psi_7(\overline{\alpha})}{\partial\alpha_i}$	$(i = 1, 2, \ldots 6)$	PPSI7(I)	(I = 1, 2, . . . 6)

Subroutines

Seven subroutines are used in addition to the main program. The purpose of each is given in table 2:

TABLE 2.- SUBROUTINE LISTING

Subroutine	Purpose
DERSUB	Evaluates all differential equations to be solved
CHSUB	Tests for the end of a trajectory
COMP	Evaluates the position and rate of $\overline{R}_s(t)$, the vector from the origin to the target
ITERAT	Computes and applies the correction $\delta\overline{\alpha}$ to the initial $\overline{\alpha}$
BLOCK DATA . . .	Initializes $\frac{\partial x_i(\overline{\alpha})}{\partial \alpha_j}$, $\frac{\partial \psi_i(\overline{\alpha})}{\partial \alpha_j}$, and $\frac{\partial \psi_7(\overline{\alpha})}{\partial \alpha_j}$ $(i = 1, 2, \ldots 6;\ j = 1, 2, \ldots 6)$
INT2	Numerically integrates the differential equations with a fixed-step size method by employing a fourth-order Adams-Bashforth predictor formula and a fourth-order Adams-Moulton corrector formula
MATINV	Obtains the inverse of the matrix $\left[\frac{\partial \bar{e}'}{\partial \overline{\alpha}} B \frac{\partial \bar{e}}{\partial \overline{\alpha}} + \lambda I\right]$

Input

Input is of the form shown in table 3:

TABLE 3.- INPUT DATA

Card number	FORTRAN variable name	Description	FORTRAN format
1	NO	Case number	I20
2	SØMEG, BETA, C, TF	ω, β, c, $t_f = \alpha_7$	4E20.8
3	PHIVO, THETVO, RV	$\varphi_v{}^o$, $\theta_v{}^o$, $R_v(t_o)$	3E20.8
4	DRVXO, DRVYO, DRVZO	$\dot{R}_{v_x}(t_o)$, $\dot{R}_{v_y}(t_o)$, $\dot{R}_{v_z}(t_o)$	3E20.8
5	PHIO, THETAO, IO, RS	φ_o, θ_o, ι_o, R_s	4E20.8
6	VAR(1), VAR(8), MU	t_o, m_o, μ	3E20.8
7	VAR(9) to VAR(12)	α_1 to α_4	4E20.8
8	VAR(13) to VAR(15)	α_5, α_6, $\psi_7(t_o)$	3E20.8
9	CI, SPEC	Computing interval, printing frequency (see "Output" section)	2E20.8
10	IPRINT, IERØR, IMAT	(See "Output" section)	3I20
11	LAMBDA, CRIT, MAXIT	λ, stopping criterion, maximum number of iterations (see "Output" section)	2E20.8, I20
12	B(1) to B(4)	b_1 to b_4	4E20.8
13	B(5) to B(7)	b_5 to b_7	3E20.8

All the input variables are dimensionalized and angles are in radians; α_i $(i = 1, 2, \ldots 6)$, b_i $(i = 1, 2, \ldots 7)$, and ψ_i $(i = 0, 1, \ldots 8)$ are considered dimensionless.

Output

The program offers several options for output. Regardless of the options, the input data are always printed initially. Afterwards, output is presented at each iteration according to the following input variables: SPEC, IPRINT, IERØR, IMAT.

SPEC specifies how often results are to be printed. If SPEC = 10^{10}, output will be printed only at $t = t_o$ and $t = t_f$. If SPEC = nCI, where CI is the integration computing interval and n is a positive integer, results will be printed every n integration step. At $t = t_o$, the variables that are printed are t_o, $\psi_i(t_o)$ $(i = 1, 2, \ldots 7)$, and u_i $(i = 1, 2, 3)$. At any other time t, determined by SPEC, the variables that are printed are t, $\psi_i(t)$ $(i = 1, 2, \ldots 7)$, $\overline{R}_s(t)$, $\dot{\overline{R}}_s(t)$, $\overline{R}_v(t)$, $\dot{\overline{R}}_v(t)$, x_i $(i = 1, 2, \ldots 7)$, u_i $(i = 1, 2, 3)$, $\dot{x}_2(t)$, $\dot{x}_4(t)$, $\dot{x}_6(t)$, β, $\left\|\dot{\overline{R}}_v(t)\right\|$, and the relative distances and velocities $\sqrt{{x_1}^2 + {x_3}^2 + {x_5}^2}$ and $\frac{d}{dt}\sqrt{{x_1}^2 + {x_3}^2 + {x_5}^2}$. At $t = t_f$, $E\left[\bar{e}(\overline{\alpha})\right]$ and $\delta\overline{\alpha}$ are also printed.

IPRINT provides the option for printing the partial derivatives $\frac{\partial\psi_i(t)}{\partial\alpha_j}$, $\frac{\partial x_i(t)}{\partial\alpha_j}$, and $\frac{\partial\psi_7}{\partial\alpha_j}$ $(i = 1, 2, \ldots 6;\ j = 1, 2, \ldots 6)$. If IPRINT = 0, the partial derivatives are not printed; if IPRINT = 1, the partial derivatives are printed.

IERØR provides the option for printing the truncation errors for the differential equations. If IERØR = 0, truncation errors for the differential equations are not printed; if IERØR = 1, the truncation errors are printed.

IMAT provides the option for printing the matrix $\left[\frac{\partial\bar{e}'(\overline{\alpha})}{\partial\overline{\alpha}} B \frac{\partial\bar{e}(\overline{\alpha})}{\partial\overline{\alpha}} + \lambda I\right]$, its inverse, and the product of the two. If IMAT = 0, the matrices are not printed; if IMAT = 1, the matrices are printed.

The program terminates when convergence is reached $\left(E\left[\bar{e}(\overline{\alpha})\right] \leqq \text{CRIT}\right)$ or when the maximum number of iterations (MAXIT) has been reached.

EXAMPLE COMPUTATIONS

The use of the foregoing procedures is demonstrated by the problem of a space vehicle launched from the surface of the moon to rendezvous, in a minimum of flight time, with a target in a circular orbit. The vehicle has a bounded thrust magnitude which, along with the thrust direction, acts as a control variable. There is no terminal mass constraint. The system constants used in this study are given in table 4.

TABLE 4.- SYSTEM CONSTANTS

Initial time, t_o, sec	0	
Upper bound on thrust magnitude, β, lbm (kg)	3504	(1589.4)
Initial mass, m_o, slugs (kg)	285.5	(4166.3)
Effective exhaust velocity, c, ft/sec (m/sec)	9853.2	(279.86)
$R_v(t_o)$ set equal to radius of moon, ft (km)	5.707×10^6	(1739.4)
R_s set equal to radius of 80-nautical-mile satellite circular orbit, ft (km)	6.1934×10^6	(1887.7)
Universal gravitational constant multiplied by the moon mass, μ, ft^3/sec^2 (m^3/sec^2)	1.727×10^{14}	(48.90×10^{11})
Angular velocity of moon about axis of rotation, ω, rad/sec	2.66×10^{-6}	

The satellite orbital plane was placed in the xy-plane of the rotating system (fig. A-1). Studies were made with the vehicle launched from rest from the surface of the moon with the launch site both in and out of this plane (planar and nonplanar case, respectively).

It was found that a workable set of b_i $(i = 1, 2, \ldots 7)$ and λ for convergence was $b_1 = b_3 = b_5 = b_7 = 1$, $b_2 = b_4 = b_6 = 10$, and $\lambda = 10$ or $\lambda = 1$. These values were used throughout. It was observed that increasing the value of λ yielded a slower converging process, while a decrease was apt to produce divergence. It was also observed that by increasing a particular b_i, greater influence could be applied to the correction of the error e_i; that is, this error would be corrected more quickly than before, but at the expense of the other errors.

It was found that in the planar case, with the vehicle launched such that the satellite lead angle $\varphi_o - \varphi_v{}^o$ was 9^o (with $\varphi_o = 89^o$), the set of values

$$\alpha_1 = -100\ddot{R}_{s_x}(t_o) = 7.8565$$

$$\alpha_2 = \dot{R}_{s_x}(t_o) = 5.296 \times 10^3$$

$$\alpha_3 = \ddot{R}_{s_y}(t_o) = -4.5016$$

$$\alpha_4 = \dot{R}_{s_y}(t_o) = -92.44$$

$$\alpha_5 = \ddot{R}_{s_z}(t_o) = 0$$

$$\alpha_6 = \dot{R}_{s_z}(t_o) = 0$$

$$\alpha_7 = 500 \text{ seconds}$$

$(\psi_7(t_o)$ having been normalized at $-1.184 \times 10^5)$ leads to convergence with the values of b_i previously mentioned and $\lambda = 10$ in 12 iterations. This procedure gave a solution to be used as a nominal, or guessed, solution for neighboring lead angles. Table 5 shows the planar results obtained. For each lead angle the iteration was stopped when

$$\bar{x}(t_f) \cdot \bar{x}(t_f) + \psi_7^2(t_f) \leqq 1$$

with $\psi_7(t_o)$ normalized at -1.184×10^5. Since the results are planar, $\psi_5(t) \equiv \psi_6(t) \equiv 0$.

It can be seen from table 5 that near the lead angle 13.7^o, the smallest value of t_f and hence the largest terminal mass occur. Figure 1 shows a graph of these results.

TABLE 5.- TIME-OPTIMAL PLANAR RESULTS

Lead angle, $\varphi_o - \varphi_v^o$, deg	Unknown parameters					Percent initial mass at t_f
	$\psi_1(t_o)$	$\psi_2(t_o)$	$\psi_3(t_o)$	$\psi_4(t_o)$	t_f, sec	
8	11.662	5577.3	5.6883	2051.8	547.9	31.7
9	12.929	5948.3	7.4703	2638.8	524.8	34.6
10	14.120	6239.4	10.026	3422.5	499.5	37.8
12	12.530	5464.5	17.992	5427.7	453.0	43.6
13	6.9429	3774.3	20.577	5758.3	443.3	44.8
13.7	2.4694	2500.4	20.507	5501.0	442.3	44.9
14	.73674	2012.3	20.108	5315.8	443.0	44.8
16	-6.2567	-42.810	15.897	4005.4	454.8	43.3
18	-8.4150	-882.65	12.256	3110.3	471.7	41.2
22	-8.5773	-1472.0	7.8560	2155.4	507.3	36.8

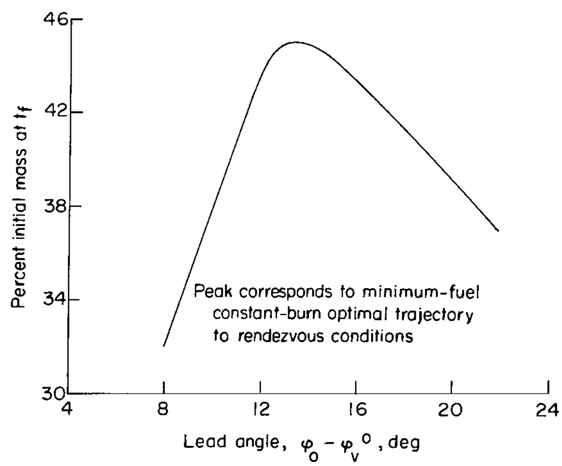

Figure 1.- Percentage of initial mass at rendezvous as a function of lead angle $\varphi_o - \varphi_v^o$. Planar case; $\theta_v^o = 0^o$.

Schematic views of the flight paths for planar solutions with lead angles 8°, 13.7°, and 22° are shown in figure 2. From this figure an idea can be gained as to the different maneuvers required for different lead angles. Arrows placed along the trajectories indicate the true direction of the thrust vector at 50-second intervals. The spatial coordinates used in this plot are to different scales. The xy-plane is viewed as being inertial since the total rotation of the moon was less than 0.1° for the longest flight time.

Examples of out-of-plane results were obtained by holding φ_0 and $\varphi_v{}^{o}$ fixed at 93.7° and 80°, respectively, and allowing nonzero values of $\theta_v{}^{o}$. For $\theta_v{}^{o} = 2^{\circ}$, the planar solution for $\varphi_0 - \varphi_v{}^{o} = 13.7^{\circ}$ was used as a nominal. The results for this nonplanar case exemplify a typical sequence of iterations, and this sequence is tabulated in table 6. For $\theta_v{}^{o} = 2^{\circ}$, $b_1 = b_3 = b_5 = b_7 = 1$, $b_2 = b_4 = b_6 = 10$, and $\lambda = 1$. Table 7 shows other out-of-plane results for the fixed lead angle of 13.7°. A graph of the percentage of initial mass at rendezvous as a function of out-of-plane angle $\theta_v{}^{o}$ is shown in figure 3.

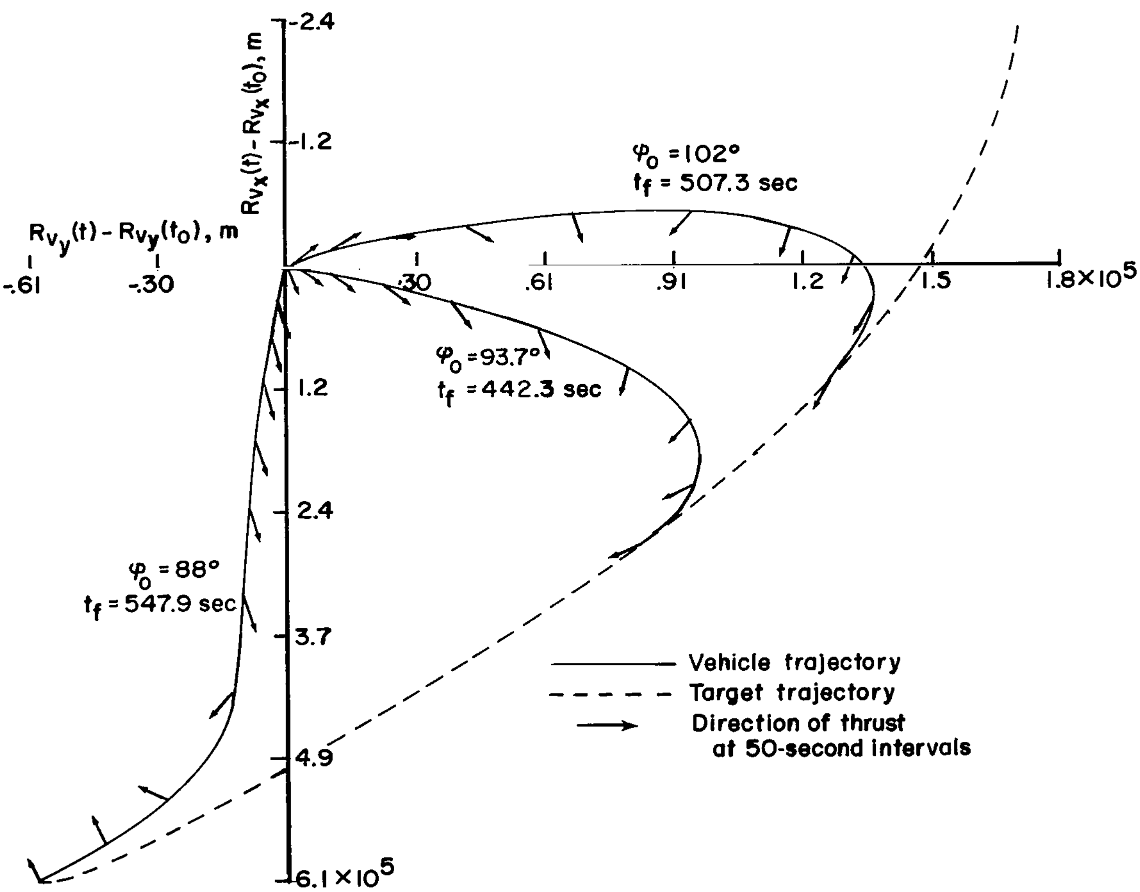

Figure 2.- Comparison of planar time-optical trajectories. $\varphi_v{}^{o} = 80^{\circ}$; $\varphi_0 = 88^{\circ}$, 93.7°, and 102°; $\theta_v{}^{o} = 0^{\circ}$. (0.3048 meter = 1 foot)

TABLE 6.- TYPICAL TIME-OPTIMAL ITERATION SEQUENCE

$\left[\varphi_o = 93.7^o, \quad \varphi_v{}^o = 80^o; \quad \theta_v{}^o = 2^o \quad \text{(except for nominal results)}\right]$

Iteration	Unknown parameters $\psi_1(t_o)$	$\psi_2(t_o)$	$\psi_3(t_o)$	$\psi_4(t_o)$	$\psi_5(t_o)$	$\psi_6(t_o)$	t_f, sec	$E\left[\bar{e}(\bar{\alpha})\right]$
Nominal[a]	2.4694	2500.4	20.507	5501.0	0	0	442.3	1.75×10^{10}
1	2.6048	2526.5	20.415	5498.3	-4.6177	-1340.9	443.2	2.30×10^{8}
2	3.5498	2718.7	19.445	5369.0	-4.8561	-1409.4	448.5	2.23×10^{6}
3	3.6100	2732.8	19.461	5371.7	-4.8560	-1412.6	448.8	9.29×10
4	3.6263	2736.9	19.469	5374.8	-4.8587	-1413.6	448.8	9.55
5	3.6325	2738.6	19.474	5376.2	-4.8596	-1413.9	448.8	1.61
[b]6	3.6351	2739.2	19.476	5376.8	-4.8599	-1414.0	448.8	.271

[a] $\theta_v{}^o = 0$; $x_1(t_f) = -582.7$, $x_2(t_f) = -0.03386$, $x_3(t_f) = -3275.5$, $x_4(t_f) = 0.15997$, $x_5(t_f) = 1.8728 \times 10^5$, $x_6(t_f) = -27.581$, and $\psi_7(t_f) = 9.9489$.

[b] $x_1(t_f) = -0.0017$, $x_2(t_f) = 0.230$, $x_3(t_f) = -0.006$, $x_4(t_f) = -0.026$, $x_5(t_f) = 0.001$, $x_6(t_f) = 0.015$, and $\psi_7(t_f) = -0.042$ for a computing time of 23 seconds.

TABLE 7.- TIME-OPTIMAL OUT-OF-PLANE RESULTS

$\left[\varphi_o = 93.7^o, \quad \varphi_v{}^o = 80^o\right]$

$\theta_v{}^o$, deg	Unknown parameters $\psi_1(t_o)$	$\psi_2(t_o)$	$\psi_3(t_o)$	$\psi_4(t_o)$	$\psi_5(t_o)$	$\psi_6(t_o)$	t_f, sec	Percent initial mass at t_f
0	2.4694	2500.4	20.507	5501.0	0	0	442.3	44.9
2	3.6351	2739.2	19.476	5376.8	-4.8599	-1414.0	448.8	44.1
4	5.6559	3154.0	16.644	4913.9	-7.8427	-2482.2	466.8	41.8
6	6.6424	3323.6	13.215	4201.8	-8.6545	-3010.3	492.2	38.7
8	6.5482	3215.7	10.282	3495.9	-8.2675	-3132.6	520.0	35.2
10	5.9679	2970.9	8.0805	2914.0	-7.4739	-3048.8	546.6	31.9

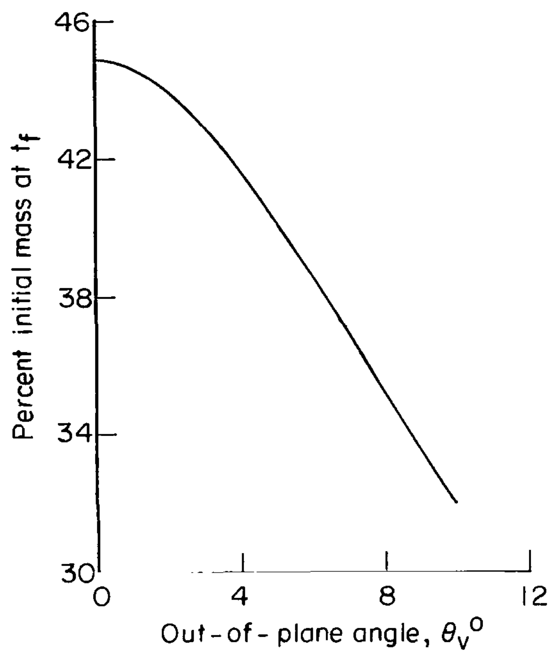

Figure 3.- Percentage of initial mass at rendezvous as a function of out-of-plane angle $\theta_v{}^o$ for fixed lead angle $\varphi_0 - \varphi_v{}^o$ of 13.7^o.

The computational time for obtaining both planar and nonplanar trajectories was less than 1 minute.

CONCLUDING REMARKS

A technique has been presented for obtaining three-dimensional rendezvous trajectories between a one-stage rocket vehicle and a target in a circular Keplerian orbit. The trajectories satisfy the necessary conditions of the Pontryagin maximum principle for time-optimal rendezvous in which no terminal mass constraint is placed on the rocket.

The use of Pontryagin's theory leads to a two-point boundary-value problem. Certain initial conditions on a set of differential equations introduced by the maximum principle had to be found such that certain boundary conditions were met. A digital program was given for the solution of this problem based on an iteration method. Given assumed values of the initial conditions, which do not yield rendezvous, the program attempts to correct these values in such a way that rendezvous is more closely attained. Iterative use of this procedure gives a sequence of trajectories converging to one yielding rendezvous.

The program was successfully applied to a problem in which a space vehicle was launched from the surface of the moon and required to rendezvous with a target in an 80-nautical-mile circular orbit. Both planar and nonplanar trajectories were obtained with equal ease in less than 1 minute of computational time on the Control Data series 6000 computer systems.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., November 8, 1968,
125-17-05-10-23.

APPENDIX A

FORMULATION OF DYNAMIC EQUATIONS

The dynamic equations for a space vehicle which is required to rendezvous with a target in a circular Keplerian orbit about a rotating body are derived in reference 5. The formulation of these equations is summarized in this appendix. The vehicle is a one-stage rocket, treated as a point mass, with bounded thrust magnitude. The controls are the magnitude and direction of the thrust vector.

Let x, y, and z be Cartesian coordinates of a rotating axis system located in the center of the body with the z-axis through the axis of rotation of the body. The geometry is represented in figure A-1.

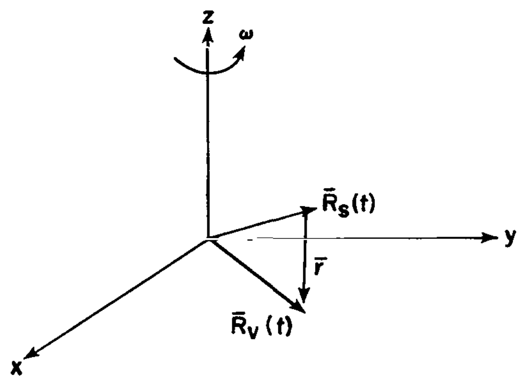

Figure A-1.- The rotating axis system.

The vector $\overline{R}_S(t)$ is from the origin to the target, $\overline{R}_V(t)$ is from the origin to the vehicle, and ω is the angular velocity of the body about its axis of rotation. The relative distance between the target and vehicle is given by

$$\bar{r}(t) = \overline{R}_V(t) - \overline{R}_S(t) \tag{A1}$$

Since the target is assumed to be in a circular orbit, it moves in its orbital plane at a constant distance R_S from the center of the body with a constant angular velocity $\Omega = \left(\mu/R_S{}^3\right)^{1/2}$, where μ is the universal gravitational constant multiplied by the body mass and where the magnitude of R_S is $\left[\overline{R}_S(t) \cdot \overline{R}_S(t)\right]^{1/2}$. Consider an inertial XYZ-axis system fixed in the center of the body such that, at the initial time t_o, it is alined with the rotating xyz-axis system. In this framework the target can be pictured as in figure A-2.

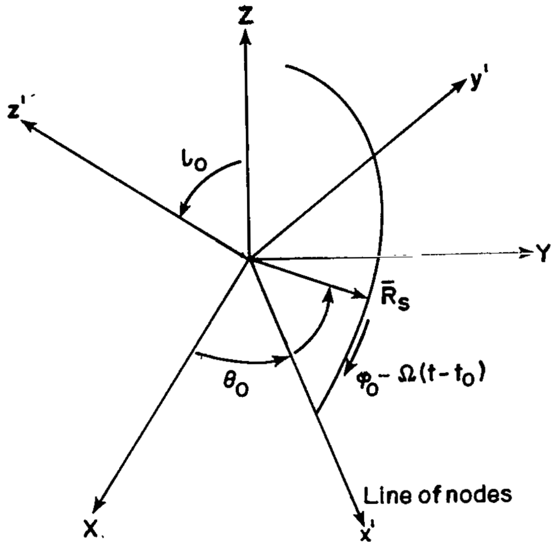

Figure A-2.- Target viewed in the inertial axis system.

The angles ι_o and θ_o define the normal and line of nodes, respectively, of the target orbital plane relative to the inertial system. The x' and y' axes therefore define the orbital plane of the target. If at t_o the target is in the position $(x',y') = \left(R_s \cos \varphi_o,\ R_s \sin \varphi_o\right)$ and moves toward the line of nodes, then

$$\overline{R}_s\left[x'(t),y'(t),z'(t)\right] = R_s\begin{Bmatrix} \cos\left[\varphi_o - \Omega(t - t_o)\right] \\ \sin\left[\varphi_o - \Omega(t - t_o)\right] \\ 0 \end{Bmatrix} \tag{A2}$$

and

$$\overline{R}_s(X,Y,Z) = T_1\overline{R}_s(x',y',z') \tag{A3}$$

where

$$T_1 = \begin{bmatrix} \cos \theta_o & -\cos \iota_o \sin \theta_o & \sin \iota_o \sin \theta_o \\ \sin \theta_o & \cos \iota_o \cos \theta_o & -\sin \iota_o \cos \theta_o \\ 0 & \sin \iota_o & \cos \iota_o \end{bmatrix} \tag{A4}$$

Since the xyz-axis system rotates about the Z-axis with a constant angular velocity ω,

$$\overline{R}_s(x,y,z) = \overline{R}_s(t) = T_2(t)\overline{R}_s(X,Y,Z) \tag{A5}$$

where

$$T_2(t) = \begin{bmatrix} \cos \omega(t - t_o) & \sin \omega(t - t_o) & 0 \\ -\sin \omega(t - t_o) & \cos \omega(t - t_o) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also

$$\frac{d\overline{R}_s(t)}{dt} = \dot{T}_2(t)\overline{R}_s(X,Y,Z) + T_2(t)\dot{\overline{R}}_s(X,Y,Z) \tag{A6}$$

where

$$\dot{T}_2(t) = \omega\begin{bmatrix} -\sin \omega(t - t_o) & \cos \omega(t - t_o) & 0 \\ -\cos \omega(t - t_o) & -\sin \omega(t - t_o) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\dot{\overline{R}}_s(X,Y,Z) = R_s\Omega T_1\begin{Bmatrix} \sin\left[\varphi_o - \Omega(t - t_o)\right] \\ -\cos\left[\varphi_o - \Omega(t - t_o)\right] \\ 0 \end{Bmatrix}$$

Thus the position and rate of $\overline{R}_s(t)$ can be obtained by specifying R_s, ι_o, θ_o, and φ_o at t_o and by using equations (A5) and (A6).

The thrust control vector $\overline{T}$ is related to the rotating axis system by

$$\overline{T} = T\begin{bmatrix} \cos \theta_c & \cos \varphi_c \\ \cos \theta_c & \sin \varphi_c \\ & \sin \theta_c \end{bmatrix} \tag{A7}$$

as shown in figure A-3.

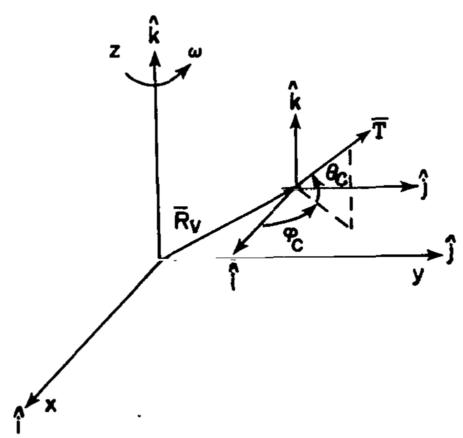

Figure A-3.- Reference axis system for control vector.

The vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors in the direction of the x, y, and z axes, respectively, and T is the magnitude of the thrust vector. Let

$$\hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos \theta_c & \cos \varphi_c \\ \cos \theta_c & \sin \varphi_c \\ & \sin \theta_c \end{bmatrix}$$

$$u_4 = T$$

$$\bar{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} R_{V_x} - R_{S_x} \\ R_{V_y} - R_{S_y} \\ R_{V_z} - R_{S_z} \end{bmatrix}$$

$$\dot{\bar{r}} = \begin{bmatrix} \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \end{bmatrix}$$

and

$x_1 = r_x$ (relative x distance)
$x_2 = \dot{r}_x$ (relative x velocity)
$x_3 = r_y$ (relative y distance)
$x_4 = \dot{r}_y$ (relative y velocity)
$x_5 = r_z$ (relative z distance)
$x_6 = \dot{r}_z$ (relative z velocity)
$x_7 = m(t)$ (instantaneous vehicle mass)
$x_8 = t$ (time)

In this framework the dynamic equations can be written as (ref. 5)

$$\left.\begin{aligned} \dot{\bar{x}} &= \frac{u_4 M\hat{u}}{x_7} + \bar{Y}(\bar{x},x_8) && \left(\bar{x}(t_o) = \bar{x}_o;\quad \bar{x}(tf) = \bar{0}\right) \\ \dot{x}_7 &= -\frac{u_4}{c} && \left(x_7(t_o) = m(t_o) = m_o\right) \\ \dot{x}_8 &= 1 && \left(x_8(t_o) = t_o\right) \end{aligned}\right\} \qquad \text{(A8)}$$

where

$$Y(\bar{x},x_8) = -\frac{\Omega^2 {R_s}^3}{\left\|A\bar{x} + R_s\right\|^3}\left[N\bar{x} + M\bar{R}_s(x_8)\right] + \Omega^2 M\bar{R}_s(x_8) + \left(N' + 2\omega K + \omega^2 L\right)\bar{x} \qquad \text{(A9)}$$

with

$\bar{x} = \mathrm{col}\left(x_1, \ . \ . \ . \ x_6\right)$

APPENDIX A

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

In order to compute the initial value of $\bar{x}(t_o)$, the initial value of $\bar{R}_V(t_o)$ can be specified by

$$\bar{R}_V(t_o) = \begin{bmatrix} \cos\theta_V{}^o & \cos\varphi_V{}^o \\ \cos\theta_V{}^o & \sin\varphi_V{}^o \\ \multicolumn{2}{c}{\sin\theta_V{}^o} \end{bmatrix}$$

where $\theta_V{}^o$ and $\varphi_V{}^o$ are as shown in figure A-4. Also

$$\dot{\bar{R}}_V(t_o) = \begin{bmatrix} \dot{R}_{V_x}(t_o) \\ \dot{R}_{V_y}(t_o) \\ \dot{R}_{V_z}(t_o) \end{bmatrix}$$

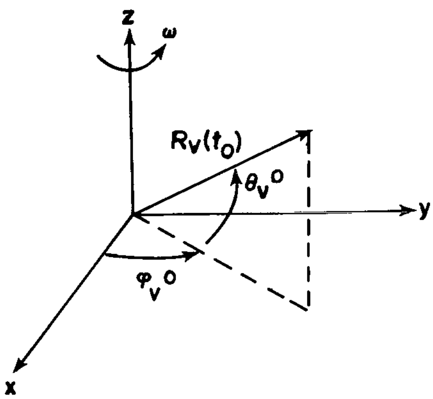

Figure A-4.- Initial orientation of vehicle with respect to the rotating axis system.

and $\bar{R}_S(t_o)$ and $\dot{\bar{R}}_S(t_o)$ are computed from equations (A2) to (A6).

The act of rendezvous requires that the vehicle and target have the same position and velocity at t_f; hence, the condition $\bar{x}(t_f) = \bar{0}$. In addition, $u_4 \leqq \beta$, the largest value obtainable for the thrust magnitude.

APPENDIX B

PROGRAM LISTING

The program presented on the following pages is written in FORTRAN IV language for the Control Data series 6000 computer systems at the Langley Research Center.

APPENDIX B

```
      PROGRAM E1257(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     TIME OPTIMAL RENDEZVOUS STUDY
C
C
C     DEFINITIONS
C
C     NO                          CASE NO.
C     SOMEG                       ANG VEL OF CENTRAL BODY ABOUT ITS OWN AXIS
C     BETA                        MAX. THRUST
C     C                           EFFECTIVE EXHAUST SPEED
C     TF                          FINAL TIME - GUESS
C     PHIVO,THETVO                INITIAL POSITION OF VEHICLE
C     RV                          MAG. OF RAD. VECTOR TO VEHICLE
C     DRVXO,DRVYO,DRVZO           RVXO DOT, RVYO DOT, RVZO DOT
C     PHIO,THETAO,IO              INITIAL POSITION OF ORBITING VEHICLE
C     RS                          MAG. OF RAD. VECTOR TO ORBITING VEHICLE
C     VAR(1)                      INITIAL TIME
C     VAR(8)                      INITIAL MASS
C     MU                          GRAVITATIONAL CONSTANT OF THE CENTRAL BODY
C     VAR(9)-VAR(15)              I.C. ON  PSI1 - PSI7
C     CI                          COMPUTING INTERVAL
C     SPEC                        SPEC=1.0E10  DO NOT PRINT AT SUB-INTERVALS
C                                 SPEC .LT. 1.0E10 USE AS PRINT INTERVAL
C     IPRINT                      1 = PRINT PARTIALS  -  0 = NO PRINT
C     IEROR                       0=DO NOT PRINT TRUNCATION ERRORS - 1=PRINT
C     IMAT                        1 = PRINT MATRICIES -  0 = NO PRINT
C     LAMBDA                      PARAMETER USED IN ITERATION SCHEME
C     CRIT                        STOPPING CRITERION
C     MAXIT                       MAX NO. OF ITERATIONS DESIRED
C     B                           DIAGONAL MATRIX OF WEIGHTING FACTORS
C
C     INPUT AS FOLLOWS
C
C     INPUT   CARD NO.            QUANTITY                  FORMAT
C
C               1                 NO                        I
C               2                 SOMEG,BETA,C,TF           E
C               3                 PHIVO,THETVO,RV           E
C               4                 DRVXO,DRVYO,DRVZO         E
C               5                 PHIO,THETAO,IO,RS         E
C               6                 VAR(1),VAR(8),MU          E
C               7                 VAR(9) - VAR(12)          E
C               8                 VAR(13) - VAR(15)         E
C               9                 CI,SPEC                   E
```

```
C             10                      IPRINT,IEROR,IMAT          I
C             11                      LAMBDA,CRIT,MAXIT          E,I
C             12                      B(1)-B(4)                  E
C             13                      B(5)-B(7)                  E
C
C
      COMMON /SPACE/
     1        VAR      ,CUVAR    ,SAVE     ,C        ,MAXIT    ,IMAT     ,
     2        E        ,PE       ,PEMAT    ,PEVEC    ,ERRVAL   ,SOMEG    ,
     3        DER      ,RS       ,BETA     ,DP1      ,TF       ,ELE1     ,
     4        RSV      ,F1       ,DP2      ,CI       ,ELE2     ,DRSV     ,
     5        F2       ,E1       ,II       ,SOMEGS   ,COMEGS   ,IPRINT   ,
     6        F3       ,E2       ,N        ,COMEG    ,EN2      ,KOUNT    ,
     7        Q1       ,E3       ,TEMPCI   ,SIO      ,PHIO     ,Q2       ,
     8        E4       ,SPEC     ,STO      ,LAMBDA   ,TEMPT    ,Q3       ,
     9        T2       ,TEMPSP   ,CIO      ,CRIT     ,EN1      ,IKOUNT   ,
     1        RVXO     ,RVYO     ,RVZO     ,DRVXO    ,DRVYO    ,DRVZO    ,
     2        T4       ,MU       ,CTO      ,B        ,XT1
C
      REAL LAMBDA,MU
      REAL IO
      LOGICAL FIRST
C
C
      DIMENSION
     1        VAR(93)             ,CUVAR(93)          ,DER(93)           ,
     2        ELE1(92)            ,ELE2(92)           ,ERRVAL(92)        ,
     3        RSV(3)              ,DRSV(3)            ,PX(6,6)           ,
     4        PPSI(6,6)           ,PPSI7(6)           ,CUPX(6,6)         ,
     5        CUPSI(6,6)          ,CUPSI7(6)          ,DRPX(6,6)         ,
     6        DRPSI(6,6)          ,DRPSI7(6)          ,ERX(6,6)          ,
     7        ERPSI(6,6)          ,ERPSI7(6)          ,E(7)              ,
     8        PE(7,7)             ,PEMAT(7,7)         ,PEVEC(7,1)        ,
     9        B(7)                ,SAVE(93)           ,XT1(3,3)
C
C
      EQUIVALENCE
     1       (VAR(16),PX(1,1))               ,(VAR(52),PPSI(1,1))         ,
     2       (VAR(88),PPSI7(1))              ,(CUVAR(16),CUPX(1,1))       ,
     3       (CUVAR(52),CUPSI(1,1))          ,(CUVAR(88),CUPSI7(1))       ,
     4       (DER(16),DRPX(1,1))             ,(DER(52),DRPSI(1,1))        ,
     5       (DER(88),DRPSI7(1))             ,(ERRVAL(15),ERX(1,1))       ,
     6       (ERRVAL(51),ERPSI(1,1))         ,(ERRVAL(87),ERPSI7(1))
C
      EXTERNAL DERSUB,CHSUB
```

```
      EQUIVALENCE(VAR(1),T1)
C
C     FORMAT STATEMENTS
C
  100 FORMAT(I20/4E20.8/3E20.8/3E20.8/4E20.8/3E20.8/4E20.8/3E20.8)
  200 FORMAT(2E20.8/3I20/2E20.8,I20/4E20.8/3E20.8)
  500 FORMAT(30H1TIME OPTIMAL RENDEZVOUS STUDY,10X 8HCASE NO. I3//////
     16H INPUT///5H BETA 2X E16.8,10X 2HRS 8X E16.8, 10X 7HOMEGA M 3X
     2E16.8,10X 6HLAMBDA 2X E16.8/2H C 5X E16.8, 10X 2HRV 8X E16.8, 10X
     37HOMEGA T 3X E16.8, 10X 2HMU 6X E16.8//)
  501 FORMAT(19H WEIGHTING ELEMENTS/7E16.7///)
  502 FORMAT(19H INITIAL CONDITIONS//3H T0,4X,E16.8,10X,6HPHI V0,4X,
     1E16.8,10X,8HRVX0 DOT,2X,E16.8,10X,4HPHI0,4X,E16.8/5H MASS,2X,E16.8
     2,10X,8HTHETA V0,2X,E16.8,10X,8HRVY0 DOT,2X,E16.8,10X,6HTHETA0,2X,
     3E16.8/5H MDOT2XE16.8,46X 8HRVZ0 DOT2X E16.8,10X2HI0 6XE16.8///)
  503 FORMAT(19H ASSUMED CONDITIONS//3H TF,4X,E16.8,10X,4HPSI1,6X,E16.8,
     110X,4HPSI2,6X,E16.8/5H PSI7,2X,E16.8,10X,4HPSI3,6X,E16.8,10X,
     24HPSI4,6X,E16.8/33X,4HPSI5,6X,E16.8,10X,4HPSI6,6X,E16.8///)
  504 FORMAT(28H COMPUTED INITIAL CONDITIONS//3H XR,4X,E16.8,10X,6HXR DO
     1T,4X,E16.8/3H YR,4X,E16.8,10X,6HYR DOT,4X,E16.8/3H ZR,4X,E16.8,
     210X,6HZR DOT,4X,E16.8//4H RSX,3X,E16.8,10X,7HRSX DOT,3X,E16.8,10X,
     33HRVX,7X,E16.8/4H RSY,3X,E16.8,10X,7HRSY DOT,3X,E16.8,10X,3HRVY,
     47X,E16.8/4H RSZ,3X,E16.8,10X,7HRSZ DOT,3X,E16.8,10X,3HRVZ,7X,E16.8
     5/53H1INTEGRATION ROUTINE USES FIXED COMPUTING INTERVAL = ,E16.8//)
C
  800 FORMAT(5H TIME 2X E16.8, 10X 4HRELD 5X E16.8,10X6HTHRUST 3X E16.8/
     15H MASS 2X E16.8, 10X 4HRELV 5X E16.8,10X2HRV 7X E16.8/)
  804 FORMAT(5H PSI1 2X E16.8,10X 4HPSI2 5X E16.8,10X 4HPSI7 5X E16.8/
     15H PSI3 2X E16.8, 10X 4HPSI4 5X E16.8/5H PSI5 2X E16.8, 10X 4HPSI6
     25X E16.8/)
  805 FORMAT(9H PARTIALS//8H X/ALPHA/6(6E18.8/)//
     110H PSI/ALPHA/6(6E18.8/)//11H PSI7/ALPHA/6E18.8//)
  806 FORMAT(3H XR,4X,E16.8,10X,6HXR DOT,3X,E16.8,10X,7HDXR DOT,2X,
     1E16.8,10X,2HUX,8X,E16.8/3H YR,4X,E16.8,10X,6HYR DOT,3X,E16.8,10X,
     27HDYR DOT,2X,E16.8,10X,2HUY,8X,E16.8/3H ZR,4X,E16.8,10X,6HZR DOT,
     33X,E16.8,10X,7HDZR DOT,2X,E16.8,10X,2HUZ,8X,E16.8/)
  808 FORMAT(4H RSX,3X,E16.8,10X,7HRSX DOT,2X,E16.8,10X,3HRVX,6X,E16.8,
     110X,7HRVX DOT,3X,E16.8/4H RSY,3X,E16.8,10X,7HRSY DOT,2X,E16.8,10X,
     23HRVY,6X,E16.8,10X,7HRVY DOT,3X,E16.8/4H RSZ,3X,E16.8,10X,
     37HRSZ DOT,2X,E16.8,10X,3HRVZ,6X,E16.8,10X,7HRVZ DOT,3X,E16.8/)
  810 FORMAT(5H TIME 2X E16.8/)
  811 FORMAT(5H PSI1 2X E16.8,10X 4HPSI2 5X E16.8,10X 4HPSI7 5X E16.8,
     110X 2HUX 8X E16.8/5H PSI3 2X E16.8, 10X 4HPSI4 5X E16.8, 45X
     22HUY 8X E16.8/5H PSI5 2X E16.8, 10X 4HPSI6 5X E16.8, 45X 2HUZ 8X
     3E16.8/)
```

```
  900 FORMAT(24 H LOCAL TRUNCATION ERRORS//2H X/6E18.8//3H X7/E18.8//
     14H PSI/6E18.8//5H PSI7/E18.8//11H P(X/ALPHA)/6(6E18.8/)//
     213H P(PSI/ALPHA)/6(6E18.8/)//14H P(PSI7/ALPHA)/
     36E18.8//)
C
 4900 FORMAT(//)
  901 FORMAT(40H  MAX NO. OF ITERATIONS HAS BEEN REACHED)
C
  903 FORMAT(9H E(ALPHA) 2X E16.8/)
C
C     START
C
C     READ INPUT
C
    1 READ(5,100) NO,SOMEG,BETA,C,TF,PHIVO,THETVO,RV,DRVXO,DRVYO,DRVZO,
     1PHIO,THETAO,IO,RS,VAR(1),VAR(8),MU,(VAR(I),I=9,15)
    2 READ(5,200)CI,SPEC,IPRINT,IEROR,IMAT,LAMBDA,CRIT,MAXIT,
     1(B(I),I=1,7)
C     COMPUTE CONSTANT TERMS AND INITIAL CONDITIONS
C
      COMEGS=MU/RS**3
      COMEG=SQRT(COMEGS)
      SOMEGS=SOMEG**2
      SIO=SIN(IO)
      STO=SIN(THETAO)
      CIO=COS(IO)
      CTO=COS(THETAO)
C
C     T1 MATRIX
C
      XT1(1,1)=CTO
      XT1(1,2)=-CIO*STO
      XT1(1,3)=SIO*STO
      XT1(2,1)=STO
      XT1(2,2)=CIO*CTO
      XT1(2,3)=-SIO*CTO
      XT1(3,1)=0.0
      XT1(3,2)=SIO
      XT1(3,3)=CIO
C
C     INITIALIZATION
C
      RVX=RV*COS(THETVO)*COS(PHIVO)
      RVY=RV*COS(THETVO)*SIN(PHIVO)
      RVZ=RV*SIN(THETVO)
```

```
      RVX0=RVX
      RVY0=RVY
      RVZ0=RVZ
      CALL COMP(0.0)
      VAR(2)=RVX-RSV(1)
      VAR(3)=DRVX0-DRSV(1)
      VAR(4)=RVY-RSV(2)
      VAR(5)=DRVY0-DRSV(2)
      VAR(6)=RVZ-RSV(3)
      VAR(7)=DRVZ0-DRSV(3)
      DER(8)=-BETA/C
      WRITE(6,500) NO,BETA,RS,SOMEG,LAMBDA,C,RV,COMEG,MU
      WRITE(6,501) (B(I),I=1,7)
      WRITE(6,502) VAR(1),PHIV0,DRVX0,PHI0,VAR(8),THETV0,DRVY0,THETA0,
     1DER(8),DRVZ0,I0
      WRITE(6,503) TF,VAR(9),VAR(10),VAR(15),VAR(11),VAR(12),VAR(13),
     1VAR(14)
      WRITE(6,504) (VAR(I),I=2,7),RSV(1),DRSV(1),RVX,RSV(2),DRSV(2),RVY,
     2RSV(3),DRSV(3),RVZ,CI
C
C      INITIALIZATION FOR INTEGRATION ROUTINE
C
C
      DO 17 I=1,15
   17 SAVE(I)=VAR(I)
      DO 14 I=16,93
   14 VAR(I)=SAVE(I)
C
      TEMPCI=CI
      TEMPSP=SPEC
      IKOUNT=0
 1010 KOUNT=0
      II=0
      FIRST=.TRUE.
      CI=TEMPCI
      SPEC=TEMPSP
   10 CALL INT2(II,N,NT,CI,SPEC,CIMAX,IERR,VAR,CUVAR,DER,ELE1,ELE2,ELT,
     1ERRVAL,DERSUB,CHSUB,ITEXT)
C
C      RETURN FROM INTEGRATION ROUTINE
   11 IF ((ABS(T1-TF) .LE. 1.0E-06) .OR. FIRST .OR.
     1((SPEC .LT. 1.0E10) .AND. (SPEC .NE. 0.0))) GO TO 16
      GO TO 20
C
C
```

```
C     WRITE OUTPUT
C
   16 RELD=SQRT(VAR(2)**2+VAR(4)**2+VAR(6)**2)
      RELV=SQRT(VAR(3)**2+VAR(5)**2+VAR(7)**2)
      RVX=VAR(2)+RSV(1)
      RVY=VAR(4)+RSV(2)
      RVZ=VAR(6)+RSV(3)
      RVMAG=SQRT(RVX**2+RVY**2+RVZ**2)
      DRVX=VAR(3)+DRSV(1)
      DRVY=VAR(5)+DRSV(2)
      DRVZ=VAR(7)+DRSV(3)
C
      U4=BETA
      UX=VAR(10)/EN1
      UY=VAR(12)/EN1
      UZ=VAR(14)/EN1
      IF (FIRST) GO TO 60
      WRITE(6,800) VAR(1),RELD,U4,VAR(8),RELV,RVMAG
      WRITE(6,804) VAR(9),VAR(10),VAR(15),VAR(11),VAR(12),
     1VAR(13),VAR(14)
      WRITE(6,806) VAR(2),VAR(3),DER(3),UX,VAR(4),VAR(5),DER(5),UY,
     1VAR(6),VAR(7),DER(7),UZ
      WRITE(6,808) RSV(1),DRSV(1),RVX,DRVX,RSV(2),DRSV(2),RVY,DRVY,
     1RSV(3),DRSV(3),RVZ,DRVZ
      GO TO 61
   60 WRITE(6,810) VAR(1)
      WRITE(6,811) VAR(9),VAR(10),VAR(15),UX,VAR(11),VAR(12),UY,
     1VAR(13),VAR(14),UZ
   61 IF (IPRINT .NE. 1) GO TO 50
      WRITE(6,805) ((PX(I,J),J=1,6),I=1,6),
     1((PPSI(I,J),J=1,6),I=1,6),(PPSI7(I),I=1,6)
C
   50 IF(IEROR .EQ. 1)GO TO 9
      GO TO 49
C
    9 WRITE(6,900)(ERRVAL(I),I=1,14),((ERX(I,J),J=1,6),I=1,6),
     1((ERPSI(I,J),J=1,6),I=1,6),(ERPSI7(I),I=1,6)
   49 WRITE(6,4900)
   20 IF (FIRST) FIRST=.FALSE.
      IF (ABS(T1-TF) .LE. 1.0E-06) GO TO 13
      IF ((T1+CI) .LE. TF) GO TO 10
      CI=TF-T1
      II=0
      SPEC=0.0
      GO TO 10
```

```
   13 IKOUNT=IKOUNT+1
      IF(IKOUNT .GT. MAXIT) GO TO 15
C
C     COMPUTE (E.B*E)/2
C
      EDP=0.0
      DO 19 I=2,7
   19 EDP=EDP+B(I-1)*VAR(I)**2
      EDP=.5*(EDP+B(7)*VAR(15)**2)
      WRITE(6,903) EDP
      IF (EDP .LE. CRIT) GO TO 1
      CALL ITERAT
      DO 18 I=1,93
   18 VAR(I)=SAVE(I)
      GO TO 1010
   15 WRITE(6,901)
      GO TO 1
      END
      SUBROUTINE DERSUB
      COMMON /SPACE/
     1           VAR      ,CUVAR    ,SAVE     ,C        ,MAXIT    ,IMAT     ,
     2           E        ,PE       ,PEMAT    ,PEVEC    ,ERRVAL   ,SOMEG    ,
     3           DER      ,RS       ,BETA     ,DP1      ,TF       ,ELE1     ,
     4           RSV      ,F1       ,DP2      ,CI       ,ELE2     ,DRSV     ,
     5           F2       ,E1       ,II       ,SOMEGS   ,COMEGS   ,IPRINT   ,
     6           F3       ,E2       ,N        ,COMEG    ,EN2      ,KOUNT    ,
     7           Q1       ,E3       ,TEMPCI   ,SIO      ,PHIO     ,Q2       ,
     8           E4       ,SPEC     ,STO      ,LAMBDA   ,TEMPT    ,Q3       ,
     9           T2       ,TEMPSP   ,CIO      ,CRIT     ,EN1      ,IKOUNT   ,
     1           RVXO     ,RVYO     ,RVZO     ,DRVXO    ,DRVYO    ,DRVZO    ,
     2           T4       ,MU       ,CTO      ,B        ,XT1
C
      REAL LAMBDA,MU
C
C
      DIMENSION
     1           VAR(93)            ,CUVAR(93)          ,DER(93)            ,
     2           ELE1(92)           ,ELE2(92)           ,ERRVAL(92)         ,
     3           RSV(3)             ,DRSV(3)            ,PX(6,6)            ,
     4           PPSI(6,6)          ,PPSI7(6)           ,CUPX(6,6)          ,
     5           CUPSI(6,6)         ,CUPSI7(6)          ,DRPX(6,6)          ,
     6           DRPSI(6,6)         ,DRPSI7(6)          ,ERX(6,6)           ,
     7           ERPSI(6,6)         ,ERPSI7(6)          ,E(7)               ,
     8           PE(7,7)            ,PEMAT(7,7)         ,PEVEC(7,1)         ,
     9           B(7)               ,SAVE(93)           ,XT1(3,3)
```

```
C
C
      EQUIVALENCE
     1         (VAR(16),PX(1,1))              ,(VAR(52),PPSI(1,1))         ,
     2         (VAR(88),PPSI7(1))             ,(CUVAR(16),CUPX(1,1))       ,
     3         (CUVAR(52),CUPSI(1,1))         ,(CUVAR(88),CUPSI7(1))       ,
     4         (DER(16),DRPX(1,1))            ,(DER(52),DRPSI(1,1))        ,
     5         (DER(88),DRPSI7(1))            ,(ERRVAL(15),ERX(1,1))       ,
     6         (ERRVAL(51),ERPSI(1,1))        ,(ERRVAL(87),ERPSI7(1))
C
      EQUIVALENCE (CUVAR(1),TJ)
C
      TCOMP=TJ-SAVE(1)
      CALL COMP(TCOMP)
      Q1=CUVAR(2)+RSV(1)
      Q2=CUVAR(4)+RSV(2)
      Q3=CUVAR(6)+RSV(3)
      EN1=SQRT(CUVAR(10)**2+ CUVAR(12)**2+ CUVAR(14)**2)
      EN2=SQRT(Q1**2+Q2**2+Q3**2)
      DP1=CUVAR(2)*CUVAR(10)+CUVAR(4)*CUVAR(12)+CUVAR(6)*CUVAR(14)
      DP2=RSV(1)*CUVAR(10)+RSV(2)*CUVAR(12)+RSV(3)*CUVAR(14)
      F2=(COMEGS*RS**3)/EN2**3
      F3=(3.0*F2/EN2**2)*(DP1+DP2)
      E1=1.0/(CUVAR(8)*EN1)
      E2=E1/CUVAR(8)
      E3=E1/EN1**2
      E4=EN1/CUVAR(8)**2
      T2=(3.0*F2)/EN2**2
      T4=5.0*F3/EN2**2
C
C     STATE VARIABLES
C
C
      DER(9)=F2*CUVAR(10)-F3*Q1-SOMEGS*CUVAR(10)
      DER(10)=-CUVAR(9)+2.0*SOMEG*CUVAR(12)
      DER(11)=F2*CUVAR(12)-F3*Q2-SOMEGS*CUVAR(12)
      DER(12)=-2.0*SOMEG*CUVAR(10)-CUVAR(11)
      DER(13)=F2*CUVAR(14)-F3*Q3
      DER(14)=-CUVAR(13)
      F1=BETA/(CUVAR(8)*EN1)
   51 DER(2)=CUVAR(3)
      DER(3)=F1*CUVAR(10)-F2*Q1+COMEGS*RSV(1)+SOMEGS*CUVAR(2)+2.0*SOMEG*
     1CUVAR(5)
      DER(4)=CUVAR(5)
      DER(5)=F1*CUVAR(12)-F2*Q2+COMEGS*RSV(2)-2.0*SOMEG*CUVAR(3)+SOMEGS*
```

```
     1CUVAR(4)
      DER(6)=CUVAR(7)
      DER(7)=F1*CUVAR(14)-F2*Q3+COMEGS*RSV(3)
      DER(8)=-BETA/C
      DER(15)=(BETA*EN1)/CUVAR(8)**2
C
C
C     PARTIALS
C
      DO 60 I=1,6
      DRPX(1,I)=CUPX(2,I)
C
      DRPX(2,I)=BETA*((E1-E3*CUVAR(10)**2)*CUPSI(2,I)-E3*CUVAR(10)*
     1CUVAR(12)*CUPSI(4,I)-E3*CUVAR(10)*CUVAR(14)*CUPSI(6,I))+(T2*Q1**2-
     2F2+SOMEGS)*CUPX(1,I)+T2*Q1*Q2*CUPX(3,I)+2.0*SOMEG*CUPX(4,I)+T2*Q1*
     3Q3*CUPX(5,I)
C
      DRPX(3,I)=CUPX(4,I)
C
      DRPX(4,I)=BETA*(-E3*CUVAR(10)*CUVAR(12)*CUPSI(2,I)+(E1-E3*
     1CUVAR(12)**2)*CUPSI(4,I)-E3*CUVAR(12)*CUVAR(14)*CUPSI(6,I))+T2*Q1*
     2Q2*CUPX(1,I)-2.0*SOMEG*CUPX(2,I)+(T2*Q2**2-F2+SOMEGS)*CUPX(3,I)+
     3T2*Q2*Q3*CUPX(5,I)
C
      DRPX(5,I)=CUPX(6,I)
C
      DRPX(6,I)=BETA*(-E3*CUVAR(10)*CUVAR(14)*CUPSI(2,I)-E3*
     1CUVAR(12)*CUVAR(14)*CUPSI(4,I)+(E1-E3*CUVAR(14)**2)*CUPSI(6,I))+
     2T2*Q1*Q3*CUPX(1,I)+T2*Q2*Q3*CUPX(3,I)+(T2*Q3**2-F2)*CUPX(5,I)
C
      DRPSI(1,I)=(F2-T2*Q1**2-SOMEGS)*CUPSI(2,I)-T2*Q1*Q2*CUPSI(4,I)-T2*
     1Q3*Q1*CUPSI(6,I)+(-2.0*T2*CUVAR(10)*Q1-F3+T4*Q1**2)*CUPX(1,I)+(-T2
     2*(CUVAR(10)*Q2+CUVAR(12)*Q1)+T4*Q1*Q2)*CUPX(3,I)+(-T2*(CUVAR(10)*
     3Q3+CUVAR(14)*Q1)+T4*Q1*Q3)*CUPX(5,I)
C
      DRPSI(2,I)=-CUPSI(1,I)+2.0*SOMEG*CUPSI(4,I)
C
      DRPSI(3,I)=-T2*Q1*Q2*CUPSI(2,I)+(F2-T2*Q2**2-SOMEGS)*CUPSI(4,I)-
     1T2*Q2*Q3*CUPSI(6,I)+(-T2*(CUVAR(12)*Q1+CUVAR(10)*Q2)+T4*Q1*Q2)*
     2CUPX(1,I)+(-2.0*T2*CUVAR(12)*Q2-F3+T4*Q2**2)*CUPX(3,I)+(-T2*(
     3 CUVAR(12)*Q3+CUVAR(14)*Q2)+T4*Q2*Q3)*CUPX(5,I)
C
      DRPSI(4,I)=-2.0*SOMEG*CUPSI(2,I)-CUPSI(3,I)
C
      DRPSI(5,I)=-T2*Q1*Q3*CUPSI(2,I)-T2*Q2*Q3*CUPSI(4,I)+(F2-T2*Q3**2)*
```

```
     1CUPSI(6,I)+(-T2*(CUVAR(14)*Q1+CUVAR(10)*Q3)+T4*Q1*Q3)*CUPX(1,I)+
     2(-T2*(CUVAR(14)*Q2+CUVAR(12)*Q3)+T4*Q2*Q3)*CUPX(3,I)+(-2.0*T2*
     3CUVAR(14)*Q3-F3+T4*Q3**2)*CUPX(5,I)
C
      DRPSI(6,I)=-CUPSI(5,I)
C
      DRPSI7(I)=BETA*E2*(CUVAR(10)*CUPSI(2,I)+CUVAR(12)*CUPSI(4,I)
     1+CUVAR(14)*CUPSI(6,I))
C
   60 CONTINUE
      RETURN
      END
      SUBROUTINE CHSUB
      COMMON /SPACE/
     1          VAR       ,CUVAR     ,SAVE      ,C         ,MAXIT     ,IMAT      ,
     2          E         ,PE        ,PEMAT     ,PEVEC     ,ERRVAL    ,SOMEG     ,
     3          DER       ,RS        ,BETA      ,DP1       ,TF        ,ELE1      ,
     4          RSV       ,F1        ,DP2       ,CI        ,ELE2      ,DRSV      ,
     5          F2        ,E1        ,II        ,SOMEGS    ,COMEGS    ,IPRINT    ,
     6          F3        ,E2        ,N         ,COMEG     ,EN2       ,KOUNT     ,
     7          Q1        ,E3        ,TEMPCI    ,SIO       ,PHIO      ,Q2        ,
     8          E4        ,SPEC      ,STO       ,LAMBDA    ,TEMPT     ,Q3        ,
     9          T2        ,TEMPSP    ,CIO       ,CRIT      ,EN1       ,IKOUNT    ,
     1          RVXO      ,RVYO      ,RVZO      ,DRVXO     ,DRVYO     ,DRVZO     ,
     2          T4        ,MU        ,CTO       ,B         ,XT1
C
      REAL LAMBDA,MU
C
C
      DIMENSION
     1          VAR(93)             ,CUVAR(93)           ,DER(93)             ,
     2          ELE1(92)            ,ELE2(92)            ,ERRVAL(92)          ,
     3          RSV(3)              ,DRSV(3)             ,PX(6,6)             ,
     4          PPSI(6,6)           ,PPSI7(6)            ,CUPX(6,6)           ,
     5          CUPSI(6,6)          ,CUPSI7(6)           ,DRPX(6,6)           ,
     6          DRPSI(6,6)          ,DRPSI7(6)           ,ERX(6,6)            ,
     7          ERPSI(6,6)          ,ERPSI7(6)           ,E(7)                ,
     8          PE(7,7)             ,PEMAT(7,7)          ,PEVEC(7,1)          ,
     9          B(7)                ,SAVE(93)            ,XT1(3,3)
C
C
      EQUIVALENCE
     1         (VAR(16),PX(1,1))              ,(VAR(52),PPSI(1,1))            ,
     2         (VAR(88),PPSI7(1))             ,(CUVAR(16),CUPX(1,1))          ,
     3         (CUVAR(52),CUPSI(1,1))         ,(CUVAR(88),CUPSI7(1))          ,
```

```
     4        (DER(16),DRPX(1,1))              ,(DER(52),DRPSI(1,1))          ,
     5        (DER(88),DRPSI7(1))              ,(ERRVAL(15),ERX(1,1))         ,
     6        (ERRVAL(51),ERPSI(1,1))          ,(ERRVAL(87),ERPSI7(1))
C
      EQUIVALENCE (CUVAR(1),TI)
C
      IF ((TI+CI) .LE. TF) GO TO 78
      II=3
   78 RETURN
      END
      SUBROUTINE COMP(DT)
      COMMON /SPACE/
     1          VAR      ,CUVAR    ,SAVE     ,C        ,MAXIT    ,IMAT     ,
     2          E        ,PE       ,PEMAT    ,PEVEC    ,ERRVAL   ,SOMEG    ,
     3          DER      ,RS       ,BETA     ,DP1      ,TF       ,ELE1     ,
     4          RSV      ,F1       ,DP2      ,CI       ,ELE2     ,DRSV     ,
     5          F2       ,E1       ,II       ,SOMEGS   ,COMEGS   ,IPRINT   ,
     6          F3       ,E2       ,N        ,COMEG    ,EN2      ,KOUNT    ,
     7          Q1       ,E3       ,TEMPCI   ,SIO      ,PHIO     ,Q2       ,
     8          E4       ,SPEC     ,STO      ,LAMBDA   ,TEMPT    ,Q3       ,
     9          T2       ,TEMPSP   ,CIO      ,CRIT     ,EN1      ,IKOUNT   ,
     1          RVXO     ,RVYO     ,RVZO     ,DRVXO    ,DRVYO    ,DRVZO    ,
     2          T4       ,MU       ,CTO      ,B        ,XT1
C
      REAL LAMBDA,MU
C
C
      DIMENSION
     1          VAR(93)               ,CUVAR(93)             ,DER(93)             ,
     2          ELE1(92)              ,ELE2(92)              ,ERRVAL(92)          ,
     3          RSV(3)                ,DRSV(3)               ,PX(6,6)             ,
     4          PPSI(6,6)             ,PPSI7(6)              ,CUPX(6,6)           ,
     5          CUPSI(6,6)            ,CUPSI7(6)             ,DRPX(6,6)           ,
     6          DRPSI(6,6)            ,DRPSI7(6)             ,ERX(6,6)            ,
     7          ERPSI(6,6)            ,ERPSI7(6)             ,E(7)                ,
     8          PE(7,7)               ,PEMAT(7,7)            ,PEVEC(7,1)          ,
     9          B(7)                  ,SAVE(93)              ,XT1(3,3)
C
C
      EQUIVALENCE
     1        (VAR(16),PX(1,1))                ,(VAR(52),PPSI(1,1))           ,
     2        (VAR(88),PPSI7(1))               ,(CUVAR(16),CUPX(1,1))         ,
     3        (CUVAR(52),CUPSI(1,1))           ,(CUVAR(88),CUPSI7(1))         ,
     4        (DER(16),DRPX(1,1))              ,(DER(52),DRPSI(1,1))          ,
     5        (DER(88),DRPSI7(1))              ,(ERRVAL(15),ERX(1,1))         ,
```

```
      6       (ERRVAL(51),ERPSI(1,1))        ,(ERRVAL(87),ERPSI7(1))
       DIMENSION XTO(3),XT2(3,3),XT3(3),XTD2(3,3),RSXYZ(3),TDRS(3),
      1RSXYZD(3),TRSDOT(3)
       PHIOMT=PHIO-COMEG*DT
C
C      TO MATRIX
C
       XTO(1)=RS*COS(PHIOMT)
       XTO(2)=RS*SIN(PHIOMT)
       XTO(3)=0.0
C
C      T2 MATRIX
C
       SOT=SIN(SOMEG*DT)
       COT=COS(SOMEG*DT)
       XT2(1,1)=COT
       XT2(1,2)=SOT
       XT2(1,3)=0.0
       XT2(2,1)=-SOT
       XT2(2,2)=COT
       XT2(2,3)=0.0
       XT2(3,1)=0.0
       XT2(3,2)=0.0
       XT2(3,3)=1.0
C
C      T2 DOT MATRIX
C
       XTD2(1,1)=-SOMEG*SOT
       XTD2(1,2)=SOMEG*COT
       XTD2(1,3)=0.0
       XTD2(2,1)=-SOMEG*COT
       XTD2(2,2)=-SOMEG*SOT
       XTD2(2,3)=0.0
       XTD2(3,1)=0.0
       XTD2(3,2)=0.0
       XTD2(3,3)=0.0
C
C      T3 MATRIX
C
       XT3(1)=COMEG*XTO(2)
       XT3(2)=-COMEG*XTO(1)
       XT3(3)=0.0
C
C      T1TO MATRIX
C
```

```
      DO 100 I=1,3
      RSXYZ(I)=0.0
      DO 100 J=1,3
  100 RSXYZ(I)=RSXYZ(I)+XT1(I,J)*XT0(J)
C
C     RSV MATRIX
C
      DO 101 I=1,3
      RSV(I)=0.0
      DO 101 J=1,3
  101 RSV(I)=RSV(I)+XT2(I,J)*RSXYZ(J)
      DO 102 I=1,3
      TDRS(I)=0.0
      DO 102 J=1,3
  102 TDRS(I)=TDRS(I)+XTD2(I,J)*RSXYZ(J)
      DO 103 I=1,3
      RSXYZD(I)=0.0
      DO 103 J=1,3
  103 RSXYZD(I)=RSXYZD(I)+XT1(I,J)*XT3(J)
      DO 104 I=1,3
      TRSDOT(I)=0.0
      DO 104 J=1,3
  104 TRSDOT(I)=TRSDOT(I)+XT2(I,J)*RSXYZD(J)
C
C     RSV DOT MATRIX
C
      DO 105 I=1,3
  105 DRSV(I)=TDRS(I)+TRSDOT(I)
      RETURN
      END
      SUBROUTINE ITERAT
      COMMON /SPACE/
     1          VAR      ,CUVAR    ,SAVE     ,C        ,MAXIT    ,IMAT     ,
     2          E        ,PE       ,PEMAT    ,PEVEC    ,ERRVAL   ,SOMEG    ,
     3          DER      ,RS       ,BETA     ,DP1      ,TF       ,ELE1     ,
     4          RSV      ,F1       ,DP2      ,CI       ,ELE2     ,DRSV     ,
     5          F2       ,E1       ,II       ,SOMEGS   ,COMEGS   ,IPRINT   ,
     6          F3       ,E2       ,N        ,COMEG    ,EN2      ,KOUNT    ,
     7          Q1       ,E3       ,TEMPCI   ,SIO      ,PHIO     ,Q2       ,
     8          E4       ,SPEC     ,STO      ,LAMBDA   ,TEMPT    ,Q3       ,
     9          T2       ,TEMPSP   ,CIO      ,CRIT     ,EN1      ,IKOUNT   ,
     1          RVXO     ,RVYO     ,RVZO     ,DRVXO    ,DRVYO    ,DRVZO    ,
     2          T4       ,MU       ,CTO      ,B        ,XT1
C
      REAL LAMBDA,MU
```

```
C
C
      DIMENSION
     1          VAR(93)              ,CUVAR(93)            ,DER(93)            ,
     2          ELE1(92)             ,ELE2(92)             ,ERRVAL(92)         ,
     3          RSV(3)               ,DRSV(3)              ,PX(6,6)            ,
     4          PPSI(6,6)            ,PPSI7(6)             ,CUPX(6,6)          ,
     5          CUPSI(6,6)           ,CUPSI7(6)            ,DRPX(6,6)          ,
     6          DRPSI(6,6)           ,DRPSI7(6)            ,ERX(6,6)           ,
     7          ERPSI(6,6)           ,ERPSI7(6)            ,E(7)               ,
     8          PE(7,7)              ,PEMAT(7,7)           ,PEVEC(7,1)         ,
     9          B(7)                 ,SAVE(93)             ,XT1(3,3)
C
C
      EQUIVALENCE
     1          (VAR(16),PX(1,1))               ,(VAR(52),PPSI(1,1))            ,
     2          (VAR(88),PPSI7(1))              ,(CUVAR(16),CUPX(1,1))          ,
     3          (CUVAR(52),CUPSI(1,1))          ,(CUVAR(88),CUPSI7(1))          ,
     4          (DER(16),DRPX(1,1))             ,(DER(52),DRPSI(1,1))           ,
     5          (DER(88),DRPSI7(1))             ,(ERRVAL(15),ERX(1,1))          ,
     6          (ERRVAL(51),ERPSI(1,1))         ,(ERRVAL(87),ERPSI7(1))
C
      DIMENSION IPIVOT(7),INDEX(7,2)
C
C
      DIMENSION SAVMAT(7,7),UNIT(7,7)
C
C     FORM MATRIX OF PARTIALS
C
      DO 25 I=1,6
      DO 25 J=1,6
      PE(I,J)=PX(I,J)
   25 CONTINUE
C
C
      J=1
      DO 26 I=1,5,2
      J=J+2
      PE(I,7)=VAR(J)
      PE(I+1,7)=DER(J)
   26 CONTINUE
      DO 27 I=1,6
   27 PE(7,I)=PPSI7(I)
C
      PE(7,7)=DER(15)
```

```
C
C     FORM E VECTOR
C
      DO 38 I=1,6
   38 E(I)=VAR(I+1)
      E(7)=VAR(15)
C
C             T
C     FORM PE *B*PE+LAMBDA*I MATRIX
C
C
      DO 40 I=1,7
      DO 40 J=1,7
      PEMAT(I,J)=0.0
      DO 40 K=1,7
      PEMAT(I,J)=PEMAT(I,J)+PE(K,I)*B(K)*PE(K,J)
   40 CONTINUE
      DO 41 I=1,7
   41 PEMAT(I,I)=PEMAT(I,I)+LAMBDA
C
C                                 T                                T
C     SOLVE THE MATRIX EQ (PE *B*PE+LAMBDA*I)DELTA=PE *B*E
C
      DO 50 I=1,7
      PEVEC(I,1)=0.0
      DO 50 J=1,7
   50 PEVEC(I,1)=PEVEC(I,1)-PE(J,I)*B(J)*E(J)
      IF(IMAT   .NE. 1) GO TO 13
      DO 14 I=1,7
      DO 14 J=1,7
   14 SAVMAT(I,J)=PEMAT(I,J)
      IF(IMAT .EQ. 1) WRITE(6,11) ((PEMAT(I,J),J=1,7),I=1,7)
   11 FORMAT(6H PEMAT/7(7E16.7/)//)
   13 CALL MATINV(PEMAT,7,PEVEC,1,DETERM,IPIVOT,INDEX,7,ISCALE)
C
      IF(IMAT .EQ. 1) WRITE(6,12)((PEMAT(I,J),J=1,7),I=1,7)
   12 FORMAT(8H INVERSE/7(7E16.7/)//)
      IF(IMAT   .NE. 1) GO TO 15
      DO 16 I=1,7
      DO 16 J=1,7
      UNIT(I,J)=0.0
      DO 16 K=1,7
   16 UNIT(I,J)=UNIT(I,J)+SAVMAT(I,K)*PEMAT(K,J)
      WRITE(6,17) ((UNIT(I,J),J=1,7),I=1,7)
   17 FORMAT(9H IDENTITY/7(7E16.7/)//)
```

APPENDIX B

```
15 WRITE(6,10) IKOUNT,PEVEC(7,1),(PEVEC(I,1),I=1,6)
10 FORMAT(34H1CORRECTIONS ON INITIAL CONDITIONS,10X,13HITERATION NO.,
  12X,I3///9H DELTA TF,4X,E16.8,                       10X,10HDELTA P
  2SI1,2X,E16.8,10X,10HDELTA PSI2,2X,E16.8/39X,10HDELTA PSI3,2X,
  3E16.8,10X,10HDELTA PSI4,2X,E16.8/39X,10HDELTA PSI5,2X,E16.8,10X,
  410HDELTA PSI6,2X,E16.8//)
C
   TF=TF+PEVEC(7,1)
   DO51I=1,6
   SAVE(I+8)=SAVE(I+8)+PEVEC(I,1)
51 CONTINUE
C
   RETURN
   END
   BLOCK DATA
   COMMON /SPACE/
  1         VAR       ,CUVAR     ,SAVE      ,C         ,MAXIT     ,IMAT      ,
  2         E         ,PE        ,PEMAT     ,PEVEC     ,ERRVAL    ,SOMEG     ,
  3         DER       ,RS        ,BETA      ,DP1       ,TF        ,ELE1      ,
  4         RSV       ,F1        ,DP2       ,CI        ,ELE2      ,DRSV      ,
  5         F2        ,E1        ,II        ,SOMEGS    ,COMEGS    ,IPRINT    ,
  6         F3        ,E2        ,N         ,COMEG     ,EN2       ,KOUNT     ,
  7         Q1        ,E3        ,TEMPCI    ,SIO       ,PHIO      ,Q2        ,
  8         E4        ,SPEC      ,STO       ,LAMBDA    ,TEMPT     ,Q3        ,
  9         T2        ,TEMPSP    ,CIO       ,CRIT      ,EN1       ,IKOUNT    ,
  1         RVXO      ,RVYO      ,RVZO      ,DRVXO     ,DRVYO     ,DRVZO     ,
  2         T4        ,MU        ,CTO       ,B         ,XT1
C
   REAL LAMBDA,MU
C
C
   DIMENSION
  1         VAR(93)              ,CUVAR(93)           ,DER(93)             ,
  2         ELE1(92)             ,ELE2(92)            ,ERRVAL(92)          ,
  3         RSV(3)               ,DRSV(3)             ,PX(6,6)             ,
  4         PPSI(6,6)            ,PPSI7(6)            ,CUPX(6,6)           ,
  5         CUPSI(6,6)           ,CUPSI7(6)           ,DRPX(6,6)           ,
  6         DRPSI(6,6)           ,DRPSI7(6)           ,ERX(6,6)            ,
  7         ERPSI(6,6)           ,ERPSI7(6)           ,E(7)                ,
  8         PE(7,7)              ,PEMAT(7,7)          ,PEVEC(7,1)          ,
  9         B(7)                 ,SAVE(93)            ,XT1(3,3)
C
C
   EQUIVALENCE
  1        (VAR(16),PX(1,1))                ,(VAR(52),PPSI(1,1))                ,
```

```
      2        (VAR(88),PPSI7(1))              ,(CUVAR(16),CUPX(1,1))       ,
      3        (CUVAR(52),CUPSI(1,1))          ,(CUVAR(88),CUPSI7(1))       ,
      4        (DER(16),DRPX(1,1))             ,(DER(52),DRPSI(1,1))        ,
      5        (DER(88),DRPSI7(1))             ,(ERRVAL(15),ERX(1,1))       ,
      6        (ERRVAL(51),ERPSI(1,1))         ,(ERRVAL(87),ERPSI7(1))
C
      DATA N/92/
      DATA (SAVE(I),I=16,93)/78*0.0/
      DATA SAVE(52)/1.0/,SAVE(59)/1.0/,SAVE(66)/1.0/,SAVE(73)/1.0/,
     1SAVE(80)/1.0/,SAVE(87)/1.0/
      END
      SUBROUTINE INT2(II,N,NT,CI,SPEC,CIMAX, IERR ,VAR,CUVAR,DER,ELE1,
     1ELE2,ELT,ERRVAL,DERSUB,CHKSUB,ITEXT)
      DIMENSION VAR(93),CUVAR(93)
      DIMENSION DER(93),ELE1(92),ELE2(92),ELT(92),ERRVAL(92)
      DIMENSION TEMP(92),DER1(92),DER2(92),DER3(92)
      DIMENSION SIVAR(93)
      IF(II)1,1,2
C     INITIALIZATION SECTION
    1 IF(CI) 3,4,3
    4 WRITE(6,1000)
 1000 FORMAT(11HOCI=0   STOP)
      CALL EXIT
C     SAVE CI
    3 H=CI
   18 IERR=1
      TO=SPEC+VAR(1)
      MODE=1
      II= 1
      N1=N+1
      DO 5 J=1,N1
      CUVAR(J)=VAR(J)
    5 CONTINUE
C     EVALUATION SECTION HERE
    8 CALL DERSUB
      IF(MODE.LE.1)     GO TO 6
      IF(II-3)36,36,7
   36 CALL CHKSUB
      IF(II.EQ.2) GO TO 1
   37 DO 38 J=1,N1
   38 VAR(J)=CUVAR(J)
      IF(II-3)6,7,7
    7 RETURN
    6 IF(SPEC) 9,7,9
    9 DEL=VAR(1)-TO
```

```
      DELP=DEL*(1.+1.0E-6)
      IF(ABS(DELP)-ABS(SPEC)) 2,10,10
   10 TO=VAR(1)
      GO TO 7
    2 II=1
      IF(MODE-4) 11,12,12
C     RUNGE-KUTTA
   11 DO 20 J=2,N1
      DER3(J-1)=DER2(J-1)
      DER2(J-1)=DER1(J-1)
      DER1(J-1)=DER(J)
      ELE1(J-1)=DER(J)
      CUVAR(J)=0.0D+00
      DELT=0.4*ELE1(J-1)*H
      S1VAR(J)=VAR(J)
      CUVAR(J)=S1VAR(J)+DELT
   20 CONTINUE
      S1VAR(1)=VAR(1)
      CUVAR(1)=S1VAR(1)+0.4*H
      CALL DERSUB
      IF(II-3)23,23,7
   23 CUVAR(1)=S1VAR(1)+0.45573725*H
      DO 24 J=2,N1
      ELE2(J-1)=DER(J)
      DELT=(0.29697761*ELE1(J-1)+0.15875964*ELE2(J-1))*H
      CUVAR(J)=S1VAR(J)+DELT
   24 CONTINUE
      CALL DERSUB
      IF(II-3)25,25,7
   25 CUVAR(1)=S1VAR(1)+H
      DO 26 J=2,N1
      TEMP(J-1)=DER(J)
      DELT=(0.21810040*ELE1(J-1)-3.05096516*ELE2(J-1)
     1+3.83286476*TEMP(J-1))*H
      CUVAR(J)=S1VAR(J)+DELT
   26 CONTINUE
      CALL DERSUB
      IF(II-3)27,27,7
   27 DH=H
      CUVAR(1)=VAR(1)+DH
      DO 28 J=2,N1
      DOUB=0.17476028*ELE1(J-1)-0.55148066*ELE2(J-1)
     1+1.20553560*TEMP(J-1)+0.17118478*DER(J)
      CUVAR(J)=VAR(J)+DH*DOUB
   28 CONTINUE
```

```
      MODE=MODE+1
      GO TO 8
C     ADAMS-MOULTON
C     ADAMS-BASHFORTH PREDICTOR
   12 CUVAR(1)=VAR(1)+H
      DH=H/24.0
      DO 13 J=2,N1
      DOUB=55.0*DER(J)-59.0*DER1(J-1)+37.0*DER2(J-1)-9.0*DER3(J-1)
      CUVAR(J)=VAR(J)+DH*DOUB
   13 CONTINUE
      DO 14 J=1,N
      DER3(J   )=DER2(J)
      DER2(J   )=DER1(J)
   14 DER1(J   )=DER(J+1)
      CALL DERSUB
      IF(II-3)15,15,7
C     ADAMS-MOULTON CORRECTOR
   15 DO 16 J=2,N1
      TEMP=CUVAR(J)
      DOUB=9.0*DER(J)+19.*DER1(J-1)-5.0*DER2(J-1)+DER3(J-1)
      CUVAR(J)=VAR(J)+DH*DOUB
   16 ERRVAL(J-1)=(TEMP-CUVAR(J))/14.210526
   19 GO TO 8
      END
      SUBROUTINE MATINV(A,N,B,M,DETERM,IPIVOT,INDEX,NMAX,ISCALE)
C        **********                                 REVISED   08/01/68
C
C     MATRIX INVERSION WITH ACCOMPANYING SOLUTION OF LINEAR EQUATIONS
C
      DIMENSION IPIVOT(N),A(NMAX,N),B(NMAX,M),INDEX(NMAX,2)
      EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM), (AMAX, T, SWAP)
C
C     INITIALIZATION
C
    5 ISCALE=0
    6 R1=10.0**100
    7 R2=1.0/R1
   10 DETERM=1.0
   15 DO 20 J=1,N
   20 IPIVOT(J)=0
   30 DO 550 I=1,N
C
C     SEARCH FOR PIVOT ELEMENT
C
   40 AMAX=0.0
```

```
   45 DO 105 J=1,N
   50 IF (IPIVOT(J)-1) 60, 105, 60
   60 DO 100 K=1,N
   70 IF (IPIVOT(K)-1) 80, 100, 740
   80 IF (ABS(AMAX)-ABS(A(J,K)))85,100,100
   85 IROW=J
   90 ICOLUM=K
   95 AMAX=A(J,K)
  100 CONTINUE
  105 CONTINUE
      IF (AMAX) 110,106,110
  106 DETERM=0.0
      ISCALE=0
      GO TO 740
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
C
C     INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
  130 IF (IROW-ICOLUM) 140, 260, 140
  140 DETERM=-DETERM
  150 DO 200 L=1,N
  160 SWAP=A(IROW,L)
  170 A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
  205 IF(M) 260, 260, 210
  210 DO 250 L=1, M
  220 SWAP=B(IROW,L)
  230 B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
  260 INDEX(I,1)=IROW
  270 INDEX(I,2)=ICOLUM
  310 PIVOT=A(ICOLUM,ICOLUM)
      IF (PIVOT) 1000,106,1000
C
C     SCALE THE DETERMINANT
C
 1000 PIVOTI=PIVOT
 1005 IF(ABS(DETERM)-R1)1030,1010,1010
 1010 DETERM=DETERM/R1
      ISCALE=ISCALE+1
      IF(ABS(DETERM)-R1)1060,1020,1020
 1020 DETERM=DETERM/R1
      ISCALE=ISCALE+1
      GO TO 1060
 1030 IF(ABS(DETERM)-R2)1040,1040,1060
```

```
 1040 DETERM=DETERM*R1
      ISCALE=ISCALE-1
      IF(ABS(DETERM)-R2)1050,1050,1060
 1050 DETERM=DETERM*R1
      ISCALE=ISCALE-1
 1060 IF(ABS(PIVOTI)-R1)1090,1070,1070
 1070 PIVOTI=PIVOTI/R1
      ISCALE=ISCALE+1
      IF(ABS(PIVOTI)-R1)320,1080,1080
 1080 PIVOTI=PIVOTI/R1
      ISCALE=ISCALE+1
      GO TO 320
 1090 IF(ABS(PIVOTI)-R2)2000,2000,320
 2000 PIVOTI=PIVOTI*R1
      ISCALE=ISCALE-1
      IF(ABS(PIVOTI)-R2)2010,2010,320
 2010 PIVOTI=PIVOTI*R1
      ISCALE=ISCALE-1
  320 DETERM=DETERM*PIVOTI
C
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
  330 A(ICOLUM,ICOLUM)=1.0
  340 DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT
  355 IF(M) 380, 380, 360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT
C
C     REDUCE NON-PIVOT ROWS
C
  380 DO 550 L1=1,N
  390 IF(L1-ICOLUM) 400, 550, 400
  400 T=A(L1,ICOLUM)
  420 A(L1,ICOLUM)=0.0
  430 DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
  455 IF(M) 550, 550, 460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550 CONTINUE
C
C     INTERCHANGE COLUMNS
C
  600 DO 710 I=1,N
```

```
610 L=N+1-I
620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630
630 JROW=INDEX(L,1)
640 JCOLUM=INDEX(L,2)
650 DO 705 K=1,N
660 SWAP=A(K,JROW)
670 A(K,JROW)=A(K,JCOLUM)
700 A(K,JCOLUM)=SWAP
705 CONTINUE
710 CONTINUE
740 RETURN
    END
```

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