IN-32-CR

P-38

BANDWIDTH EFFICIENT CODING

FOR

ERROR CONTROL

Technical Report
to
NASA
Goddard Space Flight Center
Greenbelt, Maryland 20771

Grant Number NAG 5-931

Report Number NASA 89-001

Shu Lin
Principal Investigator
Department of Electrical Engineering
University of Hawaii at Manoa
Honolulu, Hawaii 96822

August 1, 1989

BANDWIDTH EFFICIENT CODING FOR ERROR CONTROL

Shu Lin
Department of Electrical Engineering
University of Hawaii at Manoa
Honolulu, Hawaii 96822

ABSTRACT

One of the dramatic developments in bandwidth-efficient communications over the past few years is the introduction and rapid applications of combined coding and bandwidth efficient modulation, known as coded modulation, for error control. Using coded modulation, reliable data transmission can be attained without compromising bandwidth efficiency. In this report, we present the basic concepts of coded modulation. We use two simple examples to demonstrate how significant coding gains can be achieved without bandwidth expansion. This report serves as an introduction to subsequent technical reports on coded modulation.

1. Introduction

Conventionally in digital communications, channel coding and modulation are designed and performed separately. Error control is achieved by adding structured redundant bits (or parity check bits) to each transmitted message. Transmission of these redundant bits results in either a bandwidth expansion or reduction of data rate. This type of channel coding is suitable for power limited channels without bandwidth constraint, such as deep space and wideband satellite channels. Coding gain is achieved at the expense of bandwidth expansion. However, for many channels, such as voiceband telephone, terrestrial microwave, mobile and some satellite channels, bandwidth is a very precious resource, and hence bandwidth expansion is not desirable or even not possible. For this reason, conventional channel coding is rarely used for error control in these bandlimited channels.

An obvious approach to conserve bandwidth is to enlarge the signal set of the modulation. However, if coding and modulation are still designed and performed independently, this approach gives very disappointing results. For example, we compare the performance of an uncoded QPSK modulation system with that of an 8-PSK modulation system using a rate-2/3 convolutional code of constraint length $\nu = 6$ and free distance $d_{\text{free}} = 7$ which is decoded with a Viterbi decoder. Suppose the uncoded QPSK system has an output bit-error-rate (BER) of 10^{-5} at some signal-to-noise ratio (SNR). At the same SNR, the raw BER of an 8-PSK modulation system is 10^{-2} because the Euclidean distance between signal points becomes smaller. When we use the rate-2/3 convolutional code with Viterbi decoder in conjunction with this 8-PSK modulation, the resultant output BER of the overall coded system is 10^{-5}. After all the effort expended in coding and in building a rather complex 64-state Viterbi decoder, the error performance of 8-PSK modulation only breaks even with that of an uncoded QPSK system. This example simply demonstrates that, if we want to achieve coding gain while preserving bandwidth, coding and modulation should not be performed independently. This simple fact motivated communication system designers to combine coding and modulation as an integral part to achieve coding gain without sacrificing bandwidth. In 1982, Ungerboek [1] showed that it is indeed possible to design a system by combining coding and modulation to attain signifficant coding gain without bandwidth expansion. His work has stirred a rapid research and development in this area of combining coding and modulation which is now known as coded modulation.

The basic concept of coded modulation is to encode information symbols onto an **expanded** channel signal set (relative to that needed for uncoded modulation). The channel signal set expansion provides the needed redundancy for error control without increasing the bandwidth requirements, while coding is used to produce a certain **interdependency** between succesive channel signals, such that only certain sequences of channel signals are permitted. Using properly designed coded modulation, reliable data transmission can be achieved without compromising the bandwith efficiency. Coded modulation can be classified based on either the structure of the code being used or the type of modulation constellation being used. Based on the code structure, coded modulation can be classified into two categories:

(1) **trellis coded modulation (TCM)** in which convolutional (or trellis) codes are used, and

(2) **block coded modulation (BCM)** in which block codes are used.

Based on the modulation signaling, coded modulations are divided into two types:

(1) **constant-envepole-type** coded modulation in which the modulation signals have constant envelope, e.g., MPSK, and

(2) **lattice-type** coded modulation in which the signal constellation has lattice structure, e.g., QASK.

In this report, we present the basic concepts of coded modulation. We use two simple examples to illustrate these concepts and show how coding gain can be achieved without bandwidth expansion. This report serves as an introduction to subsequent technical reports on coded modulation.

2. Modulation Codes

In a coded modulation system, information sequences are coded into signal sequences over a certain modulation signal set (e.g., the 8-PSK signal set). These signal sequences form a modulation code. In order to achieve good error performance, modulation codes are generally decoded with a soft-decision decoding algorithm. For an additive white Gaussian noise (AWGN) channel, the error performance (or coding gain) of a modulation code is largely determined by its **minimum squared Euclidean distance.**

Let $(X(s), Y(s))$ denote a point s in a two dimensional Euclidean space R^2, where $X(s)$ and $Y(s)$ are the $x-$ and $y-$ coordinates of s. The squared Euclidean distance between two points, s and s', in R^2, denoted $d(s, s')$, is defined as follows:

$$d(s,s') \triangleq (X(s) - X(s'))^2 + (Y(s) - Y(s'))^2. \tag{2.1}$$

Let $\overline{v} = (s_1, s_2, \cdots, s_n)$ and $\overline{v}' = (s_1', s_2', \cdots, s_n')$ be two n-tuples over R^2. The squared Euclidean distance between $\overline{v}$ and $\overline{v}'$, denoted $|\overline{v} - \overline{v}'|^2$, is defined as follows:

$$|\overline{v} - \overline{v}'|^2 \triangleq \sum_{j=1}^{n} (X(s_j) - X(s_j'))^2 + (Y(s_j) - Y(s_j'))^2. \tag{2.2}$$

Consider a modulation code C with signals from a two-dimensional modulation signal set S. The **minimum squared Euclidean distance** of C, denoted $D[C]$, is defined as follows:

$$D[C] \triangleq \min \{|\overline{v} - \overline{v}'|^2 \ : \ \overline{v}, \overline{v}' \in C \text{ and } \overline{v} \neq \overline{v}'\}. \tag{2.3}$$

For a trellis modulation code, $D[C]$ becomes the squared free Euclidean distance, denoted d^2_{free}. Suppose k information bits are encoded into a sequence of n signals based on C. The effective rate of C, denoted $R[C]$, is defined as

$$R[C] \triangleq \frac{k}{2n} \tag{2.4}$$

which is simply the average number of information bits transmitted by C per dimension. For $R[C] \geq 1$, coding is achieved without compromising the bandwidth efficiency or reducing the data rate. With a properly designed modulation code (block or trellis), it is possible to attain significant coding gain without bandwidth expansion. The effectiveness of a modulation code is measured by its effective rate, minimum squared Euclidean distance and the ease of its decoding implementation. Just like in the design of conventional coding, for a given effective rate $R[C]$ and code length n (or constraint length for TCM), we want to design a modulation code with a minimum squared Euclidean distance as large as possible and a structure which makes maximum likelihood decoding practical.

Suppose a modulation code C is used for an AWGN channel and all the code sequences (or codewords) in C are equally likely to be transmitted. Let $\overline{r} = (x_1, y_1, x_2, y_2, \cdots)$ be the output sequence of the demodulator at the receiving end, where x_i and y_i are the two ouputs of the demodulator corresponding to the i-th received signal. The sequence $\overline{r}$ is called the received sequence. Note that x_i and y_i are two real numbers. The squared Euclidean distance between $\overline{r}$ and a code sequence $\overline{v} = (s_1, s_2, \cdots, s_j, \cdots)$ in C is given by,

$$|\overline{r} - \overline{v}|^2 = \sum_j (x_j - X(s_j))^2 + (y_j - Y(s_j))^2. \tag{2.5}$$

For maximum likelihood decoding (MLD), the received sequence $\overline{r}$ is decoded into the code sequence $\overline{v}_\ell$ for which the following condition holds,

$$|\overline{r} - \overline{v}_\ell| \leq |\overline{r} - \overline{v}_i|, \tag{2.6}$$

for $i \neq \ell$. To perform a soft-decision MLD, it is desirable for a modulation code to have a trellis structure so that the Viterbi decoding algorithm can be applied.

3. Signal Set Partition, Labeling and Distance Structure

The design of a coded modulation system depends on the choice of the modulation signal set, signal set partition, signal labeling, choice of coding and code bits-to-signal mapping. In this section, we present the concepts of signal set partitioning, signal labeling, and distance properties among the signal points in a set of a partition and between the sets of a partition.

In the following, we use an example to illustrate the signal set partitioning process. Consider the 8-PSK signal set (or constellation) as shown in Figure 1, where each signal point is labeled with an integer i from the set $S = \{0,1,2,3,4,5,6,7\}$. Each integer $i \in S$ can be represented in the following binary form:

$$i = b_0 + b_1 \cdot 2 + b_2 \cdot 2^2$$

with $b_j = 0$ or 1 for $0 \leq j \leq 2$. The binary sequence (3- tuple), (b_0, b_1, b_2), is the binary representation of the integer i, where b_0 is the least significant bit and b_2 is the most significant bit. The binary representations for the eight integers in S are:

$$\begin{array}{ll} 0 \Longleftrightarrow (000) & 1 \Longleftrightarrow (100) \\ 2 \Longleftrightarrow (010) & 3 \Longleftrightarrow (110) \\ 4 \Longleftrightarrow (001) & 5 \Longleftrightarrow (101) \\ 6 \Longleftrightarrow (011) & 7 \Longleftrightarrow (111) \end{array}$$

The squared Euclidean distance between two signal points, s and s', in the 8-PSK signal set S is given by:

$$d(s, s') \triangleq 4\sin^2\left(\frac{(s-s')\pi}{8}\right). \tag{3.1}$$

From (3.1), we find that

$$d(1,0) = d(2,1) = 0.586$$
$$d(2,0) = d(5,3) = 2$$
$$d(4,0) = d(5,1) = 4.$$

The minimum and maximum squared Euclidean distances between signal points in the 8-PSK signal set S are 0.586 and 4 respectively.

Let X be a subset of the signal set S. Define the minimum squared Euclidean distance of X as follows:

$$d[X] \triangleq \min\{d(x,x') \ : \ x,x' \in X\}. \tag{3.2}$$

where $d(x,x')$ is given by (3.1). This distance is called the **intra-set distance** of S. The minimum squared Euclidean distance between X and Y, denoted $d(X,Y)$, is defined as follows:

$$d[X,Y] \triangleq \min\{d(x,y) \ : \ x \in X \text{ and } y \in Y\}.$$

This distance is a measure of the **separation** of two sets of signals.

Now we form a chain of partitions of the signal set S with increasing number of subsets, increasing intra-set distances and set separations as shown in Figure 2. First we partition S into two subsets, $S_0 = \{0,2,4,6\}$ and $S_1 = \{1,3,5,7\}$ where, for $k = 0$ or 1, S_k consists of those integers in S with $b_0 = k$ in their binary representations. The intra-set distance of S is $d[S] = 0.586$. The first partition of S produces two subsets S_0 and S_1, both subsets having intra-set distances equal to 2, i.e., $d[S_0] = d[S_1] = 2$. We see that there is an increase in intra-set distance from the signal set S to the first partition $\{S_0, S_1\}$. The separation between S_0 and S_1 is $d[S_0, S_1] = 0.586$. From Figures 1 and 3,we see that S_0 and S_1 are actually two QPSK signal sets.

At the second step of partitioning, each subset S_k in the first partition with $k = 0$ or 1 is partitioned into two subsets, S_{k0} and S_{k1}, where for $j = 0$ or 1, S_{kj} consists of those integers in S with $b_0 = k$ and $b_1 = j$ in their binary representations. The partitioning process results in the following subsets:

$$S_{00} = \{0, 4\} \qquad S_{01} = \{2, 6\}$$
$$S_{10} = \{1, 5\} \qquad S_{11} = \{3, 7\}$$

The intra-set distance of each of the above 4 subsets is 4, i.e., $d[S_{kj}] = 4$ for $0 \leq k \leq 1$ and $0 \leq j \leq 1$. The separations between these subsets are:

$$\begin{aligned} d(S_{00}, S_{01}) &= d(S_{10}, S_{11}) = 2 \\ d(S_{00}, S_{10}) &= d(S_{00}, S_{11}) = d(S_{10}, S_{01}) \\ &= d(S_{01}, S_{11}) = 0.586 \end{aligned}$$

We see that the second partitioning of the signal set results in an increase of the intra-set distance from 2 to 4. Furthermore, we see from Figure 3 that each subset S_{kj} in the second partition is a BPSK signal set. Two BPSK signal sets which are contained in the same QPSK signal set are separated by a squared Euclidean distance of 2. Two BPSK signal sets from two different QPSK signal sets are separated by a squared Euclidean distance at least 0.586.

At the last step of partitioning, we partition each subset S_{kj} in the second partition into two subsets, S_{kj0} and S_{kj1}, where S_{kji} consists of only one integer whose binary representation is $(b_0, b_1, b_2) = (k, j, i)$. Hence,

$$S_{000} = \{0\},\ S_{001} = \{4\},\ S_{010} = \{2\},\ S_{011} = \{6\},$$
$$S_{100} = \{1\},\ S_{101} = \{5\},\ S_{110} = \{3\},\ S_{111} = \{7\}.$$

Since each subset in this last partition contains only one signal point, its intra-set distance is regarded as **infinity**. The set separations are:

$$d(S_{kj0}, S_{kj1}) = 4,$$
$$d(S_{k0i}, S_{k1i}) = 2,$$
$$d(S_{0ji}, S_{1ji}) = 0.586.$$

The above partitioning process is called **binary partition** which results in 3 partitions with increasing intra-set distances, as shown in Figure 4,

$$d[S] = 0.586, \quad d[S_k] = 2, \quad d[S_{kj}] = 4, \quad d[S_{kji}] = \infty.$$

At each partitioning stage, a subset is labeled by a binary sequence, and at the end of the partitioning process, each signal point is labeled by a binary sequence (in this case, the labeling sequence is also the binary representation of the integer which represents the signal point). Two signal points with labels different at the first position (i.e., b_0 position) are at a squared Euclidean distance at least 0.586 apart. Two signal points with labels identical at the first position but different at the second position are separated by a squared Euclidean distance at least 2 apart. Two signal points with labels identical at the first two positions but different at the last position are at a squared Euclidean distance 4 apart. For convenience, we denote the partition chain with $S/S_0/S_{00}/S_{000}$.

As another example, we consider the 16-QASK signal constellation S as shown in Figure 4, where each signal point is represented by an integer and labeled by the binary representation of the integer. Let d_1 denote the intra-set distance of S, i.e., $d_1 = d[S]$. The partition chain is shown in Figure 5. We see that the intra-set distance increases form d_1 to $8d_1$.

Signal set partitioning is used to map encoded bits onto signal set so that the channel signal sequences have largest possible minimum squared Euclidean distance.

4. Trellis Coded Modulation (TCM)

In this section, we use a simple example to illustrate the concepts and process of TCM. The code to be used is a rate-1/2 convolutional code with a 4-state trellis diagram, and the modulation signal set is the 8-PSK as shown in Figure 1. The schematic diagram of the coded modulation system is shown in Figure 6 [2].

At each unit of time, two information bits are mapped into an 8-PSK signal through convolutional encoding. At time l, the input information bits are $c_l^{(2)}$ and $c_l^{(1)}$, and the corresponding output coded bits are $v_l^{(2)}$, $v_l^{(1)}$ and $v_l^{(0)}$. The information bit $c_l^{(2)}$ is uncoded and $v_l^{(2)} = c_l^{(2)}$. The information bit $c_l^{(1)}$ is encoded into two bits, $v_l^{(1)}$ and $v_l^{(0)}$, based on a rate-1/2 convolutional code of memory $m = 2$. The parity equations are:

$$\begin{aligned} v_l^{(1)} &= c_l^{(1)}, \\ v_l^{(0)} &= c_{l-1}^{(1)} + v_{l-2}^{(0)}. \end{aligned}$$

The code has a 4-state trellis diagram as shown in Figure 7. The free (Hamming) distance of the code is 3. When two paths diverge at a certain node, it takes at least three branches for them to remerge. A branch coming out from an l-th order node is labeled by an output pair $(v_l^{(0)}, v_l^{(1)})$ corresponding to an input bit $c_l^{(1)}$.

Now we need to include the uncoded information bit $c_l^{(2)}$ in the trellis diagram to make the overall code a rate-2/3 trellis code. This can be achieved by splitting each branch in the trellis into two parallel branches, one corresponding to $c_l^{(2)} = 0$ and the other corresponding to $c_l^{(2)} = 1$. The resultant trellis is shown in Figure 8. Now each branch is labeled with a triplet, $(v_l^{(0)}, v_l^{(1)}, v_l^{(2)})$, with $v_l^{(2)} = c_l^{(2)}$ and $v_l^{(1)} = c_l^{(1)}$. Two parallel branches are labeled with $(v_l^{(0)}, v_l^{(1)}, 0)$ and $(v_l^{(0)}, v_l^{(1)}, 1)$. The new trellis has the following structure:

(1) There are two parallel branches connecting two adjacent nodes (or states);

(2) There are two groups of two parallel branches diverging from each node;

(3) There are two groups of two parallel branches merging into a node.

Every path in the trellis corresponds to a binary code sequence.

Now we need to map each triplet $(v_l^{(0)}, v_l^{(1)}, v_l^{(2)})$ on a branch into an 8-PSK signal. The mapping should be done in such a way that the resultant signal sequences in the trellis have largest possible minimum squared Euclidean distance (or squared free Euclidean distance). The mapping is done based on the set partitioning. Obviously, the two triplets on two parallel branches should be mapped into two signal points in a set S_{kj} with the largest intra-set distance. For this case, $d[S_{kj}] = 4$. For example, $S_{00} = \{0, 4\}$. The two groups of two parallel branches diverging from or merging into a code should be assigned to signal points from two subsets, S_{k0} and S_{k1}, with largest set separation, $d[S_{k0}, S_{k1}]$. In this case, $d[S_{k0}, S_{k1}] = 2$. For example, $S_{00} = \{0, 4\}$ and $S_{01} = \{2, 6\}$.

Using the above mapping rule, we obtain a 4-state trellis with signal symbols from the 8-PSK signal set as shown in Figure 9. Note that mapping is done simply by using the binary representation of each signal point, i.e., the triplet $(v_l^{(0)}, v_l^{(1)}, v_l^{(2)})$ is simply the binary representation of a signal point. For two paths in the trellis which diverge from the same node through two non-parallel branches, it will take at least 3 branches before they can remerge into a later node in the trellis. Since the free Hamming distance of the rate-

1/2 convolutional code being used is 3, it guarantees that two diverging and remerging paths must be separated by a squared Euclidean distance at least

$$d(S_{k0}, S_{k1}) + d(S) + d(S_{k0}, S_{k1}) = 2 + 0.586 + 2 = 4.586.$$

Two paths diverging and remerging through two parallel branches are separated by a squared Euclidean distance of 4. As a result, two signal sequences in the trellis are separated by a squared Euclidean distance at least 4. Hence the squared free distance d^2_{free} is 4.

In the above coded modulation system, every two information bits are encoded into a signal point in a two-dimensional 8-PSK signal set. Hence the information rate per dimension is 1 and the effective rate $R[C]$ is 1.

Now we evaluate the coding gain of the above trellis coded 8-PSK system over an uncoded reference modulation system. Consider a QPSK system without coding. Every two information bits are encoded into a signal point in a 2-dimensional QPSK signal set as shown in Figure 10. The effective rate of this system is also 1 bit per dimension. Two QPSK signal sequences may only differ in one place with two symbols separated by the smallest squared Euclidean distance, 2. Therefore, the minimum squared Euclidean distance of an uncoded QPSK system is 2. The coded 8-PSK system described above also has effective rate 1 but minimum squared Euclidean distance 4. Both systems have the same signaling rate. Hence, for the same SNR, the coded 8-PSK modulation system provides smaller output BER than the uncoded QPSK without bandwidth expansion or reducing data rate. For the same output BER, the coded 8-PSK modulation system requires less SNR than the uncoded QPSK system. This reduction, usually expressed in decibels, in the required SNR to achieve a specified BER of the coded system over an uncoded system is called the **coding gain**.

The coding gain is related to the ratio of the minimum squared Euclidean distance of a coded system to the minimum squared Euclidean distance of a reference system. The asymptotic coding gain of a coded system over an uncoded reference system is given by [1,2]

$$\gamma = 10 \log_{10} \frac{(\text{MSED})_{\text{coded system}}}{(\text{MSED})_{\text{reference system}}} \tag{4.1}$$

where MSED stands for minimum squared Euclidean distance.

The asymptotic gain of the above coded 8-PSK system over the uncoded QPSK system is

$$\gamma = 10 \log_{10} \frac{4}{2} = 3 \text{ dB}.$$

The real coding gain at 10^{-5} BER is 2.6 dB (see Figure 11). Note that this coding gain is achieved without bandwidth expansion. We see that simple coded 8-PSK modulation system achieves quite significant coding gain with a little additional decoding complexity. Due to the trellis structure, the simple coded 8-PSK modulation code can be decoded with a 4-state Viterbi decoder using a soft-decision MLD algorithm.

Coded modulation is based on the signal constellation, signal set partitioning and labeling, code selection and bits-to-signal mapping. The TCM illustrated in the above example can be applied to other types of modulation (e.g., 16-PSK, 16-QASK) and convolutional codes of higher rates and larger number of states. A general TCM system is shown in Figure 12. Larger coding gains can be achieved with larger number of trellis states. Roughly speaking, it is possible to gain 3 dB with 4 states, 4 dB with 8 states, nearly 5 dB with 16 states and up to 6 dB with 128 or more states.

5. Block Coded Modulation (BCM)

Again we use an example to illustrate the design process of BCM. The example given here is a block coded 8-PSK modulation system. The two-dimensional 8-PSK signal set is shown in Figure 1, where each signal point is represented by an integer in $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

Like the design of a TCM system, the design of a BCM system is also based on code selection, set partitioning and bits-to-signal mapping. In our example, we choose three binary block codes of length $n = 8$ as follows: (1) C_{b1} is the $(8, 1)$ repetition code which consists of the all-zero and all-one vectors,and hence its minimum Hamming distance is $\delta_1 = 8$; (2) C_{b2} is the $(8, 7)$ code with all the even-weight vectors and has minimum Hamming distance $\delta_2 = 2$; and (3) C_{b3} is the $(8, 8)$ code which consists of all the binary 8-tuples and has minimum Hamming distance $\delta_3 = 1$.

Consider the following three subsets of S:

$$B_1 = \{0, 1\}, \quad B_2 = \{0, 2\} \quad \text{and} \quad B_3 = \{0, 4\}.$$

These subsets have increasing intra-set distances,

$$d[B_1] = 0.586, \quad d[B_2] = 2, \quad d[B_3] = 4.$$

Note that

$$S = \{b_3 + b_2 + b_1 \ : \ b_i \in B_i \quad \text{for} \quad 1 \leq i \leq 3\}.$$

From the binary code C_{bi} we construct a block code C_i with symbols from B_i as follows: every 1-component in a codeword $\overline{v}$ in C_{bi} is repleaced by the symbol b_i. Denote C_i with b_iC_{bi}. Then the minimum squared Euclidean distance of C_i is $D[C_i] = \delta_i d[B_i]$ for $1 \leq i \leq 3$. The above transformation results in three codes,

$$C_1 = C_{b1}, \quad C_2 = 2C_{b2}, \quad C_3 = 4C_{B3},$$

with minimum squared Euclidean distances,

$$D[C_1] = 8 \times 0.586 = 4.688, \quad D[C_2] = 2 \times 2 = 4, \quad \text{and} \quad D[C_3] = 4.$$

respectively. Now take the direct-sum of C_1, C_2 and C_3 as follows:

$$C = \{\overline{v}_1 + \overline{v}_2 + \overline{v}_3 : \overline{v}_i \in C_i \quad \text{for } 1 \leq i \leq 3\}. \tag{5.1}$$

C is then a block code of length $n = 8$ with symbols from $S = \{0,1,2,3,4,5,6,7\}$. If each code symbol is mapped to its corresponding signal point in the 8-PSK signal set as shown in Figure 1, we then obtain a block 8-PSK modulation code of length 8. Note that each codeword in this 8-PSK modulation code is a sequence of eight 8-PSK signals.

From the construction process, we see that a total of $16 = 1 + 7 + 8$ information bits are encoded into a sequence of eight 8-PSK signals. Hence, on average, each signal carries two information bits and the effecitve rate is $R[D] = 1$. The overall coded modulation system is shown in Figure 13. The minimum squared Euclidean distance of this block 8-PSK modulation code is

$$\begin{aligned} D[C] &= \min\{D[C_1], D[C_2], d[C_3]\} \\ &= \min\{\delta_1 d[B_1], \delta_2 d[B_2], \delta_3 d[B_3]\} \\ &= \min\{4.688, 4, 4\} = 4 \end{aligned} \tag{5.2}$$

Now we consider the coding gain of the above block 8-PSK modulation code over an uncoded reference modulation system. Suppose an uncoded QPSK system is used. Then 16 information bits are encoded into a sequence of 8 QPSK signals. For this uncoded system, the effective rate is $R[C]_{\text{ref}} = 1$ and the minimum squared Euclidean distance is $D[C]_{\text{ref}} = 2$. Hence the simple block coded 8-PSK modulation system achieves a

$$\gamma = 10 \log_{10} \frac{4}{2} = 3 \text{ dB}$$

asymptotic coding gain over the uncoded QPSK system without bandwidth expansion. The real coding gain is shown in Figure 14. We see that, at 10^{-6} decoded block-error-rate, the block coded 8-PSK system achieves a 2.42 dB coding gain over the uncoded QPSK.

Next we consider the decoding of our example code. Note that each binary component code has a trellis structure as shown in Figure 15. Then the block 8-PSK modulation code has a 4-state 8-section trellis diagram as shown in Figures 16 and 17 which is the direct product of three trellis diagrams of the component codes shown in Figure 15. Hence the code can be decoded with a 4-state Viterbi decoder using a soft-decision MLD algorithm. From Figure 17, we see that the trellis consists of two identical parallel 2-state trellis sub-diagrams without cross connections between them. This structure suggests that the decoding can be done with two 2-state Viterbi decoders to process the two trellis sub-diagrams in parallel. This not only speeds up the decoding process but also simplifies the implementation. Furthermore, this block 8-PSK modulation code is phase invariant under multiples of 45° rotation. That is, if there is a 45° phase rotation, a codeword becomes another codeword in the code. This phase invariant property is useful in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after temporary loss of synchronization.

The block 8-PSK modulation code given above is an equivalent to the trellis 8-PSK modulation code given in Section 4. Except the code structure, they are similar in many ways. The block 8-PSK modulation code has advantages in decoding complexity and decoding speed. The trellis 8-PSK modulation code is phase-invariant only under 180° rotation, however it has slightly larger coding gain (less than 0.2 dB) than the block 8-PSK modulation code. The method used in the construction of the example block 8-PSK modulation code can be used to construct any type of modulation codes (MPSK or QASK modulation codes). This method is known as the multi-level construction [3-6]. The resultant code has trellis structure if the component codes have trellis structure.

6. Conclusion

In this report, we have introduced the basic concepts of coded modulation. We used two simple coded modulation systems to demonstrate that coding gain can be attained without compromising the bandwidth efficiency (or bandwidth expansion).

In our next two reports, we will present the multi-level construction of modulation codes. We will show that using the multi-level method, modulation codes with arbitrary large minimum squared Euclidean distance can be constructed.

References

[1] G. Ungerboek, "Channel Coding with Multilevel/Phase Signals," *IEEE Trans. on Information Theory*, Vol.IT-28, No.1, pp.55-67, January 1982.

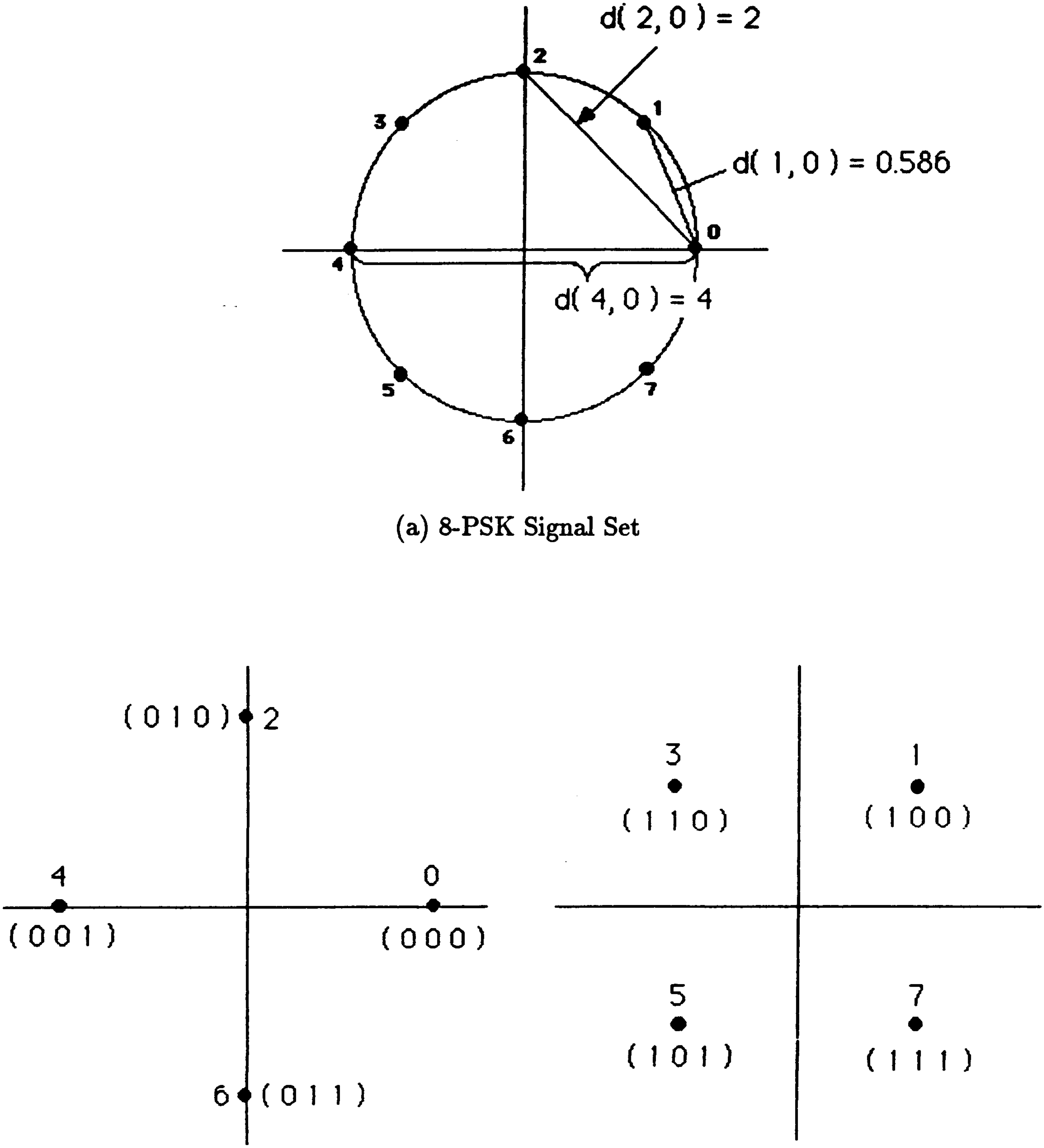
[2] G. Ungerboek, "Trellis-Coded Modulation with Redundant Signal Sets, Part I: Introduction," *IEEE Communications Magazine*, Vol.25, No.2, pp.5-11, February 1987.

[3] H. Imai and S. Hirakawa, "A New Multilevel Coding Method Using Error Correcting Codes," *IEEE Trans. on Information Theory*, Vol.IT-23, No.3, pp.371-376, May 1977.

[4] V.V. Ginzburg, "Multidimensional Signals for a Continuous Channel," *Problem Peredachi Informatsii*, Vol.20, No.1, pp.28-46, 1984.

[5] R.M. Tanner, "Algebraic Construction of Large Euclidean Distance Combined Coding Modulation Systems," *Abstract of Papers, 1986 IEEE International Symposium on Information Theory*, Ann Harbor, October 6-9, 1986, also *IEEE Trans. on Information Theory* to be published.

[6] T. Kasami, T. Takata, T. Fujiwara and S. Lin, "A Concatenated Coded Modulation Scheme for Error Control," *IEEE Trans. on Communications*, to be published.



(a) 8-PSK Signal Set

(b) QPSK

(c) QPSK

Figure 1

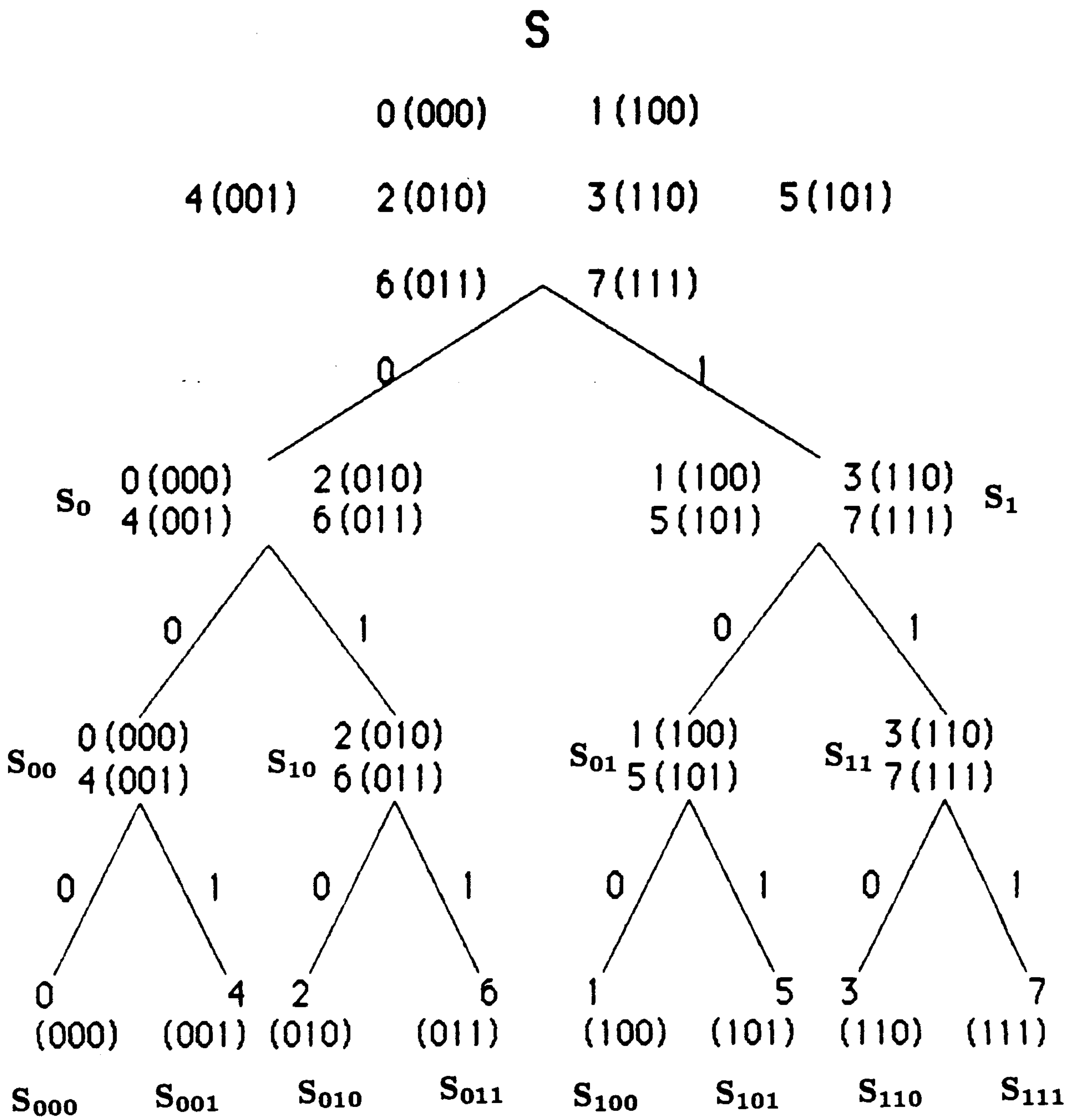


Figure 2. Partition chain of the 8-PSK signal set

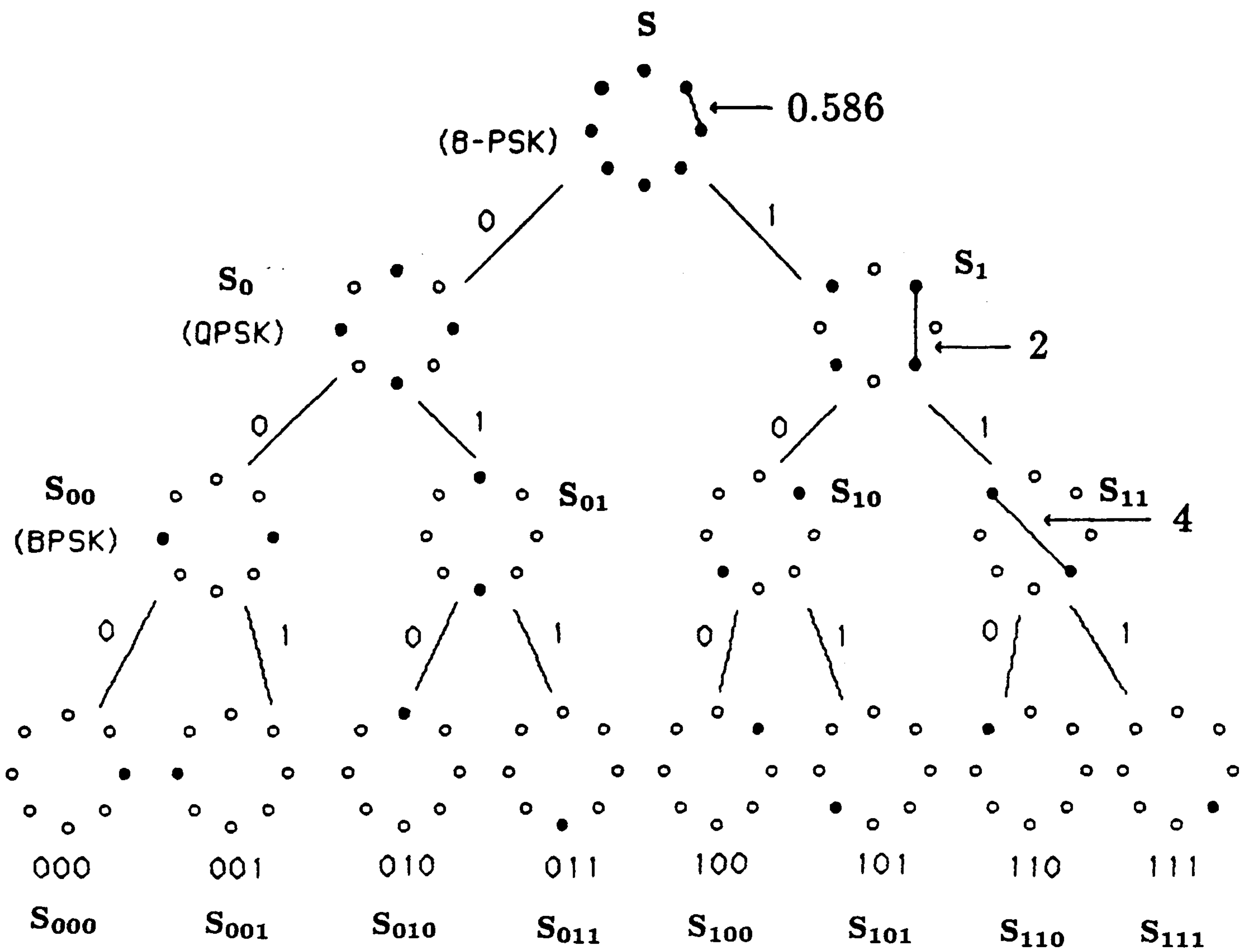


Figure 3. 8-PSK/QPSK/BPSK signal constellation chain

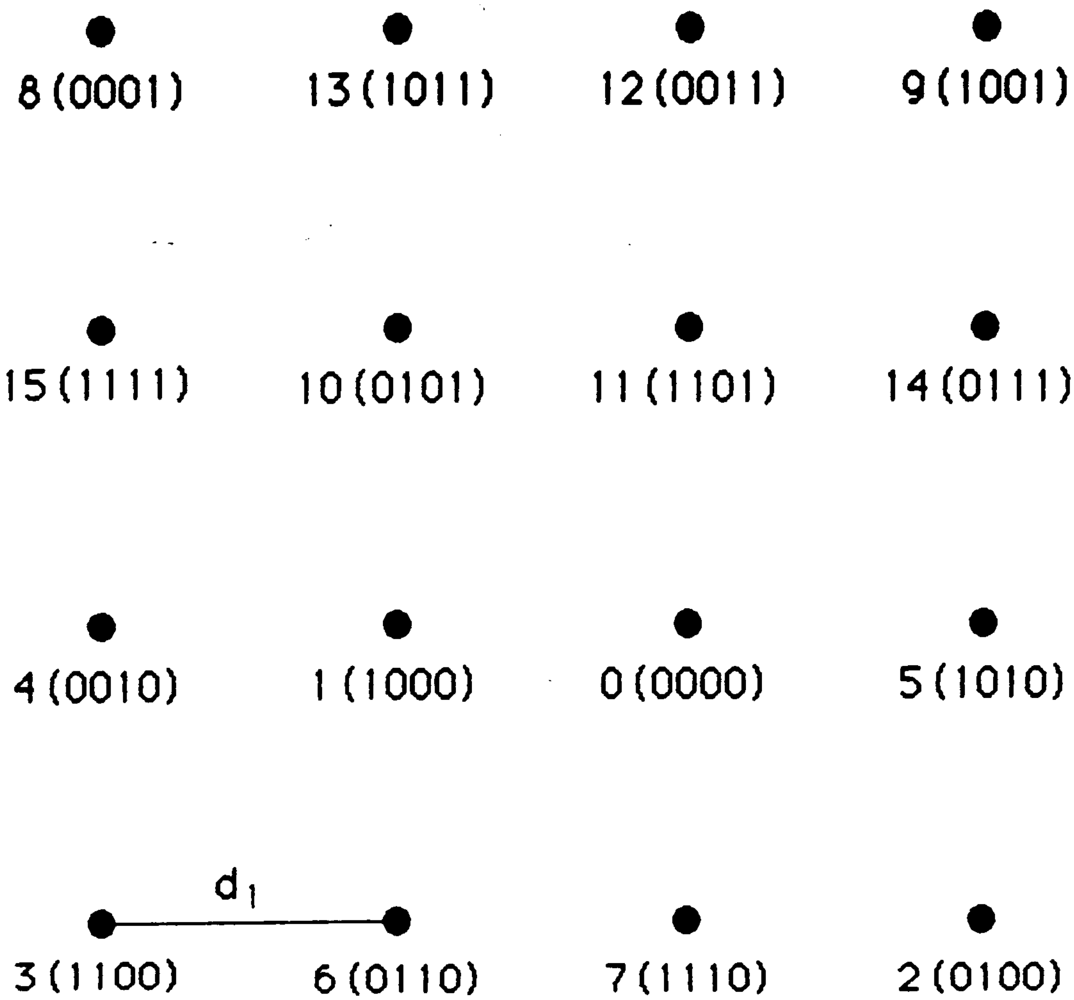


Figure 4. A 16-QASK signal set

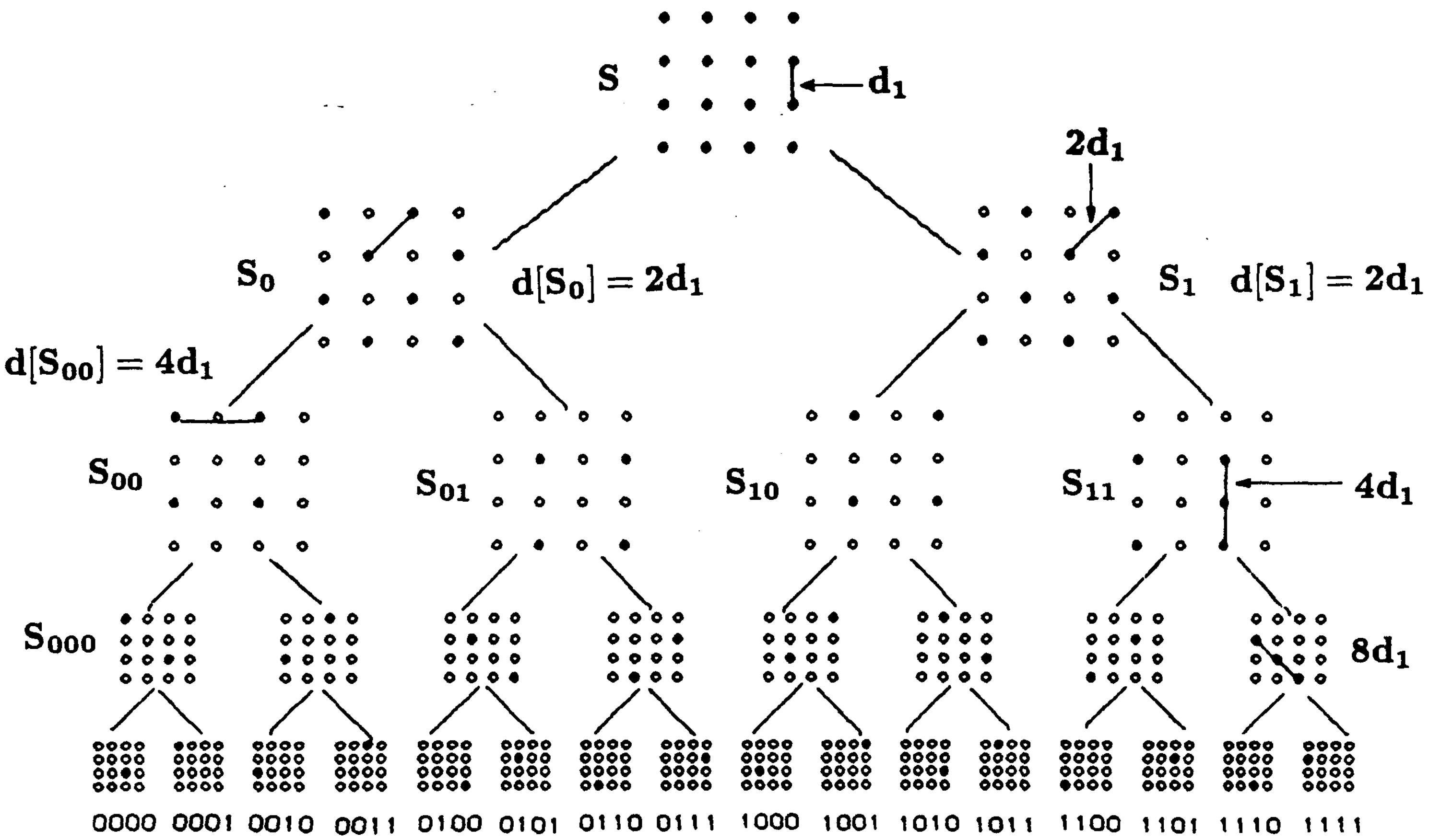


Figure 5. Partition chain of the 16-QASK signal set

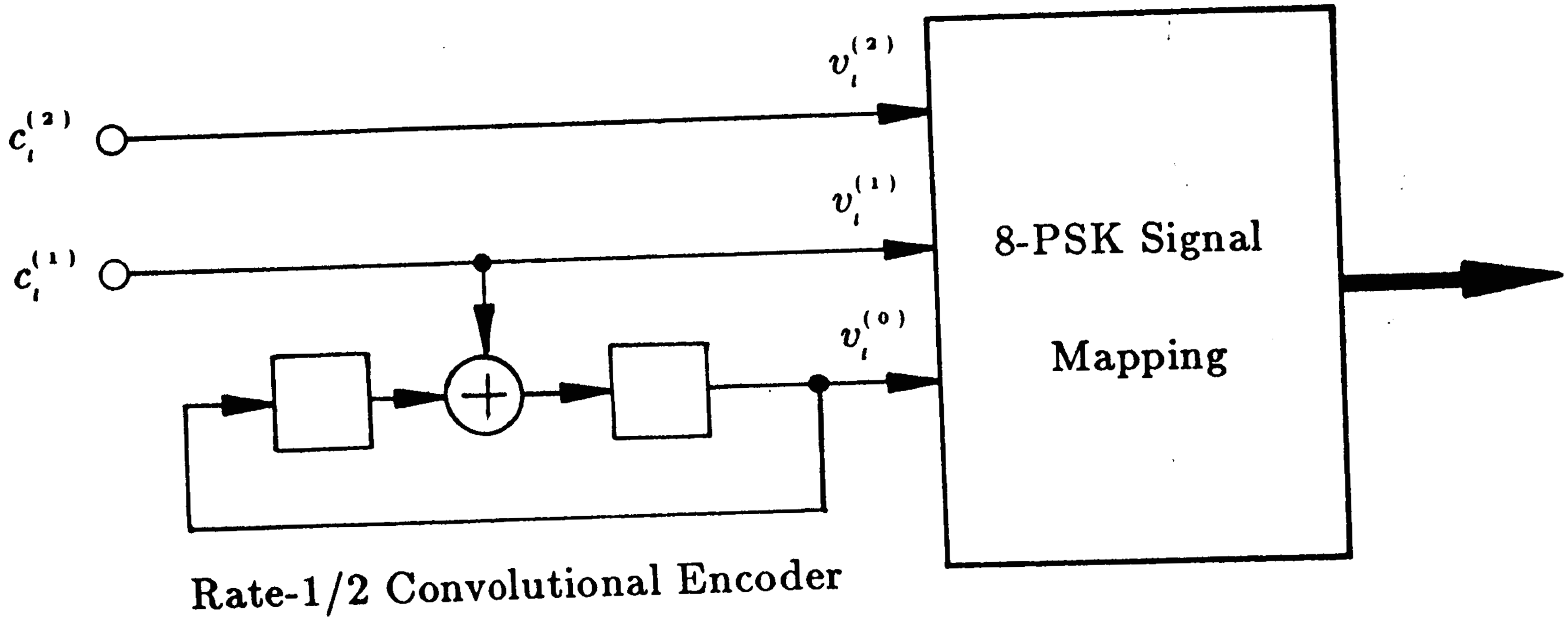


Figure 6. A 4-state trellis coded 8-PSK modulation system

$$v_l^{(0)} = c_{l-1}^{(1)} + v_{l-2}^{(0)}.$$

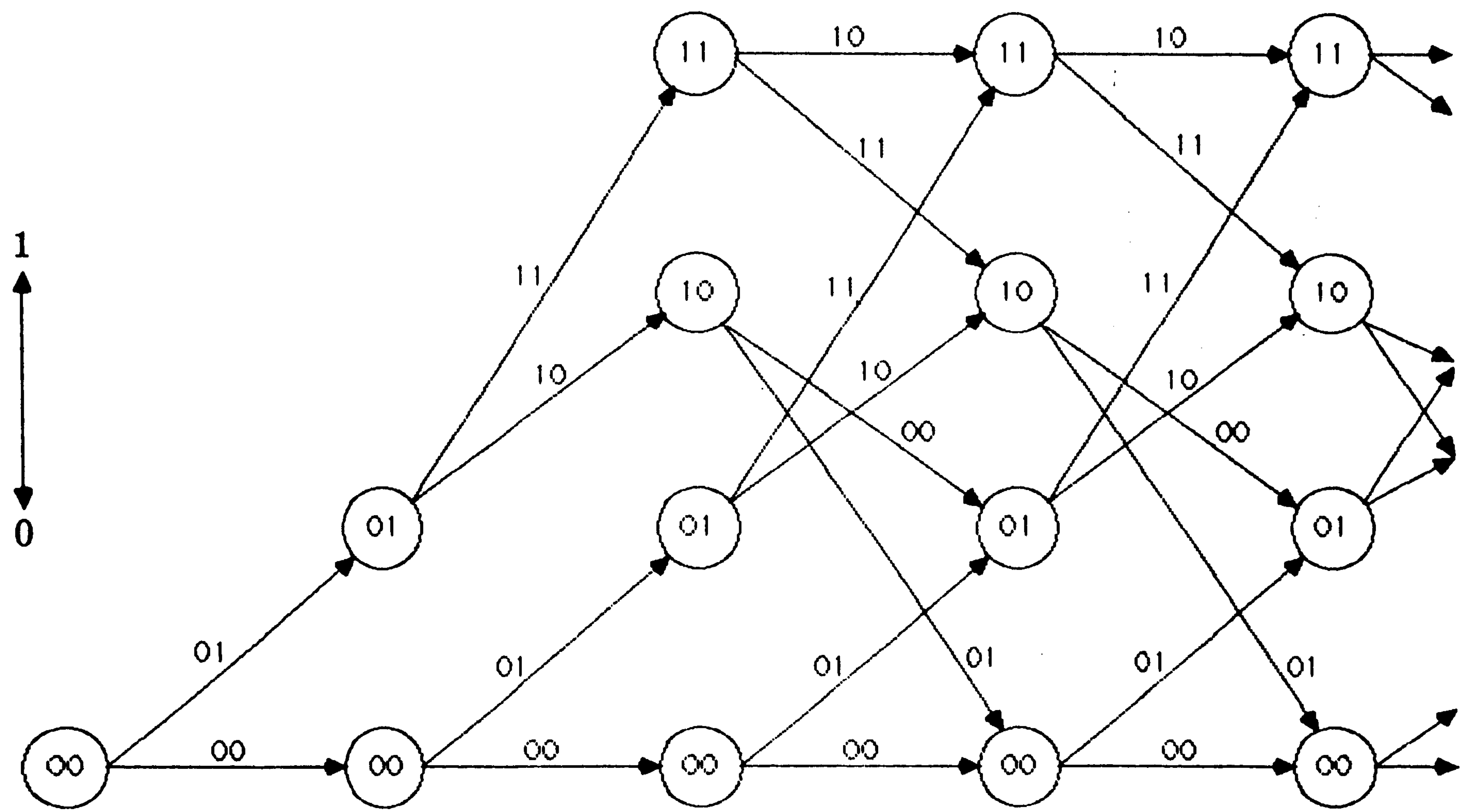


Figure 7

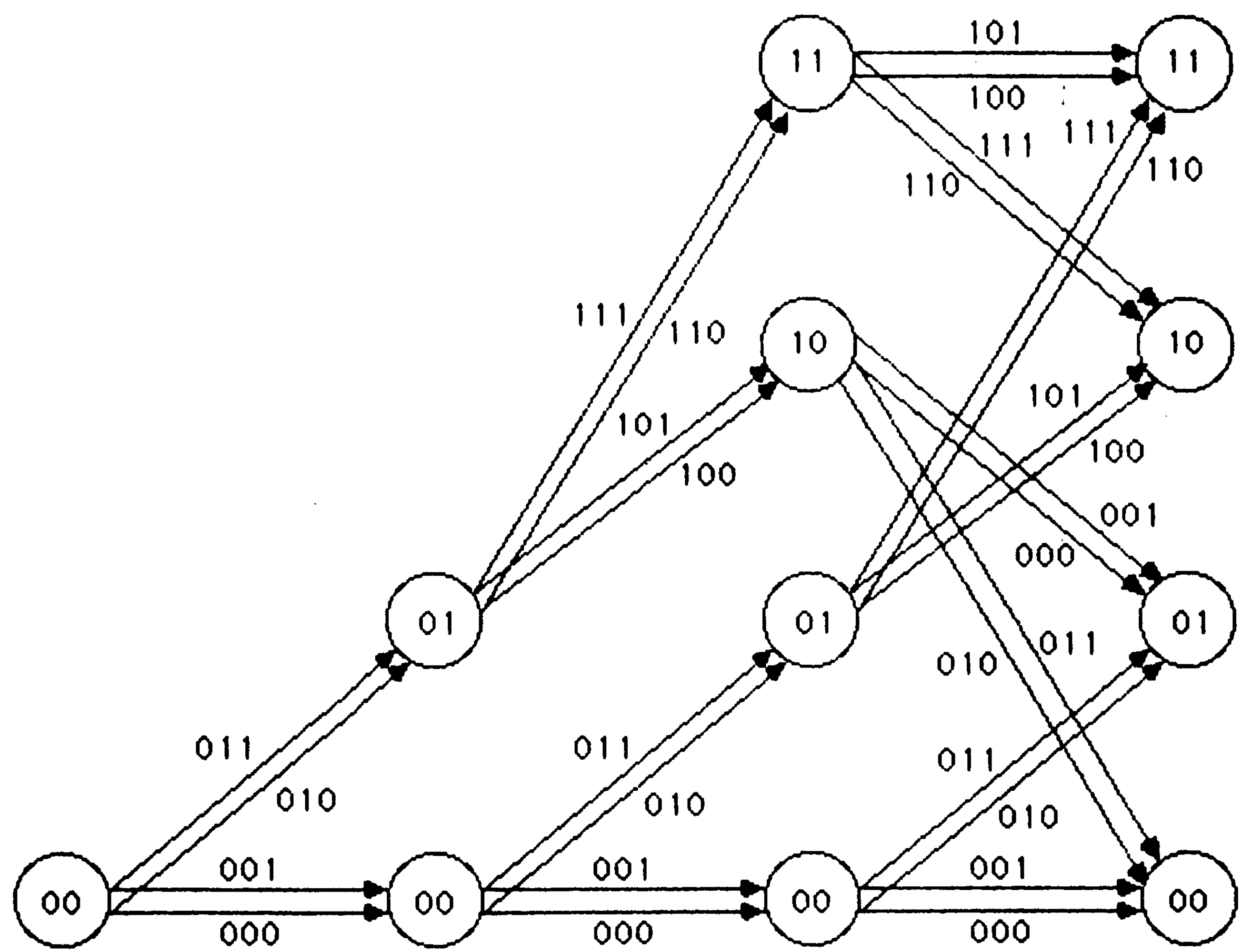


Figure 8

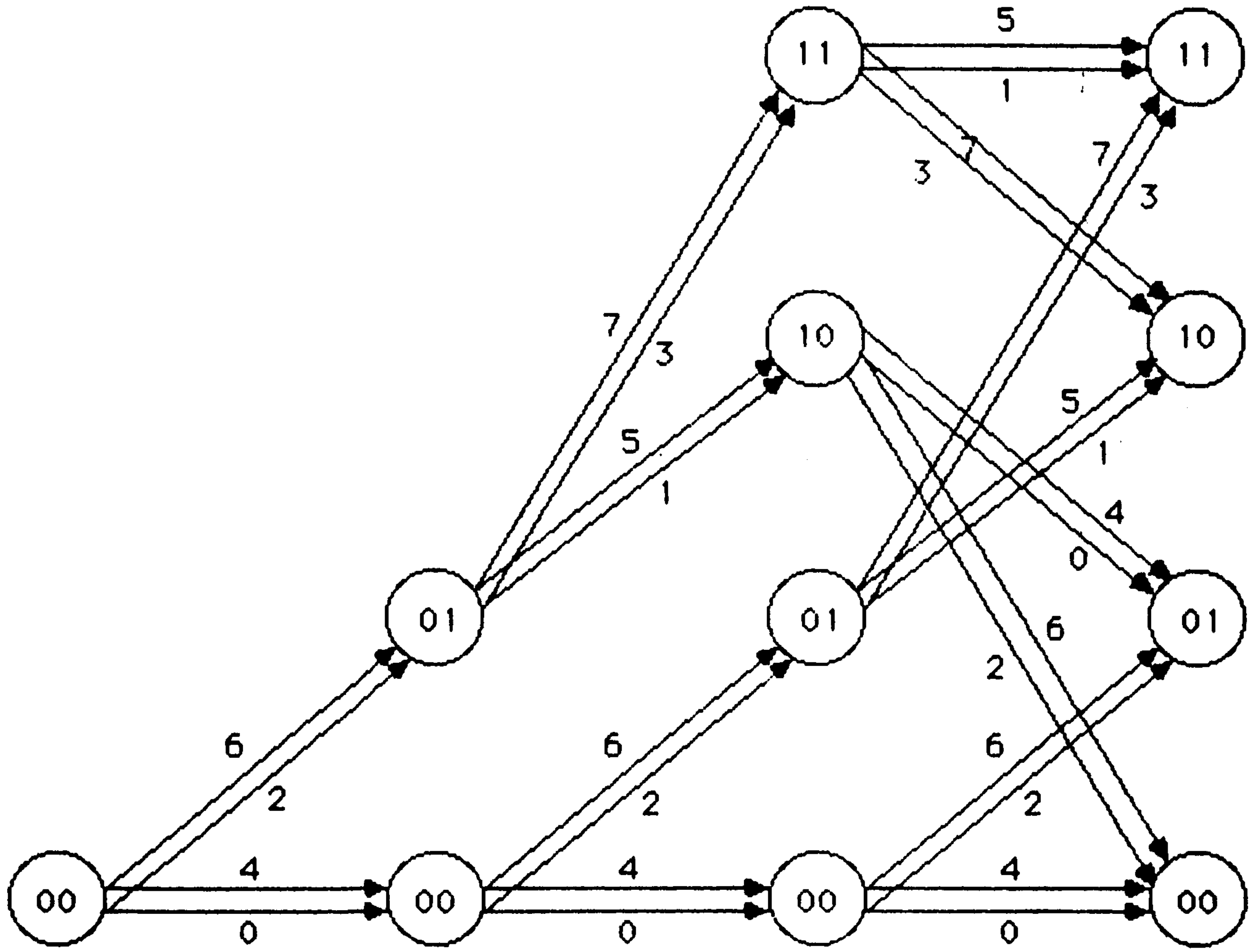


Figure 9

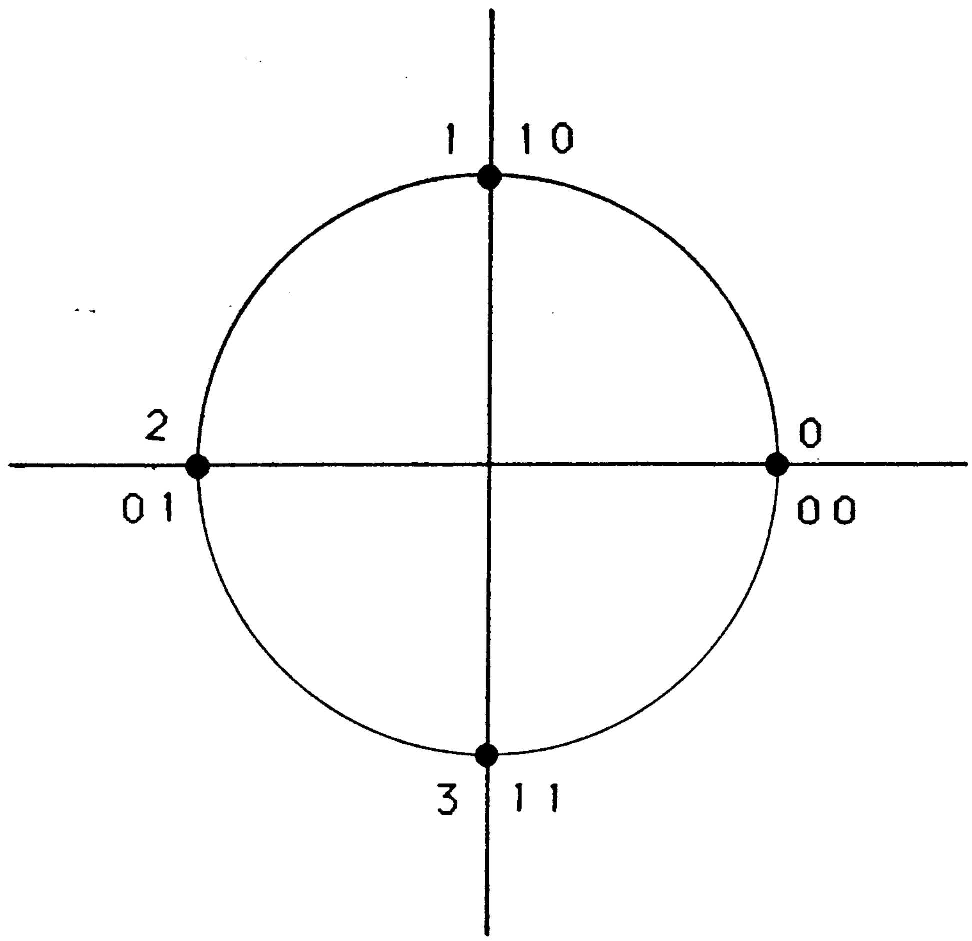


Figure 10. A QPSK signal constellation

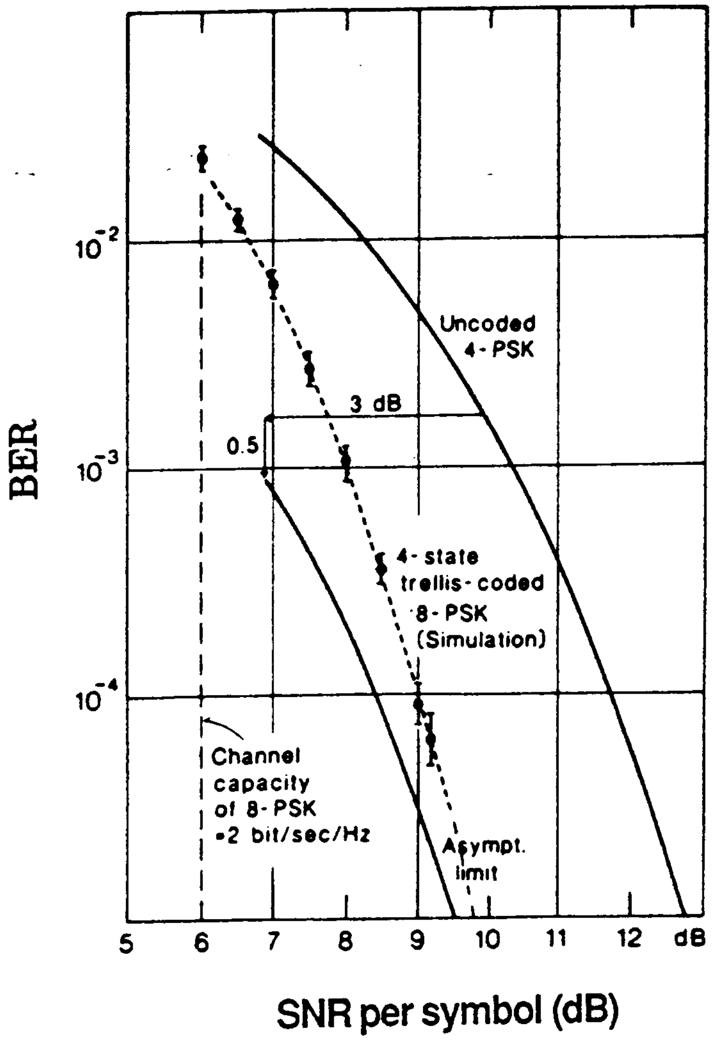


Figure 11. Coding gain over an uncoded QPSK

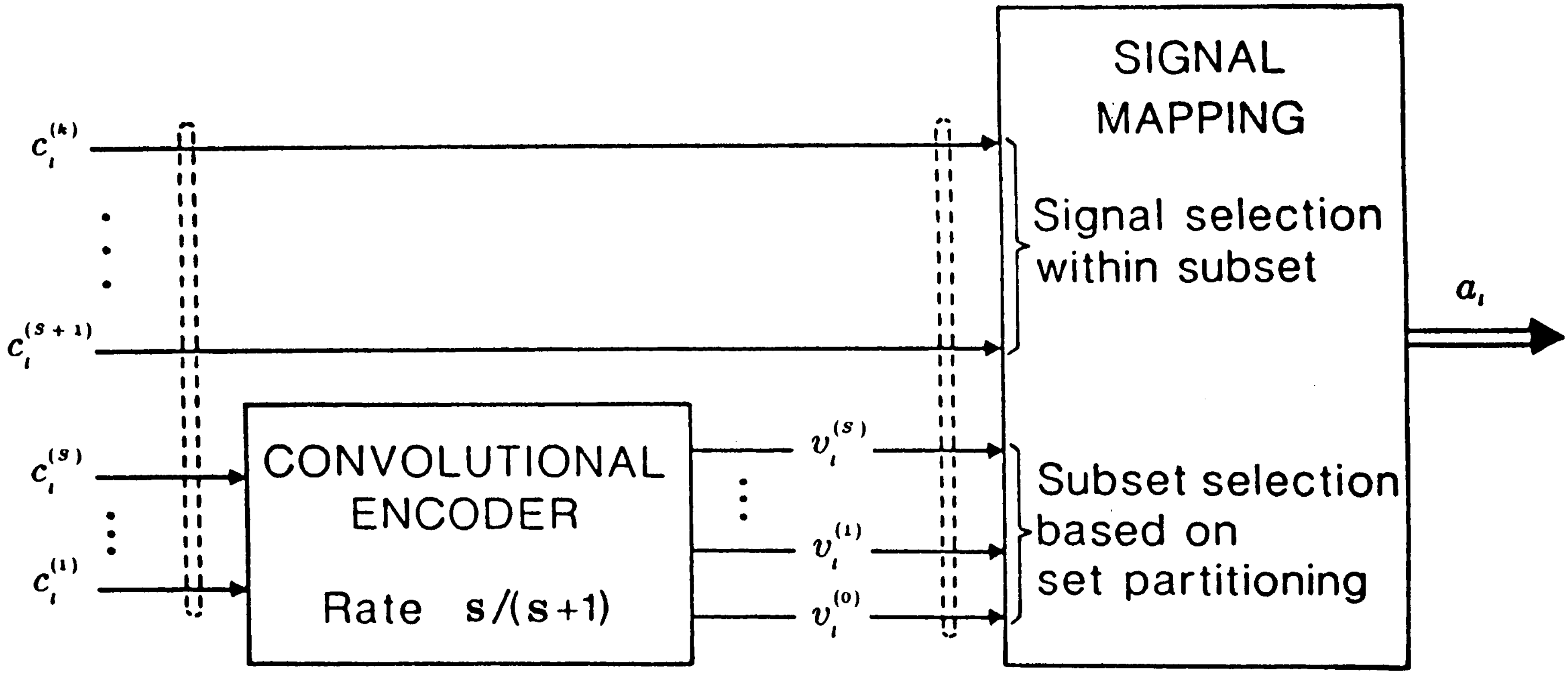


Figure 12. A general TCM system

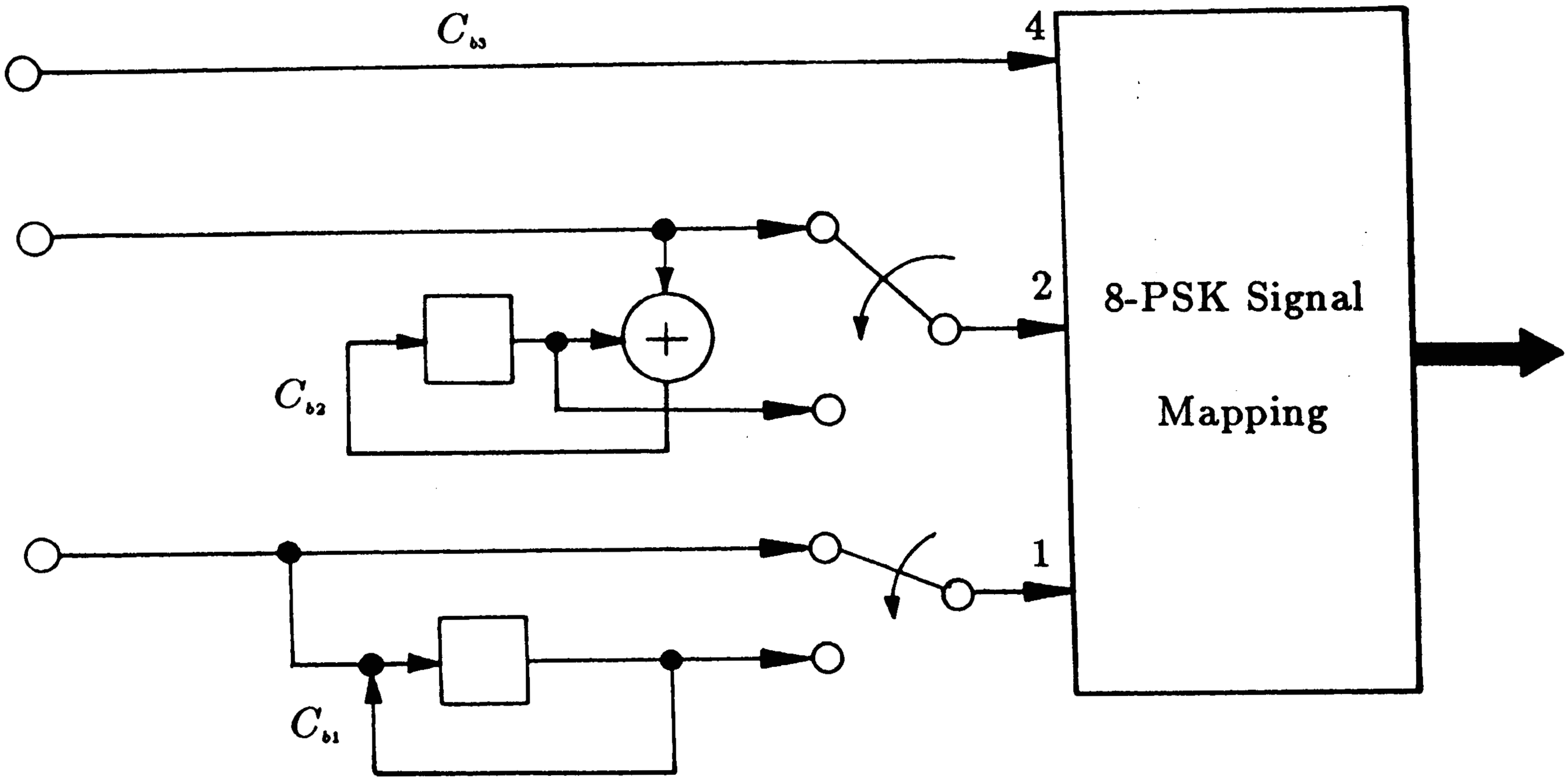


Figure 13. A block coded 8-PSK modulation system

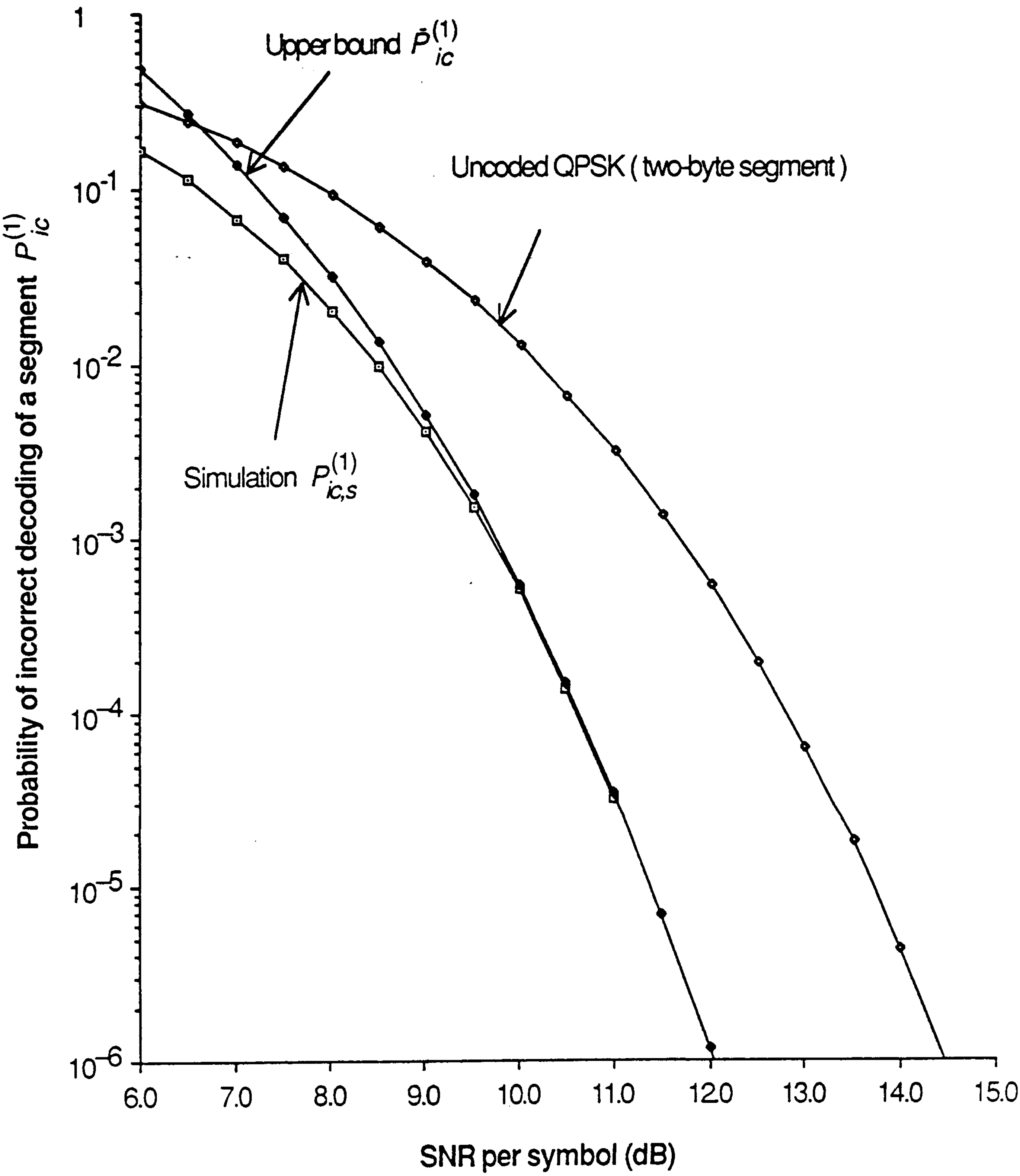


Figure 14. Error performance of the 4-state 8-PSK block code

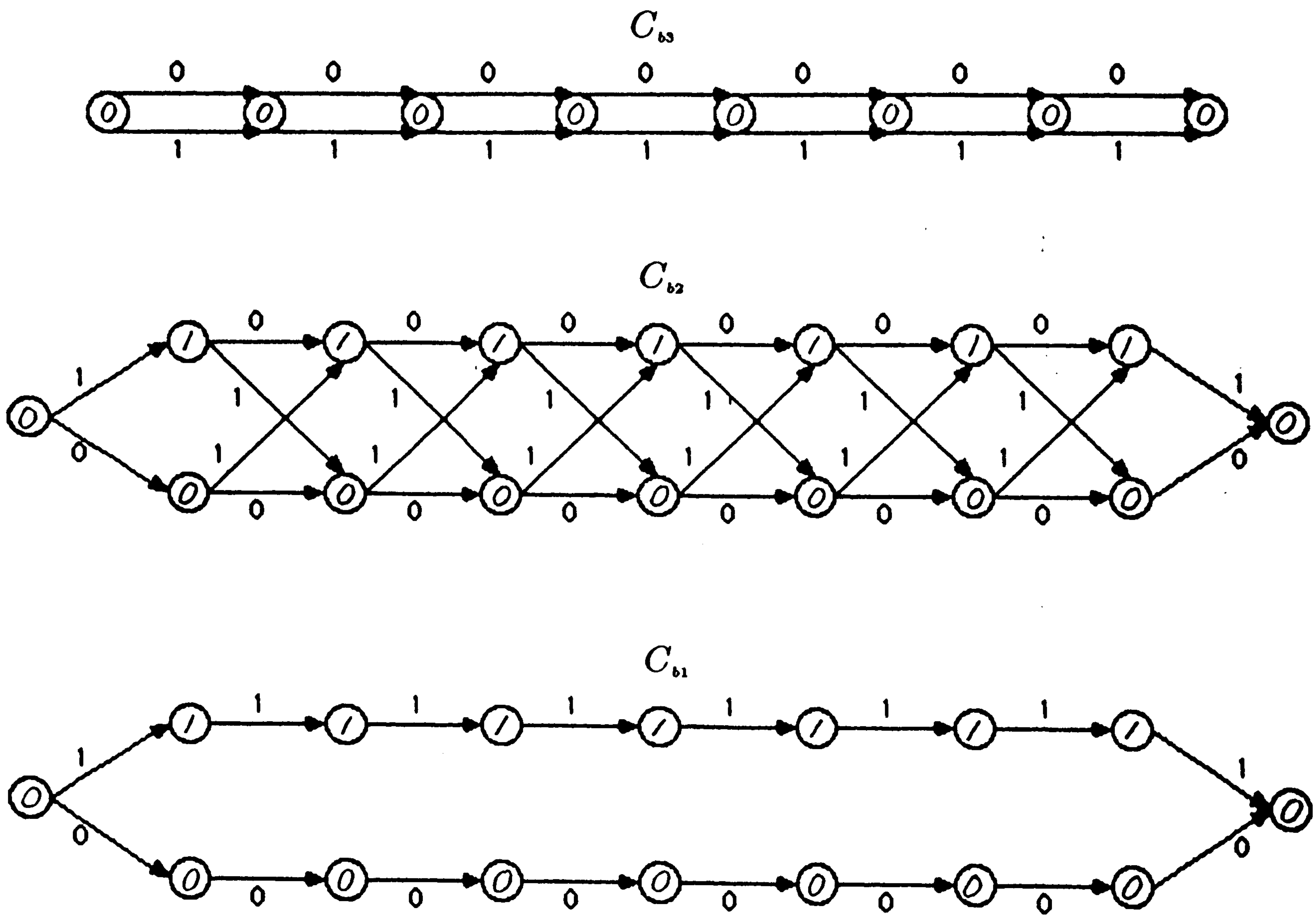
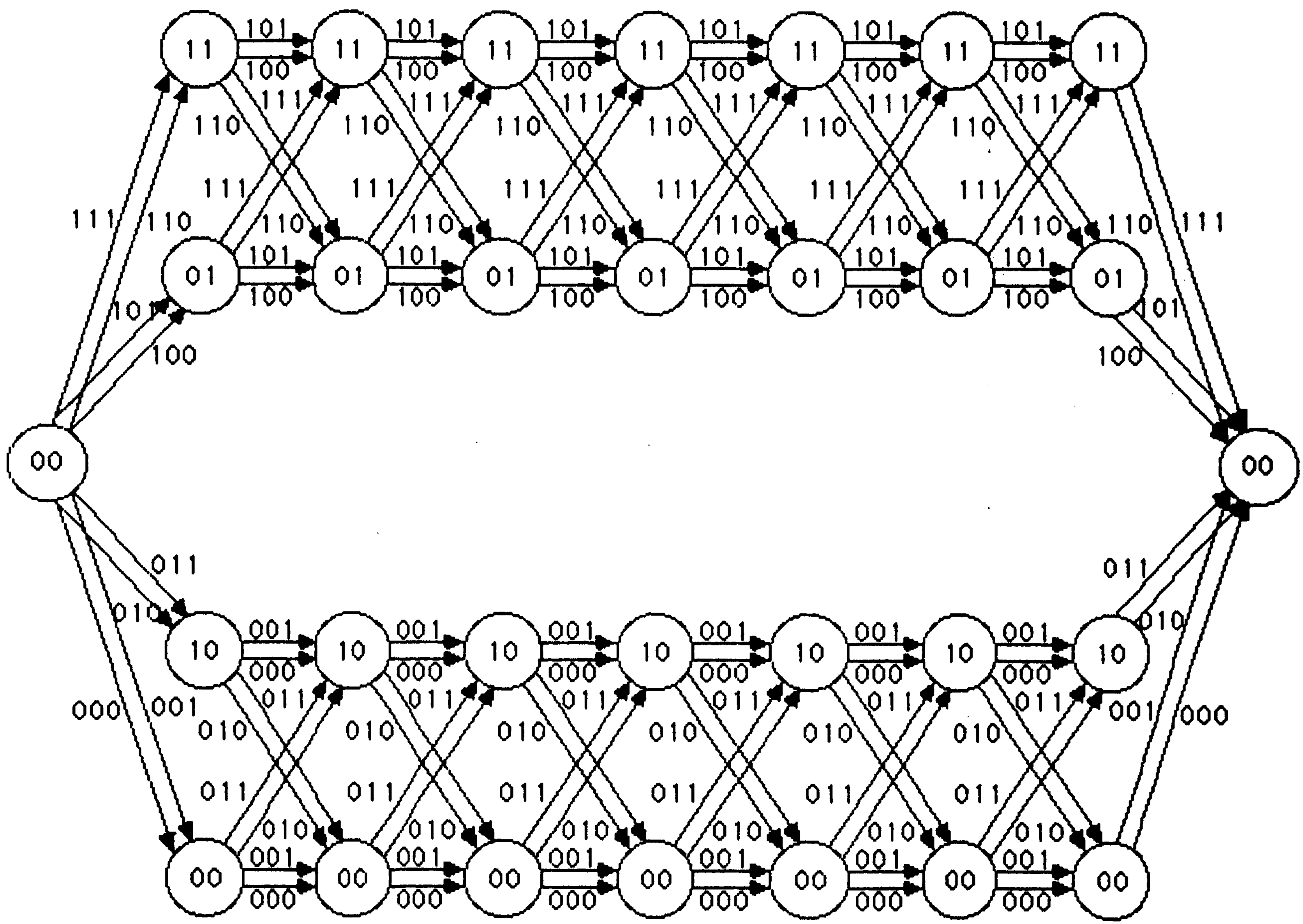


Figure 15



111
110
101
100
011
010
000
001
11 11 11 11 11 11 11
101 101 101 101 101 101
100 100 100 100 100 100
111 111 111 111 111 111
110 110 110 110 110 110
111 111 111 111 111 111
110 110 110 110 110 110
01 01 01 01 01 01 01
101 101 101 101 101 101
100 100 100 100 100 100
00
00
10 10 10 10 10 10 10
001 001 001 001 001 001
000 000 000 000 000 000
011 011 011 011 011 011
010 010 010 010 010 010
011 011 011 011 011 011
010 010 010 010 010 010
00 00 00 00 00 00 00
001 001 001 001 001 001
000 000 000 000 000 000
110
111
101
100
011
010
001
000

Figure 16

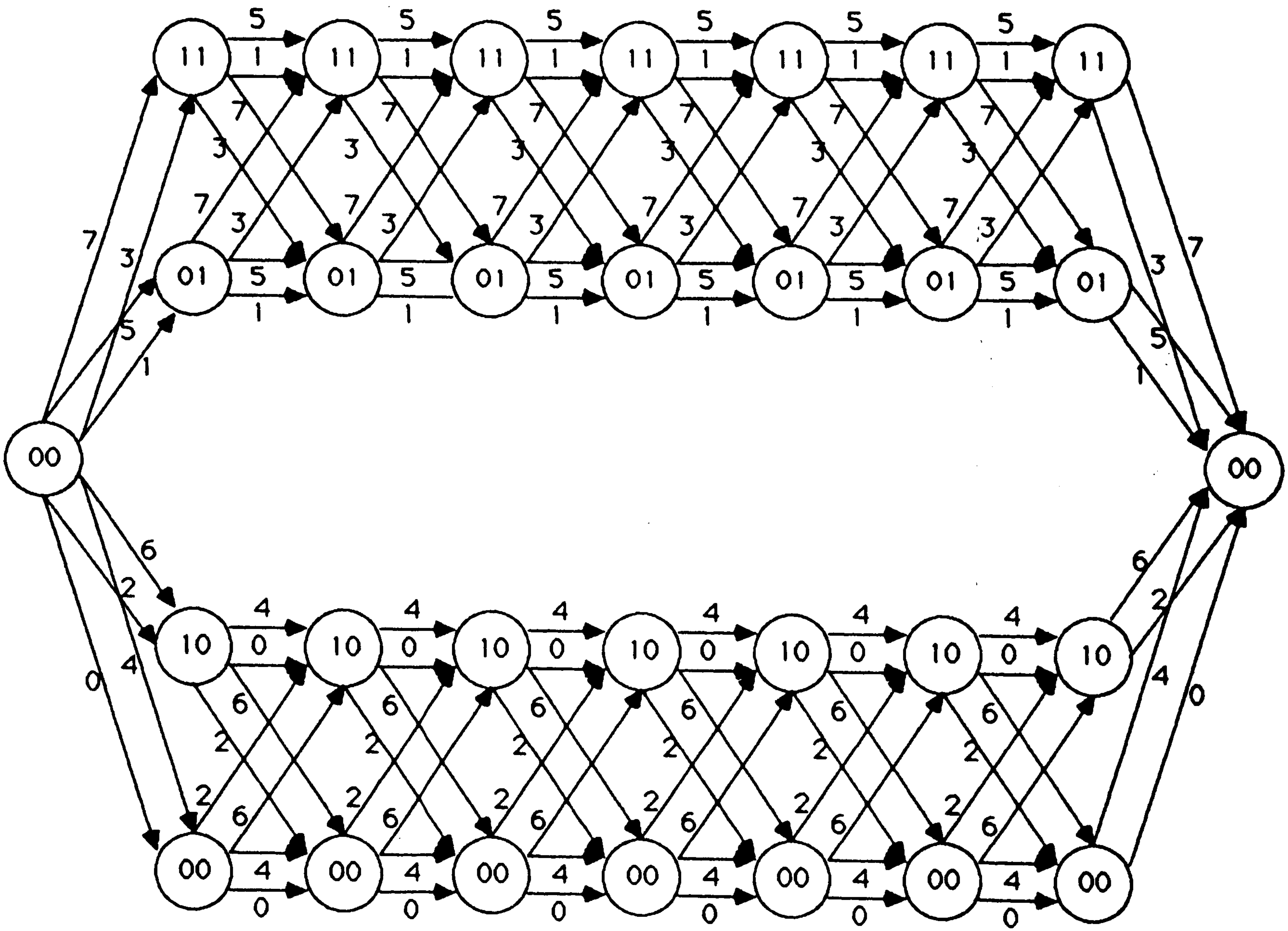


Figure 17