# A METHOD FOR CALCULATING
# LAMINAR VISCOUS, COMPRESSIBLE FLOWS
# WITH SMALL PRESSURE GRADIENTS

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16. Abstract

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    For illustration, a two-dimensional simulation of the rotating tank problem is formulated, and the numerical computation procedure to be used is indicated.

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# CONTENTS

Page

iii

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## SUMMARY

A method is presented for calculating laminar viscous, compressible flows. Pressure gradients are assumed to be negligible in energy conservation and in developing property relations, but not in momentum conservation.

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For illustration, a two-dimensional simulation of the rotating tank problem is formulated, and the numerical computation procedure to be used is indicated.

## INTRODUCTION

This report presents a method for calculating laminar viscous, compressible flows in which pressure gradients are small enough to be neglected in the governing energy equation and in certain fluid-property relations. Pressure gradients as sources of momentum change need not be negligible.

The present method was developed as an extension of a previous approximate method for calculating combined forced and contained natural convection in a tank with time-dependent rotation. For use in studying the convection processes in the cryogenic oxygen tanks in a rotating spacecraft (see ref. 1), the previous method was developed in reference 2 from the Navier-Stokes equations. The corresponding computational method is given in reference 3 and was used with evaluations of thermodynamic properties from reference 4 and special methods for analyzing thermodynamic states and simulating an internal heater from reference 5 to obtain results described in reference 6 for a square-tank configuration. Subsequent corresponding results for a circular-tank configuration are given in reference 7.

The approximate theory of reference 2 represents a small-density-variation approximation of the Navier-Stokes equations that is an extension of the Boussinesq approximation to account for all effects of time-dependent rotation. It includes the influence of the combined effects both of effective buoyancy body forces due to temperature and density stratifications (natural convection)

and of arbitrary time-dependent rotation of the container (forced convection). That method is restricted to small density and temperature variations and was considered adequate for its intended application (ref. 8): a study of convection in supercritical cryogenic oxygen in a gravitationless environment (with only very small body forces due to rotation).

The present method removes the restrictions of small temperature and density variations. It accounts for important effects of compressibility (except for sound waves) and variable properties in viscous flow. It can be regarded as a further generalization of the Boussinesq approximation, but is not restricted to either natural convection or convection in a closed container. Since the primary assumption in the method (besides laminar flow) is that pressure gradients be sufficiently small, the method should apply to any laminar flow regions in which that assumption is true (flow regions in which density changes are due mainly to changes in temperature distribution and to changes in average pressure level). This includes convection in a rotating tank with rather large density and temperature differences and, for example, some regions of separated flow over bodies (see ref. 9 for a comprehensive survey).

As in reference 2, all relevant effects of variable rotation are included in the full three-dimensional equations. The final equations and conditions are in a form similar to those of reference 2, but with significant differences. A stream function is not conveniently used in this method, but the Poisson equation for the stream function in reference 2 is just replaced by corresponding Poisson equations for the velocity components for use in one step of the computation procedure. The rotating-tank problem is formulated in a form similar to that in reference 2, and it is expected that the numerical methods of reference 3, as well as the special procedures of references 4 and 5, could be easily adapted to this formulation. The computation procedure is briefly sketched.

## NAVIER-STOKES EQUATIONS IN A TIME-VARYING

## ROTATING REFERENCE FRAME

The basic differential equations governing compressible flow of a Newtonian fluid in a noninertial system are the Navier-Stokes equations in the form (see ref. 2):

$$\nabla \cdot \underset{\sim}{V} = -\frac{1}{\rho}\frac{D\rho}{Dt} \tag{1}$$

$$\rho \frac{D\underset{\sim}{V}}{Dt} = -\nabla p + \rho \underset{\sim}{g} + \nabla \cdot [\lambda(\nabla \cdot \underset{\sim}{V}) \underset{\sim}{I}] + \nabla \cdot \left\{ \mu [\nabla \underset{\sim}{V} + (\nabla \underset{\sim}{V})_t] \right\} \tag{2}$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k\nabla T) + \mu\Phi \tag{3}$$

where: $D/Dt$ is the substantial-derivative operator,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{V} \cdot \nabla$$

the apparent body force per unit mass in the noninertial system is

$$\underset{\sim}{g} = -[\underset{\sim}{a} + \dot{\underset{\sim}{\Omega}} \times \underset{\sim}{R} + 2\underset{\sim}{\Omega} \times \underset{\sim}{V} + \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{R})] \tag{4}$$

the viscous dissipation of energy is

$$\mu\Phi \equiv \lambda(\underset{\sim}{\nabla} \cdot \underset{\sim}{V}) \underset{\sim}{I} : \underset{\sim}{\nabla}\underset{\sim}{V} + \mu[\underset{\sim}{\nabla}\underset{\sim}{V} + (\underset{\sim}{\nabla}\underset{\sim}{V})_t] : \underset{\sim}{\nabla}\underset{\sim}{V} \tag{5}$$

and

| | |
|---|---|
| $\underset{\sim}{V}$ | fluid velocity in the noninertial system |
| $\rho$ | local mass density |
| $p$ | local thermodynamic pressure |
| $T$ | temperature |
| $\mu$ | shear viscosity coefficient |
| $\lambda$ | $\kappa - (2/3)\mu$ is the second viscosity coefficient |
| $\kappa$ | bulk viscosity coefficient |
| $\underset{\sim}{I}$ | unit tensor |
| $(\underset{\sim}{\nabla}\underset{\sim}{V})_t$ | transpose of the dyadic $\underset{\sim}{\nabla}\underset{\sim}{V}$ |
| $h$ | specific enthalpy |
| $k$ | thermal conductivity in the Fourier heat conduction law for heat flux |
| $\underset{\sim}{a}$ | (possibly time-dependent) linear acceleration of the noninertial system (in excess of $\dot{\underset{\sim}{\Omega}} \times \underset{\sim}{R}$) relative to an inertial system |
| $\underset{\sim}{\Omega}$ | time-dependent angular velocity of the noninertial system relative to an inertial system |
| $\underset{\sim}{R}$ | perpendicular vector from the axis of rotation to a point of interest in the flow system (see ref. 2) |
| $\dot{\underset{\sim}{\Omega}}$ | angular acceleration of the noninertial system, $d\underset{\sim}{\Omega}/dt$ |

In equation (4), $\dot{\underset{\sim}{\Omega}} \times \underset{\sim}{R}$ is equal to a linear acceleration due to the angular acceleration $\dot{\underset{\sim}{\Omega}}$, $2\underset{\sim}{\Omega} \times \underset{\sim}{V}$ is the Coriolis acceleration, and $\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{R})$ is equal to the centripetal acceleration due to rotation of the system. The direction of the axis of rotation is assumed to be constant—

$$\underset{\sim}{\Omega}/\Omega = \underset{\sim}{e}_3 \tag{6a}$$

3

where $\Omega$ is the time-dependent magnitude of $\underset{\sim}{\Omega}$ and $\underset{\sim}{e}_3$ is a constant unit vector in a direction parallel to $\underset{\sim}{\Omega}$. Note that by the definition of $\underset{\sim}{R}$

$$\underset{\sim}{\Omega} \cdot \underset{\sim}{R} = 0 \tag{6b}$$

To make the system of equations complete, one must specify auxiliary expressions for the transport properties $\mu$, $\lambda$, and $k$, and equations of state,

$$\rho = \rho(p,T) \tag{7a}$$

$$h = h(p,T) \tag{7b}$$

## ASSUMPTIONS AND RESULTING VECTOR EQUATIONS

In flows in which pressure gradients are sufficiently small so that density changes are due mainly to changes in temperature distribution and to changes in the average pressure level at a given time, the terms $\underset{\sim}{V} \cdot \nabla p$ and $\mu\Phi$ can be omitted from the energy equation (3). Those terms are of minor importance in that equation whose primary function is to determine the changes in the temperature distribution. Refer also to Whitham (ref. 10, pp. 124–127), who shows that those terms are negligible if

$$\frac{T - T_1}{T_1} \gg M_1{}^2$$

where $T_1$ is a typical temperature in the flow and $M_1$ is a corresponding typical Mach number. The method of reference 2 also required $(T - T_1)/T_1$ to be small; the present method does not. The above conditions allow the pressure $p$ to be replaced by its instantaneous spatial average $\bar{p}(t)$ for use in the energy equation:

$$h(p,T) \approx h[\bar{p}(t), T] \tag{8}$$

$$\frac{Dp}{Dt} \approx \frac{d\bar{p}}{dt} \equiv \bar{p}'(t) \tag{9}$$

Similarly, whereas in reference 2 use was made of

$$\rho(p,T) = \rho_O + (T - T_O)\left[\left(\frac{\partial\rho}{\partial T}\right)_p\right]_O + O\,[(\Delta T)^2, \Delta p]$$

we write now the more accurate representation based on the above conditions:

$$\rho(p,T) = \rho(p_1,T) + (p - p_1)\left(\frac{\partial\rho}{\partial p}\right)_T + O[(p - p_1)^2] \approx \rho(p_1,T) + [\bar{p}(t) - p_1]\left(\frac{\partial\rho}{\partial p}\right)_T \tag{10a}$$

In the above, $\Delta T \equiv T - T_O$ and $\Delta p = p - p_O$, where $\rho_O$, $p_O$, and $T_O$ in reference 2 represented the

4

spatial average thermodynamic state; and $p_1$ is a constant typical pressure in the flow. Note that equation (10a) includes an assumption of small pressure gradients because $p$ is replaced by $\bar{p}(t)$.

Similarly, the transport properties $\mu$ and $k$, as well as the thermodynamic state properties including $(\partial h/\partial p)_T$ and

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p \tag{11}$$

can be approximated in the form (cf. (10a))

$$X(p,T) = F_1(T) + \left(\frac{\bar{p}(t) - p_1}{p_2 - p_1}\right) F_2(T) \tag{10b}$$

where $p_1$ and $p_2$ are constants and $F_1(T)$ and $F_2(T)$ are different functions to be specified for each property X to be so represented. The validity of this approximation was found from experience in the progress of studies reported in references 5 and 6. With $h$ in (7b) represented approximately by equation (8) and with the definition (11) and

$$c_t = \left(\frac{\partial h}{\partial p}\right)_T \frac{d\bar{p}}{dt} \equiv \left(\frac{\partial h}{\partial p}\right)_T \bar{p}'(t) \tag{12}$$

one obtains from (8):

$$\frac{Dh}{Dt} = c_p \frac{DT}{Dt} + c_t \tag{13}$$

In addition to the general property representation (10b), it is convenient to write specifically (for (10a))

$$\rho = \rho_O[F_1(T) + \bar{f}(t)F(T)] \tag{14}$$

in place of (7a), where $\rho_O$ is a constant, $F_1(T)$ and $F(T)$ now denote definite functions to be specified, and $\bar{f}(t)$ is the bracketed factor in (10b).

With the above assumptions the energy equation (3) yields

$$\frac{DT}{Dt} = \frac{1}{\rho c_p} \underset{\sim}{\nabla} \cdot (k \underset{\sim}{\nabla} T) - \frac{c_t}{c_p} + \frac{1}{\rho c_p} \bar{p}'(t) \tag{15}$$

It is then convenient to define

$$\alpha \equiv -\frac{1}{\rho} \frac{D\rho}{Dt} \tag{16}$$

and to use equation (14) to obtain

$$\alpha = -\frac{\rho_O}{\rho}\left\{\bar{f}'(t)F(T) + [\bar{f}(t)F'(T) + F'_1(T)]\frac{DT}{Dt}\right\}$$ (17a)

where

( )′      generally denotes the derivative of a function

$\rho_O/\rho$      is obtained from (14)

$DT/Dt$      is specified by (15)

Therefore

$$\alpha = \frac{-\rho_O}{\rho}\left\{\bar{f}'(t)F(T) + [\bar{f}(t)F'(T) + F'_1(T)]\left[\frac{1}{\rho c_p}\underset{\sim}{\nabla}\cdot(k\underset{\sim}{\nabla}T) - \frac{c_t}{c_p} + \frac{1}{\rho c_p}\bar{p}'(t)\right]\right\}$$ (17b)

in which $\rho$ is given by (14).

The resulting vector equations are now, from (1) and (16), *conservation of mass*:

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \alpha$$ (18)

where $\alpha$ is given by (17b); *conservation of momentum* given by equation (2); and from equations (15) and (18) a "divergence form" (convenient for numerical computation) of the *energy conservation* equation:

$$\frac{\partial T}{\partial t} + \underset{\sim}{\nabla}\cdot(\underset{\sim}{V}T) = \alpha T + \frac{1}{\rho c_p}\underset{\sim}{\nabla}\cdot(k\underset{\sim}{\nabla}T) - \frac{c_t}{c_p} + \frac{1}{\rho c_p}\bar{p}'(t)$$ (19)

## FORMULATION IN TERMS OF VORTICITY

To use a solution procedure similar to that described in reference 3, it is convenient to formulate the problem in terms of vorticity. In addition to the appropriateness of vorticity as a primary variable (see Lighthill, ref. 11, pp. 57−60), it is convenient to replace equation (18) by a corresponding vector Poisson equation. The formulation also eliminates from the problem both the second viscosity coefficient $\lambda$ and the pressure, except for the average pressure $\bar{p}(t)$, which must be determined from an appropriate equation of state as in reference 5 in terms of a spatial average temperature $\bar{T}$ and corresponding average density $\bar{\rho}$.

With the definition of vorticity

$$\underset{\sim}{\omega} = \underset{\sim}{\nabla}\times\underset{\sim}{V}$$ (20)

and the identity

$$\nabla \times \underset{\sim}{\omega} \equiv \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \equiv \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{V}) - \nabla^2 \underset{\sim}{V} \tag{21}$$

along with (18), one obtains a vector Poisson equation for $\underset{\sim}{V}$ in terms of $\underset{\sim}{\omega}$ and $\alpha$:

$$\nabla^2 \underset{\sim}{V} = \underset{\sim}{\nabla}\alpha - \underset{\sim}{\nabla}\times \underset{\sim}{\omega} \tag{22}$$

which would be solved for the velocity components after $\underset{\sim}{\omega}$ is determined at each time step.

The momentum equation (2) is put into the form of a vorticity equation as follows: Use the identity

$$\underset{\sim}{V} \cdot \underset{\sim}{\nabla}\underset{\sim}{V} \equiv \underset{\sim}{\nabla}\left(\tfrac{1}{2}V^2\right) - \underset{\sim}{V} \times (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \tag{23}$$

to write equation (2) as

$$\rho\left[\frac{\partial \underset{\sim}{V}}{\partial t} + \underset{\sim}{\nabla}\left(\tfrac{1}{2}V^2\right) - \underset{\sim}{V} \times \underset{\sim}{\omega}\right] = \underset{\sim}{\nabla}(-p + \lambda\alpha) + \rho\underset{\sim}{g} + \underset{\sim}{\nabla} \cdot \left\{\mu[\underset{\sim}{\nabla}\underset{\sim}{V} + (\underset{\sim}{\nabla}\underset{\sim}{V})_t]\right\} \tag{24}$$

Using the identities (for any scalar $\phi$ and vector $\underset{\sim}{A}$):

$$\underset{\sim}{\nabla} \times (\phi\underset{\sim}{A}) \equiv \phi\underset{\sim}{\nabla} \times \underset{\sim}{A} + \underset{\sim}{\nabla}\phi \times \underset{\sim}{A} \tag{25a}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{\nabla}\phi \equiv 0 \tag{25b}$$

take the curl of equation (24) to obtain

$$\rho\left[\frac{\partial \underset{\sim}{\omega}}{\partial t} - \underset{\sim}{\nabla} \times (\underset{\sim}{V} \times \underset{\sim}{\omega})\right] + \underset{\sim}{\nabla}\rho \times \frac{D\underset{\sim}{V}}{Dt} = \rho\underset{\sim}{\nabla} \times \underset{\sim}{g} + \underset{\sim}{\nabla}\rho \times \underset{\sim}{g} + \underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \cdot \left\{\mu[\underset{\sim}{\nabla}\underset{\sim}{V} + (\underset{\sim}{\nabla}\underset{\sim}{V})_t]\right\} \tag{26}$$

For use in equation (26)

$$\frac{1}{\rho}\underset{\sim}{\nabla}\rho = \frac{[\overline{f}(t)F'(T) + F_1'(T)]}{[\overline{f(t)}F(T) + F_1(T)]}\underset{\sim}{\nabla}T \tag{27}$$

The following developments are also useful for reducing equation (26). From equation (4) along with

$$\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{R}) = -\Omega^2\underset{\sim}{R} = -\underset{\sim}{\nabla}\left(\tfrac{1}{2}\Omega^2 R^2\right) \tag{28}$$

$$\underset{\sim}{a} = \underset{\sim}{\nabla}\phi \tag{29}$$

(where $\phi = \underset{\sim}{R}^* \cdot \underset{\sim}{a}(t)$ is a scalar potential for the time-dependent acceleration $\underset{\sim}{a}$ and $\underset{\sim}{R}^*$ is the position vector from an arbitrary fixed point on the axis of rotation in the noninertial system; see

7

ref. 2), one can write

$$\underset{\sim}{g} = -\left[ \underset{\sim}{\nabla}\left(\phi - \frac{1}{2}\Omega^2 R^2\right) + \underset{\sim}{\dot{\Omega}} \times \underset{\sim}{R} + 2\underset{\sim}{\Omega} \times \underset{\sim}{V} \right] \tag{30}$$

Then

$$\underset{\sim}{\nabla} \times \underset{\sim}{g} = -[\underset{\sim}{\nabla} \times (\underset{\sim}{\dot{\Omega}} \times \underset{\sim}{R}) + 2\underset{\sim}{\nabla} \times (\underset{\sim}{\Omega} \times \underset{\sim}{V})] \tag{31}$$

For arbitrary vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$, the identity

$$\underset{\sim}{\nabla} \times (\underset{\sim}{A} \times \underset{\sim}{B}) \equiv \underset{\sim}{B} \cdot \underset{\sim}{\nabla A} - \underset{\sim}{A} \cdot \underset{\sim}{\nabla B} + \underset{\sim}{A}\underset{\sim}{\nabla} \cdot \underset{\sim}{B} - \underset{\sim}{B}\underset{\sim}{\nabla} \cdot \underset{\sim}{A} \tag{32a}$$

$$\equiv \underset{\sim}{\nabla} \cdot (\underset{\sim}{BA}) - \underset{\sim}{\nabla} \cdot (\underset{\sim}{AB}) \tag{32b}$$

yields

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\dot{\Omega}} \times \underset{\sim}{R}) = \underset{\sim}{\dot{\Omega}}\underset{\sim}{\nabla} \cdot \underset{\sim}{R} \equiv 2\underset{\sim}{\dot{\Omega}} \tag{33a}$$

and

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\Omega} \times \underset{\sim}{V}) = -\underset{\sim}{\Omega} \cdot \underset{\sim}{\nabla V} + \underset{\sim}{\Omega}\alpha \tag{33b}$$

$$= -\Omega\,(\partial \underset{\sim}{V}/\partial Z) + \underset{\sim}{\Omega}\alpha \tag{33c}$$

where $Z$ is a coordinate in the direction of the unit vector $\underset{\sim}{e}_3$ defined by equation (5). Therefore, results for use in equation (26) are

$$\underset{\sim}{g} = -(\underset{\sim}{a} + \underset{\sim}{\dot{\Omega}} \times \underset{\sim}{R} + 2\underset{\sim}{\Omega} \times \underset{\sim}{V} - \Omega^2\underset{\sim}{R}) \tag{34a}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{g} = 2(\Omega\frac{\partial \underset{\sim}{V}}{\partial Z} - \underset{\sim}{\Omega}\alpha - \underset{\sim}{\dot{\Omega}}) \tag{34b}$$

Further, if the identity (32b) is used in the left side of (26) for $\underset{\sim}{\nabla} \times (\underset{\sim}{V} \times \underset{\sim}{\omega})$, the above developments yield

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + \underset{\sim}{\nabla} \cdot (\underset{\sim}{V}\omega) - \underset{\sim}{\nabla} \cdot (\underset{\sim}{\omega}V) = \left(\frac{1}{\rho}\underset{\sim}{\nabla}\rho\right) \times \left(\underset{\sim}{g} - \frac{D\underset{\sim}{V}}{Dt}\right)$$
$$+ 2\left(\Omega\frac{\partial \underset{\sim}{V}}{\partial Z} - \underset{\sim}{\Omega}\alpha - \underset{\sim}{\dot{\Omega}}\right) + \frac{1}{\rho}\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \cdot \left\{\mu[\underset{\sim}{\nabla V} + (\underset{\sim}{\nabla V})_t]\right\} \tag{35}$$

where

$\frac{1}{\rho}\underset{\sim}{\nabla}\rho$    is given by (27)

$\underset{\sim}{g}$    is given by (34a)

8

$$DV/Dt \quad = \partial V/\partial t + V \cdot \nabla V$$

$\rho$         is given by (14)

The last term in equation (35) can be reduced further for possible convenience in the solution procedure. The identities:

$$\nabla \cdot [\mu \nabla V] \equiv \mu \nabla^2 V + (\nabla \mu) \cdot \nabla V \tag{36a}$$

$$\nabla \cdot [\mu (\nabla V)_t] \equiv \mu \nabla (\nabla \cdot V) + (\nabla \mu) \cdot (\nabla V)_t \tag{36b}$$

can be used along with equation (22) to obtain

$$\nabla \cdot \left\{ \mu [\nabla V + (\nabla V)_t] \right\} = \mu [2 \nabla \alpha - \nabla \times \omega] + (\nabla \mu) \cdot [\nabla V + (\nabla V)_t] \tag{37}$$

When the curl is taken of equation (37) for use in (35), one can use the identities (25) along with the additional identities:

$$\nabla \cdot \omega \equiv \nabla \cdot (\nabla \times V) \equiv 0 \tag{38a}$$

$$\nabla \times (\nabla \times \omega) \equiv \nabla (\nabla \cdot \omega) - \nabla^2 \omega \equiv -\nabla^2 \omega \tag{38b}$$
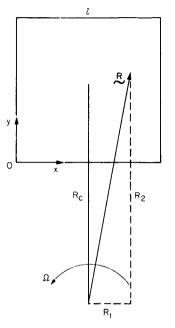
and with equation (22) to obtain for the last term in (35):

$$\frac{1}{\rho} \nabla \times \nabla \cdot \left\{ \mu [\nabla V + (\nabla V)_t] \right\} = \frac{1}{\rho} \left\{ \mu \nabla^2 \omega + (\nabla \mu) \times (\nabla \alpha + \nabla^2 V) \right\}$$
$$+ \frac{1}{\rho} \nabla \times \left\{ (\nabla \mu) \cdot [\nabla V + (\nabla V)_t] \right\} \tag{39}$$

The final results for this formulation are now:

1. Equation (14) for $\rho$, where $\overline{f}(t)$ is the bracketed factor in (10b)

2. Equation (17b) for the auxiliary variable $\alpha$

3. The vector Poisson equation (22) for $V$

4. Equation (35) for $\omega$, where the last term may be replaced by equation (39), $(1/\rho)\nabla p$ is given by (27), and $g$ is given by (34a)

5. Equation (19) for $T$

6. Equation (12) for $c_t$

These equations must be supplemented by appropriate relations in the form of (10b) for $c_p$, $(\partial h/\partial p)_T$, $\mu$, and $k$, and appropriate relations for $\overline{p} = \overline{p}(t)$ as a function of $\overline{\rho}$ and $\overline{T}$ and for $F(T)$ and $F_1(T)$ in equation (14).

9

Figure 1. – Square tank configuration.

For a two-dimensional simulation of the convection in a tank it is convenient to use a square-tank configuration with the equations and conditions in Cartesian coordinates. For convection in a plane normal to the axis of rotation, the assumed configuration is as shown in figure 1. With $R_C$ being the distance from the center of rotation to the tank center and $\underset{\sim}{R}$ the radius vector from the center of rotation, the components of $\underset{\sim}{R}$ in terms of the tank-fixed coordinates $(x, y)$ are

$$R_1 = x - \frac{1}{2}l \tag{40a}$$

$$R_2 = y - \frac{1}{2}l + R_c \tag{40b}$$

The vorticity vector is

$$\underset{\sim}{\omega} = \underset{\sim}{e}_3 \, \omega \tag{41a}$$

where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{41b}$$

and where $u$ and $v$ are the $x$ and $y$ components of $\underset{\sim}{V}$. The $x$ and $y$ components of the apparent body force per unit mass $\underset{\sim}{g}$ are, from (34a),

$$g_1 = R_1 \Omega^2 + 2\Omega v + R_2 \dot{\Omega} - a_1 \tag{42a}$$

$$g_2 = R_2 \Omega^2 - 2\Omega u - R_1 \dot{\Omega} - a_2 \tag{42b}$$

where $a_1$ and $a_2$ are the components of $\underset{\sim}{a}$.

For two dimensions we now have, from (14) and (17b):

$$\rho = \rho_O \left[ F_1(T) + \overline{f}(t)F(T) \right] \tag{43}$$

and

$$\alpha = \frac{-\rho_O}{\rho} \left( \overline{f}'(t)F(T) + [\overline{f}(t)F'(T) + F_1'(T)] \left\{ \frac{1}{\rho c_p} \left[ \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) \right] - \frac{c_t}{c_p} + \frac{1}{\rho c_p} \overline{p}'(t) \right\} \right)$$

$$\tag{44}$$

10

Equation (22) yields two scalar Poisson equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial \alpha}{\partial x} - \frac{\partial \omega}{\partial y} \tag{45a}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial \alpha}{\partial y} + \frac{\partial \omega}{\partial x} \tag{45b}$$

Denote by $A$ in equation (27) the factor

$$A = \frac{\overline{f}(t)F'(T) + F_1'(T)}{\overline{f}(t)F(T) + F_1(T)} \tag{46}$$

and by $W$ the factor multiplying $1/\rho$ in the viscous diffusion term (the last term) in equation (35). Then one obtains from (39):

$$W = \mu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) + \frac{\partial \mu}{\partial x}\left(\frac{\partial \alpha}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$
$$- \frac{\partial \mu}{\partial y}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial}{\partial x}\left[\frac{\partial \mu}{\partial x} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial \mu}{\partial y}\left(2\frac{\partial v}{\partial y}\right)\right]$$
$$- \frac{\partial}{\partial y}\left[\frac{\partial \mu}{\partial x}\left(2\frac{\partial u}{\partial x}\right) + \frac{\partial \mu}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \tag{47a}$$

or more directly from the left side of (39),

$$W = \frac{\partial^2}{\partial x^2}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial^2}{\partial x \partial y}\left[2\mu\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\right] - \frac{\partial^2}{\partial y^2}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \tag{47b}$$

In terms of $A$ and $W$ the vorticity equation (35) becomes in two dimensions:

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}(u\omega) + \frac{\partial}{\partial y}(v\omega) = A\left[\frac{\partial T}{\partial x}\left(g_2 - \frac{Dv}{Dt}\right) - \frac{\partial T}{\partial y}\left(g_1 - \frac{Du}{Dt}\right)\right] - 2(\Omega\alpha + \dot{\Omega}) + \frac{1}{\rho}W \tag{48}$$

where

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \tag{49a}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \tag{49b}$$

The two-dimensional energy equation for $T$ with the convective terms in the divergence form is, from (19),

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha T + \frac{1}{\rho c_p}\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right] - \frac{c_t}{c_p} + \frac{1}{\rho c_p}\overline{p}'(t) \tag{50}$$

11

In equations (44) and (50), $c_f$ is given by equation (12), and

$$\bar{f}(t) = \frac{\bar{p}(t) - p_1}{p_2 - p_1} \tag{51}$$

(See eq. (10b)), where $\bar{p}(t)$ can be computed by a method described in reference 5, from a knowledge of the mean fluid density $\bar{\rho}$ and the temperature distribution.

Initially one can specify the conditions,

$$t = 0: \quad \begin{matrix} u = v = \omega = 0 \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial v}{\partial t} = 0 \end{matrix} \right\} \ \text{all } x,y \tag{52}$$

if the calculation is started at an initial condition of "rigid body rotation" *before* the rotation rate is changed. Also an initial temperature distribution can be specified. One must specify $\Omega(t)$ at all times, including $t = 0$.

At all boundaries, the conditions of no slip in the tangential direction and no normal mass flow are represented by the conditions

$$u = v = 0 \quad \text{on all boundaries} \tag{53}$$

for application to the Poisson equations (45). Boundary conditions on $\omega$ needed for application to the vorticity equation (48) (with either (47a) or (47b) for the viscous diffusion) are supplied as follows. If subscript $b$ denotes a value on the boundary, subscript $\Delta x$ denotes a value at a small distance $|\Delta x|$ inside the boundaries $x = 0$ and $x = l$ (see fig. 1), and subscript $\Delta y$ denotes a value at a small distance $|\Delta y|$ inside the boundaries $y = 0$ and $y = l$, then from the definition of vorticity (41b) with the conditions (53), one has

$$\text{at } x = 0, l: \quad \omega_b = \left(\frac{\partial v}{\partial x}\right)_b \tag{54a}$$

$$\text{at } y = 0, l: \quad \omega_b = -\left(\frac{\partial u}{\partial y}\right)_b \tag{54b}$$

which can be represented for numerical approximation by:

$$x = 0 \quad \omega_b = \frac{(v)_{\Delta x}}{|\Delta x|} \tag{55a}$$

$$x = l \quad \omega_b = \frac{-(v)_{\Delta x}}{|\Delta x|} \tag{55b}$$

12

$$y = 0 \quad \omega_b = \frac{-(u)_{\Delta y}}{|\Delta y|} \tag{55c}$$

$$y = l \quad \omega_b = \frac{(u)_{\Delta y}}{|\Delta y|} \tag{55d}$$

For application to equation (50), one may specify $\partial T/\partial x$ on $x = 0$ and $x = l$ and $\partial T/\partial y$ on $y = 0$ and $y = l$. In addition, a heat source at some interior point can be specified to represent the power input from a heater (see refs. 1 and 5).

## SOLUTION PROCEDURE

For convenience, the above equations, initial conditions, and boundary conditions could be put into dimensionless form as was done in reference 2 for the approximate method. It is anticipated that the numerical solution procedure of reference 3 for the approximate equations from reference 2 could be adapted to the present formulation. The dimensionless differential equations corresponding to the equations in the above section would be converted to difference equations in much the same manner. One new aspect is the appearance of $Du/Dt$ and $Dv/Dt$ in equation (48) and of $\bar{p}(t)$ and $\bar{p}'(t)$ in several equations. The time derivatives $\partial u/\partial t$, $\partial v/\partial t$, and $\bar{p}'(t)$ in those terms are expected to be small and can presumably be backward-differenced without significantly affecting the numerical stability, because the vorticity and energy equations are being solved for $\omega$ and $T$ at each time step with $u$ and $v$ determined by Poisson equations and $\bar{p}$ from a state relation. Another new aspect is the replacement of the Poisson equation for the stream function in references 2 and 3 by two Poisson equations for $u$ and $v$ that depend on space derivatives of $\omega$ at each time step. The latter two equations must satisfy *both* the no-slip and the no-mass-flux boundary conditions, which means that the vorticity at the advanced time step is not required to satisfy any independent conditions as in the earlier procedure (cf. also the discussion of boundary conditions in ref. 7).

The suggested computational procedure for the corresponding difference equations is:

1. Set the variables initially according to equations (52) and the prescribed temperature distribution. Also set $\bar{p}(0)$ at the desired initial pressure and $p'(0) = 0$.

2. Calculate the "predicted" vorticity and temperature distributions by the predictor in MacCormack's explicit method (ref. 12; cf. ref. 3) from the difference equations corresponding to equations (48) and (50), and using conditions (55).

3. Calculate the corresponding $\bar{p}$ by a method described in reference 5 using the temperature distribution and the known mean fluid density; then calculate $\rho$ and $\alpha$ according to equations (43) and (44).

4. Calculate the corresponding "predicted" $u$ and $v$ from equations (45) using Buneman's Poisson solver (ref. 13; cf. ref. 3) with conditions (53).

5. Calculate the corrected vorticity and temperature distributions using the corrector in

MacCormack's method (cf. ref. 3) for the difference equations from (48) and (50) (and using conditions (55) with predicted velocities).

6. Calculate the corrected $\bar{p}$, $\rho$, and $\alpha$ as in step 3.

7. Calculate the corrected $u$ and $v$ as in step 4.

8. Advance the time index and repeat steps 2 through 8.

The procedure is continued for the number of time steps desired.

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14

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