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TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 762

THE FLOW OF A COMPRESSIBLE FLUID PAST A SPHERE

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SUMMARY

The flow of a compressible fluid past a sphere fixed in a uniform stream is calculated to the third order of approximation by means of the Janzen-Rayleigh method. The velocity and the pressure distributions over the surface of the sphere are computed and the terms involving the fourth power of the Mach number, neglected in Rayleigh's calculation, are shown to be of considerable importance as the local velocity of sound is approached on the sphere. The critical Mach number, that is, the value of the Mach number at which the maximum velocity of the fluid past the sphere is just equal to the local velocity of sound, is calculated for both the second and the third approximations and is found to be, respectively, $M_{cr} = 0.587$ and $M_{cr} = 0.573$.

INTRODUCTION

The irrotational flow of a compressible fluid past a circular cylinder and a sphere was first calculated by Janzen (reference 1) and by Rayleigh (reference 2). Their method consisted in obtaining a correction term to the incompressible-fluid solution, but the results were limited to the terms involving only the square of the Mach number. Recently, the author (reference 3) and Imai (reference 4) extended the calculations for the circular cylinder by including the terms involving the fourth power of the Mach number. These higher-power terms, neglected in the earlier calculations, were found to be of considerable importance as the local velocity of sound is approached on the surface of the cylinder. It has therefore been thought worth while to extend the calculations in a similar manner for the flow past a sphere.

ANALYSIS

Preliminary developments.- The flow is assumed to be uniform at a great distance from the sphere and the motion to be everywhere irrotational and steady. Then, with

$$c^2 = \frac{dp}{d\rho} \tag{1}$$

the equations of motion reduce to

$$c^2 \frac{d\rho}{\rho} = -\frac{1}{2} d(v^2) \tag{2}$$

where v is the fluid velocity; c, the local velocity of sound; p, the pressure; and ρ, the density. Then, assuming the adiabatic relationship between p and ρ, it follows by integration of equation (2) that

$$\frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{1}{2} U^2 + \frac{\gamma}{\gamma - 1} \frac{p_o}{\rho_o} \tag{3}$$

and from equation (1) that

$$c^2 = c_o^2 \left[1 + \frac{\gamma - 1}{2} M^2 \left(1 - \frac{v^2}{U^2}\right)\right] \tag{4}$$

where U is the velocity of the undisturbed stream; p_o, ρ_o, and c_o, the corresponding quantities in the undisturbed stream; M $(= U/c_o)$, the Mach number; and γ, the ratio of the specific heats of the fluid.

Since the fluid motion is irrotational, there exists a velocity potential ϕ and the equation of continuity may be written as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{2c^2}\left(\frac{\partial v^2}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial v^2}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial v^2}{\partial z}\frac{\partial \phi}{\partial z}\right) \tag{5}$$

Let r, θ, and φ denote space polar coordinates and suppose the origin of r to be at the center of a sphere of radius a and the initial line of θ to be parallel to the direction of the stream. Then, designating by r the

ratio r/a, by ϕ the ratio ϕ/Ua and, by v the ratio v/U and taking into account the fact that the flows in all meridian planes φ = constant are similar, equation (5) becomes

$$\left[1 + \frac{\gamma - 1}{2} M^2 (1 - v^2)\right] \Delta\phi = \frac{1}{2} M^2 \left(\frac{\partial\phi}{\partial r}\frac{\partial v^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial v^2}{\partial\theta}\right) \quad (6)$$

where

$$\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left[(1 - \mu^2)\frac{\partial\phi}{\partial\mu}\right] \quad (7)$$

and

$$\mu = \cos\theta$$

It is now assumed that ϕ can be developed as a power series in M^2 (reference 4) so that

$$\phi = \phi_0 + \phi_1 M^2 + \phi_2 M^4 + \ldots \quad (8)$$

Since

$$v^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2$$

then

$$v^2 = {v_0}^2 + {v_1}^2 M^2 + {v_2}^2 M^4 + \ldots \quad (9)$$

where

$${v_0}^2 = \left(\frac{\partial\phi_0}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi_0}{\partial\theta}\right)^2 \quad (9a)$$

$${v_1}^2 = 2\left(\frac{\partial\phi_0}{\partial r}\frac{\partial\phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial\phi_1}{\partial\theta}\right) \quad (9b)$$

$${v_2}^2 = 2\left(\frac{\partial\phi_0}{\partial r}\frac{\partial\phi_2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial\phi_2}{\partial\theta}\right)$$

$$+ \left[\left(\frac{\partial\phi_1}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi_1}{\partial\theta}\right)^2\right] \quad (9c)$$

. . . .

When these expressions for ϕ and v^2 are inserted into equation (6) and the coefficients of the same powers of M on both sides are equated,

$$\Delta\phi_0 = 0 \tag{10a}$$

$$\Delta\phi_1 = \frac{1}{2}\left(\frac{\partial\phi_0}{\partial r}\frac{\partial v_0{}^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial v_0{}^2}{\partial\theta}\right) \tag{10b}$$

$$\Delta\phi_2 = \frac{\gamma - 1}{2}(v_0{}^2 - 1)\Delta\phi_1 + \frac{1}{2}\left(\frac{\partial\phi_0}{\partial r}\frac{\partial v_1{}^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial v_1{}^2}{\partial\theta}\right)$$
$$+ \frac{1}{2}\left(\frac{\partial\phi_1}{\partial r}\frac{\partial v_0{}^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_1}{\partial\theta}\frac{\partial v_0{}^2}{\partial\theta}\right) \tag{10c}$$

. . . .

From these equations, any given approximation ϕ_n clearly depends only on the preceding approximations of which the first one is the solution of Laplace's equation $\Delta\phi_0 = 0$ for an incompressible fluid.

The first approximation.- Equation (10a) is the differential equation for the velocity potential ϕ_0 for the flow of an incompressible, nonviscous fluid and may be written as

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi_0}{\partial r}\right) + \frac{\partial}{\partial\mu}\left[(1 - \mu^2)\frac{\partial\phi_0}{\partial\mu}\right] = 0$$

The solution of this equation is known to be

$$\phi_0 = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\mu)$$

where $P_n(\mu)$ is Legendre's polynomial of order n and A_n and B_n are arbitrary constants. In the case of a sphere of radius a, supposed fixed in a stream of uniform velocity U, the boundary conditions to be satisfied are as follows:

$$-\frac{\partial\phi_0}{\partial r} = \text{normal velocity} = 0, \quad \text{at the surface of the sphere}$$

and

$$-\frac{\partial \phi_o}{\partial r} = -\cos\theta, \quad \text{at infinity}$$

These conditions limit the form of the solution to

$$\phi_o = \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

where $\cos\theta = P_1(\mu)$. Inserting the boundary conditions,

$$A_1 = 1 \quad \text{and} \quad B_1 = 1/2$$

and, therefore,

$$\phi_o = \left(r + \frac{1}{2r^2}\right) P_1(\mu) \tag{11}$$

<u>The second approximation</u>.- From equations (11) and (9a), it follows that

$$v_o^2 = 1 + r^{-3} + \frac{1}{4}\, r^{-6} + \left(-3r^{-3} + \frac{3}{4}\, r^{-6}\right)\mu^2$$

or since $P_o(\mu) = 1$ and $P_2(\mu) = \frac{3}{2}\,\mu^2 - \frac{1}{2}$

$$v_o^2 = \left(1 + \frac{1}{2}\, r^{-6}\right) P_o(\mu) + \left(-2r^{-3} + \frac{1}{2}\, r^{-6}\right) P_2(\mu)$$

Then, by a simple calculation, it is found from equation (10b) that

$$\Delta\phi_1 = \left(-\frac{18}{5}\, r^{-7} + \frac{9}{4}\, r^{-10}\right) P_1(\mu)$$

$$+ \left(3r^{-4} - \frac{12}{5}\, r^{-7} + \frac{3}{4}\, r^{-10}\right) P_3(\mu) \tag{12}$$

where

$$P_3(\mu) = \frac{5}{2}\,\mu^3 - \frac{3}{2}\,\mu$$

Now, a particular integral of the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \phi}{\partial \mu}\right] = r^m P_n(\mu)$$

is

$$\phi = \frac{r^{m+2}\ P_n(\mu)}{(m + 2)\ (m + 3) - n(n + 1)}$$

Hence, the solution is given by

$$\phi = \sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+1)} + \frac{r^{m+2}}{(m+2)\ (m+3)\ -n(n+1)}\right] P_n(\mu) \tag{13}$$

except when

$$m = n - 2 \quad \text{or} \quad m = -(n + 3)$$

Accordingly, the solution of equation (12) is

$$\phi_1 = (A_1 r + B_1 r^{-2})P_1(\mu) + (A_3 r^3 + B_3 r^{-4})P_3(\mu)$$
$$+ \left(-\frac{1}{5}\ r^{-5} + \frac{1}{24}\ r^{-8}\right) P_1(\mu)$$
$$+ \left(-\frac{3}{10}\ r^{-2} - \frac{3}{10}\ r^{-5} + \frac{3}{176}\ r^{-8}\right)P_3(\mu)$$

Since ϕ_0 already satisfies the necessary boundary conditions of the problem, the higher approximations ϕ_1, ϕ_2, ... must satisfy the conditions

$$\frac{\partial\phi_1}{\partial r} = 0, \quad \frac{\partial\phi_2}{\partial r} = 0, \quad \ldots$$

for both $r = 1$ and $r = \infty$. Hence, after a simple calculation,

$$A_1 = A_3 = 0 \quad \text{and} \quad B_1 = \frac{1}{3}, \quad B_3 = \frac{27}{55}$$

Therefore,

$$\phi_1 = \left(\frac{1}{3}\ r^{-2} - \frac{1}{5}\ r^{-5} + \frac{1}{24}\ r^{-8}\right) P_1(\mu)$$
$$+ \left(-\frac{3}{10}\ r^{-2} + \frac{27}{55}\ r^{-4} - \frac{3}{10}\ r^{-5} + \frac{3}{176}\ r^{-8}\right)P_3(\mu) \tag{14}$$

<u>The third approximation</u>.- Substituting from equations (9a), (9b), (11), and (12) into the right-hand side of equation (10c), it follows after a straightforward calculation that

$$
\begin{aligned}
\Delta \phi_2 = (\gamma-1) \Bigg[& \left(-\frac{27}{35} r^{-7} + \frac{9}{4} r^{-10} - \frac{351}{140} r^{-13} + \frac{117}{140} r^{-16}\right) P_1(\mu) \\
& + \left(-\frac{4}{5} r^{-7} + \frac{15}{4} r^{-10} - \frac{57}{20} r^{-13} + \frac{23}{40} r^{-16}\right) P_3(\mu) \\
& + \left(-\frac{10}{7} r^{-7} + \frac{3}{2} r^{-10} - \frac{9}{14} r^{-13} + \frac{5}{56} r^{-16}\right) P_5(\mu) \Bigg] \\
& + \left(-\frac{147}{25} r^{-7} - \frac{1944}{385} r^{-9} + \frac{28323}{1540} r^{-10} + \frac{11178}{1925} r^{-12} - \frac{27441}{1540} r^{-13} + \frac{23367}{6160} r^{-16}\right) P_1(\mu) \\
& + \left(\frac{53}{15} r^{-4} - \frac{244}{75} r^{-7} - \frac{936}{55} r^{-9} + \frac{112}{5} r^{-10} + \frac{3411}{275} r^{-12} - \frac{12631}{660} r^{-13} + \frac{929}{330} r^{-16}\right) P_3(\mu) \\
& + \left(-\frac{10}{3} r^{-4} + \frac{108}{11} r^{-6} - \frac{20}{3} r^{-7} - \frac{720}{77} r^{-9} + \frac{253}{28} r^{-10} + \frac{261}{77} r^{-12} - \frac{997}{231} r^{-13} + \frac{1615}{3696} r^{-16}\right) P_5(\mu)
\end{aligned} \tag{15}
$$

The complete solution of this equation is obtained by means of equation (13) together with the boundary conditions $\frac{\partial \phi_2}{\partial r} = 0$ for both $r = 1$ and $r = \infty$ and is as follows:

$$
\begin{aligned}
\phi_2 = (\gamma-1) \Bigg[& \left(-\frac{3}{70} r^{-5} + \frac{1}{24} r^{-8} - \frac{13}{560} r^{-11} + \frac{13}{2800} r^{-14}\right) P_1(\mu) \\
& + \left(-\frac{1}{10} r^{-5} + \frac{15}{176} r^{-8} - \frac{57}{1960} r^{-11} + \frac{23}{6800} r^{-14}\right) P_3(\mu) \\
& + \left(\frac{1}{7} r^{-5} + \frac{3}{52} r^{-8} - \frac{9}{1120} r^{-11} + \frac{5}{8512} r^{-14}\right) P_5(\mu) \Bigg] \\
& + \left(-\frac{49}{150} r^{-5} - \frac{243}{1925} r^{-7} + \frac{1049}{3080} r^{-8} + \frac{5589}{84700} r^{-10} - \frac{3049}{18480} r^{-11} + \frac{7789}{369600} r^{-14}\right) P_1(\mu) \\
& + \left(-\frac{53}{150} r^{-2} - \frac{61}{150} r^{-5} - \frac{156}{275} r^{-7} + \frac{28}{55} r^{-8} + \frac{1137}{7150} r^{-10} - \frac{12631}{64680} r^{-11} + \frac{929}{56100} r^{-14}\right) P_3(\mu) \\
& + \left(\frac{5}{42} r^{-2} - \frac{6}{11} r^{-4} + \frac{2}{3} r^{-5} - \frac{60}{77} r^{-7} + \frac{253}{728} r^{-8} + \frac{87}{1540} r^{-10} - \frac{997}{18480} r^{-11} + \frac{1615}{561792} r^{-14}\right) P_5(\mu) \\
& + B_1 r^{-2} P_1(\mu) + B_3 r^{-4} P_3(\mu) + B_5 r^{-6} P_5(\mu)
\end{aligned} \tag{16}
$$

where

$$B_1 = 0.03566(\gamma-1) + 0.32614$$

$$B_3 = 0.02268(\gamma-1) + 0.74107$$

$$B_5 = -0.18261(\gamma-1) + 0.21216$$

From equations (11), (14), and (16), it may be easily calculated that the velocity of the fluid at the surface of the sphere is given by

$$-\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{3}{2}\sin\theta + \frac{1}{7040}(989\sin\theta - 1215\sin 3\theta)M^2$$

$$+ (0.10572\sin\theta - 0.16008\sin 3\theta + 0.06434\sin 5\theta)M^4$$

$$+ (\gamma-1)(0.01168\sin\theta - 0.02475\sin 3\theta + 0.02582\sin 5\theta)M^4$$

The appropriate value of γ for air being $\gamma = 1.408$, this equation can be written

$$-\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{3}{2}\sin\theta + \frac{1}{7040}(989\sin\theta - 1215\sin 3\theta)M^2$$

$$+ (0.11048\sin\theta - 0.17018\sin 3\theta + 0.07489\sin 5\theta)M^4 + \ldots \quad (17)$$

<u>The critical value of the Mach number for the sphere</u>.- When the velocity of the fluid equals the velocity of sound at any point in the field, a change in the type of flow occurs and potential flow may no longer exist. It will, therefore, be of interest to determine the value of U/c_0 at which the velocity of the fluid just equals the local velocity of sound in the field of flow past a sphere. This velocity is first attained on the sphere at the point of minimum pressure or of maximum velocity. According to equation (17), with $\theta = \pi/2$, the variation of the maximum velocity with the Mach number $M(= U/c_0)$ is given by

$$v_{max} = 1.5 + 0.31307M^2 + 0.35555\,M^4 + \ldots \quad (18)$$

Now, designating by c^* the velocity of sound at a point where the velocity of the fluid is equal to the local velocity of sound; that is, at a point where

$$v = c = c^*$$

it follows from equation (4) that

$$c^{*2} = \frac{2c_o^2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2} M^2\right) \tag{19}$$

It is to be noted that this equation is essentially Bernoulli's equation and does not depend on the shape of the body immersed in the fluid.

In conformity with the usage in this paper, c^* is replaced by $\frac{c^*}{U}$ $(= v^*)$ and the equation (19) becomes

$$v^{*2} = \frac{2}{\gamma + 1}\frac{1}{M^2} + \frac{\gamma - 1}{\gamma + 1} \tag{20}$$

The so-called critical value of U/c_o $(= M_{cr})$ for the flow past a sphere is then obtained by putting $v_{max} = v^*$. Tables I and II show, respectively, the values of v_{max} and v^* calculated from equations (18) and (20) for various values of the Mach number. These values are also shown in figure 1, and the points at which the curves intersect give the corresponding values of the critical Mach number M_{cr}. The critical values are, respectively, for the second and the third approximations, $M_{cr} = 0.587$ and $M_{cr} = 0.573$. The corresponding critical values of M for the case of an infinitely long circular cylinder are $M_{cr} = 0.420$ and $M_{cr} = 0.409$ (reference 3).

<u>The velocity and the pressure distributions</u>.- If p_o, ρ_o, and c_o are the pressure, the density, and the velocity of sound in the undisturbed stream, then the density ρ of the fluid at a point where the velocity is v is given by

$$\frac{\rho}{\rho_o} = \left[1 + \frac{\gamma - 1}{2} M^2(1 - v^2)\right]^{\frac{1}{\gamma-1}} \tag{21}$$

and the pressure p at this point is

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} = \left[1 + \frac{\gamma - 1}{2} M^2(1 - v^2)\right]^{\frac{\gamma}{\gamma-1}} \tag{22}$$

or

$$\frac{p - p_o}{\frac{1}{2}\rho_o U^2} = \frac{1}{\frac{\gamma}{2}M^2}\left\{\left[1 + \frac{\gamma - 1}{2} M^2(1 - v^2)\right]^{\frac{\gamma}{\gamma-1}} - 1\right\} \tag{23}$$

For an incompressible fluid, $M = 0$ and expression (23) reduces to

$$\frac{p - p_o}{\frac{1}{2}\rho_o U^2} = 1 - v^2 \tag{24a}$$

and, for a compressible fluid with $\gamma = 1.408$ and $M = 0.57$,

$$\frac{p - p_o}{\frac{1}{2}\rho_o U^2} = 4.372\left\{\left[1 + 0.06628(1-v^2)\right]^{3.45} - 1\right\} \tag{24b}$$

The values of v to be used in equations (24) are obtained from equation (17). Thus, for the incompressible fluid

$$v = \frac{3}{2}\sin\theta \tag{25a}$$

and, for the second and the third approximations to the compressible fluid with $M = 0.57$,

$$v = 1.54564 \sin\theta - 0.05607 \sin 3\theta \tag{25b}$$

$$v = 1.55731 \sin\theta - 0.07404 \sin 3\theta + 0.00791 \sin 5\theta \tag{25c}$$

Table III lists the values of v computed from these formulas and table IV gives the corresponding values of $\frac{p - p_o}{\frac{1}{2}\rho_o U^2}$ obtained from equations (24). Figures 2 and 3 show, respectively, the graphs of the velocity and the pressure distributions of tables III and IV.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
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TABLE I

M	v_{max}	
	Second approximation	Third approximation
0	1.5	1.5
.1	1.50031	1.50317
.2	1.51252	1.51309
.3	1.52818	1.53106
.4	1.55009	1.55919
.5	1.57827	1.60049
.6	1.61271	1.65878
.7	1.6534	1.73877

TABLE II

M	0.1	0.2	0.3	0.4	0.5	0.6	0.7
v*	9.1228	4.5753	3.0656	2.3152	1.8686	1.5737	1.3655

TABLE III

θ (deg)	v Incompressible	Second approximation	Third approximation
0	0	0	0
5	.13074	.12021	.11991
10	.26048	.24036	.23946
15	.38823	.36039	.35835
20	.51303	.48008	.47630
30	.75000	.71675	.70857
40	.96419	.94496	.93420
50	1.14906	1.15599	1.14851
60	1.29905	1.33857	1.34183
70	1.40954	1.48046	1.49903
80	1.47722	1.57073	1.60285
85	1.49429	1.59392	1.63005
90	1.50000	1.60172	1.63925

TABLE IV

θ (deg)	$\frac{p - p_0}{\frac{1}{2} \rho_0 U^2}$		
	Incompressible	Compressible	
		Second approximation	Third approximation
0	1.00000	1.08381	1.08381
5	.98291	1.06654	1.06696
10	.93215	1.01620	1.01703
15	.84928	.93301	.93500
20	.73680	.81843	.82287
30	.43750	.50566	.51839
40	.07035	.10801	.12860
50	-.32034	-.32717	-.31098
60	-.68752	-.74203	-.74979
70	-.98679	-1.08069	-1.12579
80	-1.18216	-1.30023	-1.37912
85	-1.23289	-1.35682	-1.44579
90	-1.25000	-1.37604	-1.46833

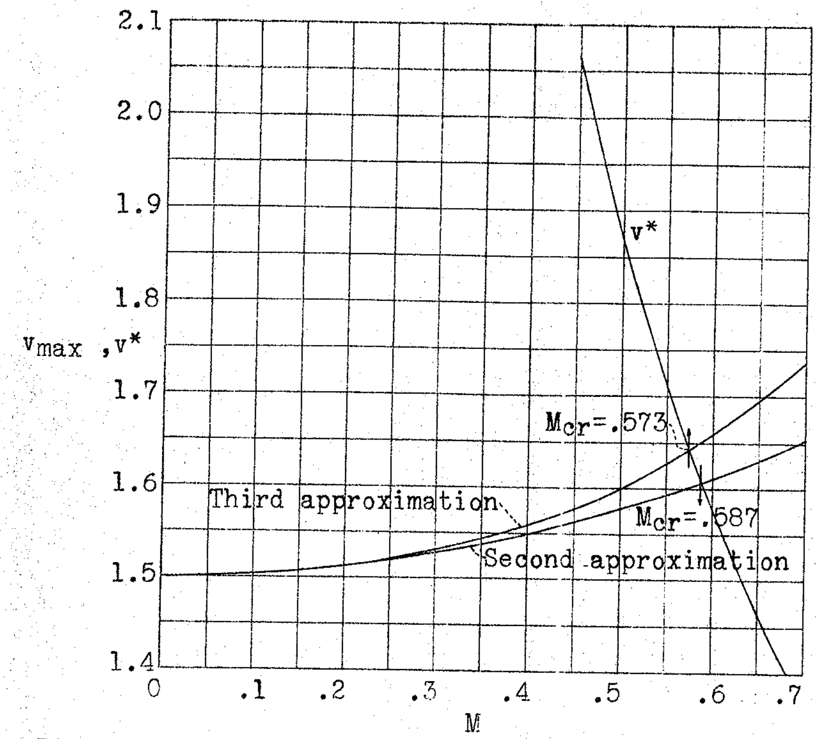

Figure 1.- Critical value of the Mach number for a sphere.

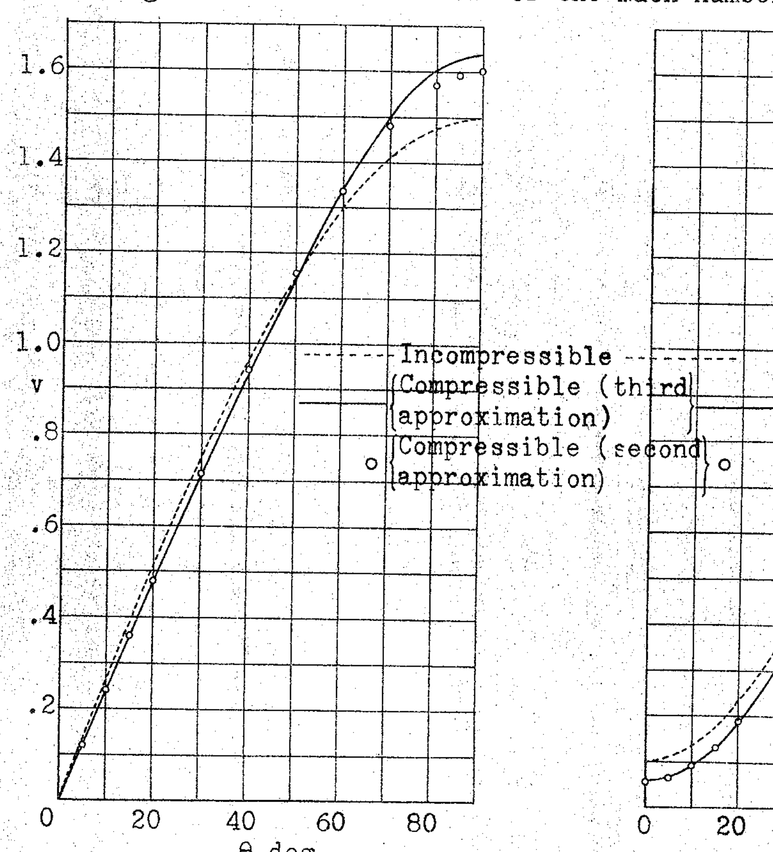

Figure 2.- Velocity distribution on the surface of a sphere.

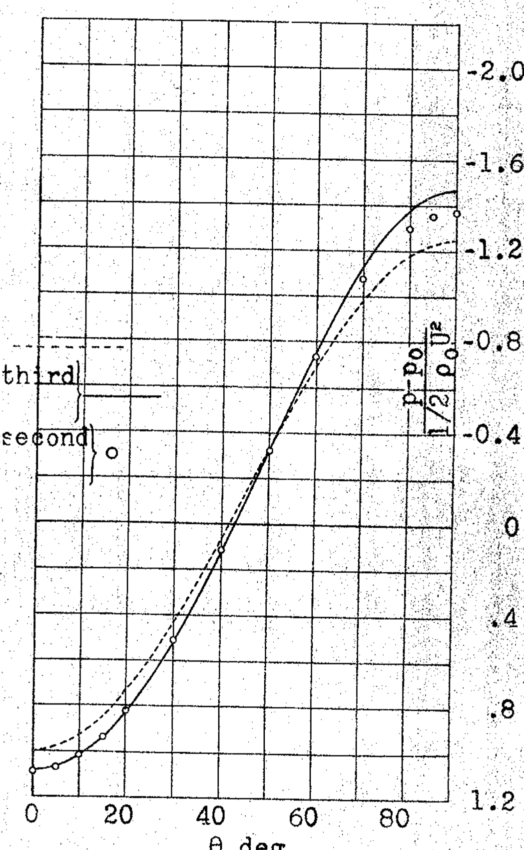

Figure 3.- Pressure distribution on the surface of a sphere.