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SUMMARY

The report is mainly concerned with the principal investigator's research on some analytical problems in the interactions between the mean-shear flows and the acoustic field in the planar and circular jets. These problems are basic in understanding the effects of coherent large structure on the generation and complications of sound in a sub-sonic jet.

Three problems have been investigated. The first problem pertains to a spatial (vs. temporal) normal-mode analysis in a planar jet. This basic problem is formulated as an eigenvalue-value problem. Since it is not of Sturm-Lioville type, the wave numbers (eigen-values) behave in a much more complex manner. This difficulty dictates the use of various methods of approximation. Here we adopted an asymptotic method, known as the WKBJ method. Eventhough the method was designed for the high frequency case, it is often found useful also for lower frequencies, as a result of analytic continuation. In reality the disturbance consists of many different frequencies. Therfore one must deal with the wave packets. Given the spectral function of the source, we showed how to determine the excited wave packet by the Fourier-method and the method of stationary phase. In the second problem, we consider a slightly divergent, planar jet. Since, from a physical viewpoint the exponential-growing wave violates the radiation condition. The unstable waves must have a bounded envelope in space. In applied mathematics, this defect is a manisfestation of a non-uniform perturbation analysis. To obtain a uniform expansion, we recommend the two-variable method. By applying this method, we derived the evelope equation for the excited waves, which is amenable to numerical solution. Thereby the effects of divergent shear flow

on the sound amplitude can be assessed. Then we extend our analysis to a parallel or non-parellel cylindrical jet. The third problem is concerned with the acoustic waves in an axisymmetrical jet. By using the cylindrical coordinates, we have shown that the analytical techniques used in treating the planar jet, such as the WKBJ method, the method of stationary phase and the two-variable method, can be applied to the parallel or non-parallel cylindrical jets as well.

In the appendix, a recent paper entitled "Reconstruction of the Mutual Coherence Functions for a Moving Source", is attached. This paper summarize a related work partially supported by this grant, on determining the acoustic source structure in a moving jet. The method is based the high dimensional radon transform. Also it was shown that, in the case of line source, the problem becomes similar to that in the X-ray tomagraphy.

ANALYSIS OF SOME ACOUSTICS-JET FLOW INTERACTION PROBLEMS*

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## Introduction

Recently many workers in jet-noise research have investigated the role of the shear mean-flow in the generation of sound. In supersonic flow, the modified Lighthill's equations had been derived by Phillips [1], Lilley [2] and Pao [3], to account for the shear-flow term in sound generation. The so-called shear noise has also been discussed by Ribner [4], Mani [5] and others. All of these studies are qualitative in nature. The experimental results of Crow-Champagne [6] has stimulated a great deal of interest in the effects of coherent shear structure on the acoustic amplification in the jet. In this paper we shall be concerned with some basic problems in the acoustic-shear flow interaction.

Perhaps, the simplest model for acoustics-flow interaction problem with physical meaning is acoustic wave propagation in a two-dimensional parallel flow of an inviscid, incompressible and non-conducting fluid. This problem is similar to that of stability of two-dimensional parallel flows. A quick reading of the recent book on hydrodynamic stability (Chapter 4) by Drazin and Reid [7] would reveal that, even in this simple situation, how little is known about the general properties of its solution. This impression has motivated us to re-examine the two-dimensional model. A number of authors had addressed to the growth of spatial waves in parallel flows. However, due to radiation condition at $\infty$, a spatially

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growing plane wave in a parallel flow seems physically meaningless. A good discussion of this point may be found in [7]. It seems more reasonable to account for the spatially growing modes by the divergence of shear flows. This is the subject of several studies, e.g. [8] - [13].

This paper constitutes a summary of our results concerning some acoustics-flow interaction problems. In section 1, the governing equations are presented. Sections 2 and 3 are concerned with acoustic waves in a planar jet of constant width. Section 4 pertains to a slowly divergent planar jet. Here we shall introduce a two-variable expansion scheme to derive an evelope equation for the spatial growth (or decay) due to the divergent flow. Finally, in Section 5, we briefly discuss how to generalize our results for planar jets to the axisymmetrical ones.

## 1. Governing Equations for the Mean Flow.

Consider the effect of mean flow on the sound propagation in an inviscid, non-heat-conducting jet flow. Let $\overline{U},\overline{u}$ denote the mean and acoustic velocities; $p_o,p$ the mean and acoustic pressures and $\rho_o,\rho$ the mean and perturbed densities, respectively. By perturbing the Euler's equation, the continuity and the energy equation and assuming that the mean pressure gradient terms are small, the following set of acoustic equations may be obtained,

$$\frac{Dp}{Dt} + \gamma p_o(\nabla \cdot \overline{u}) = q \ , \tag{1.1}$$

$$\frac{D\overline{u}}{Dt} + \frac{1}{\rho_o} \nabla p + (\overline{u} \cdot \nabla)\, \overline{v} = 0 \ . \tag{1.2}$$

where $\gamma = c_p/c_v$ is the specific heat ratio, $q$ is an applied acoustic source, and

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \bar{v} \cdot \nabla\right) \tag{1.3}$$

stands for the material derivative.

For convenience we use the notations $\bar{U} = (U,V,W)$, $\bar{u} = (u,v,w)$ and $\bar{x} = (x,y,z)$ or $(x,r,\theta)$, depending on the rectangular or cylindrical coordinates are used. The following special cases will be considered

(A) Two dimensional jet: $W = w = z = 0$

(i) Parallel flow: $V = 0$, $U = U(y)$,

(ii) Divergent flow: $V = V(x,y)$, $U = U(x,y)$.

(B) Cylindrical jet: $\bar{x} = (x,r,\theta)$

(i) Parallel flow: $V = W = 0$, $U = U(r)$,

(ii) Divergent flow: $W = 0$, $U = U(x,r)$, $V = V(x,r)$.

In what follows, we shall give a more detailed description of Case A, while Case B will be discussed only briefly.

## 2. Spatial Modal Analysis in a Planar Jet.

Let us first consider the most basic problem: the normal mode analysis in a two-dimensional jet. In components, the governing equations (1.1) - (1.2) with $q = 0$ read

$$Dp + \gamma p_o\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = q = 0 \ ,$$

$$Du + \frac{1}{\rho_o}\frac{\partial p}{\partial x} + v\frac{du}{dx} = 0 \ , -\infty < x < \infty, \quad a < y < b \tag{2.1}$$

$$Dv + \frac{1}{\rho_o}\frac{\partial p}{\partial y} = 0 \ ,$$

which is subject to appropriate boundary conditions,

where
$$D = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right), \quad U = U(y) \ . \tag{2.2}$$

According to the normal-mode stability analysis [ 7 ], we seek a solution of (2.1) in the form for $a < y < b$

$$(p,u,v) = (\tilde{p},\tilde{u},\tilde{v})(y)e^{i(kx-\omega t)} \ . \tag{2.3}$$

For a given frequency $\omega$, we are interested in the dependence of the wave number $k$ on $\omega$. If $k(\omega)$ is real, the mode is propagating or proper. Otherwise, $k$ is complex and the corresponding mode is non-propagating. If $\operatorname{Im} k \geq 0$, the mode decays and, hence, stable. However, if $\operatorname{Im} k < 0$, the mode grows exponentially as $x \to \infty$. But this means, for fixed $t$, the modal amplitude will grow along the downstream. It is physically meaningless to label such mode as unstable, since it violates the boundary conditions at $\infty$. To account for wave behavior, instead of simple mode (2.3), we shall later replace it by a wave packet by superposing modes of different frequencies.

Let us substitute (2.3) into (2.1) to get a system of ordinary differential equations in $y$

$$ik\Pi\tilde{p} + \gamma p_o\left(ik\tilde{u} + \frac{d\tilde{v}}{dy}\right) = 0 \ ,$$

$$ik\Pi\tilde{u} + \Pi'\tilde{v} + ik\tilde{p}/\rho_o = 0 \ , \quad a < y < b \tag{2.4}$$

$$ik\Pi\tilde{v} + \frac{1}{\rho_o}\frac{d\tilde{p}}{dy} = 0 \ ,$$

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where

$$\eta = U(y) - c, \quad \text{with} \quad c = (\omega/k) . \tag{2.5}$$

After eliminating $\tilde{u},\tilde{v}$ in favor of $\tilde{p}$, we obtain a second order equation

$$\eta\phi'' - 2\eta'\phi' - k^2\eta(1-\mu^2)\phi = 0, \quad a<y<b \tag{2.6}$$

where

$$\phi = \tilde{p}/\rho_o , \tag{2.7}$$

$$\mu = \eta/\alpha , \quad \alpha = (\gamma P_o/\rho_o)^{1/2} \tag{2.8}$$

and

$$(\ )' = \frac{d}{dy} .$$

By a simple change of variable

$$\phi = \eta\psi , \tag{2.9}$$

the equation (2.6) may be rewritten as

$$L(k)\psi = \psi'' - k^2(1-\mu^2)\psi - r\psi = 0 , \quad a<y<b , \tag{2.10}$$

where

$$r = \eta^{-2}(\eta'' - 2\eta'^2) . \tag{2.11}$$

Here we assume $\eta = U(y) - c \neq 0$ in $(a,b)$. Otherwise we let $c$ be complex and define $\psi$ as the principal-value as $\mathrm{Im}\, c \to 0$. Consider the homogeneous BVP (boundary-value problem) associated with the original system (2.1)

$$L(k)\psi = 0 , \tag{2.12}$$

$$B_1\psi = B_2\psi = 0 , \tag{2.13}$$

where (2.13) stands for appropriate B.C.'s (boundary conditions). Let the eigenfunctions $\psi_n$ with eigenvalues $k_n$, form a complete set in $L^2(a,b)$, the space of square-integrable functions over $(a,b)$. Suppose we multiply (2.12) by $\psi^*$, the complex conjugate of $\psi$, and integrate from $a$ to $b$ to obtain

$$k^2\int_a^b [1-(\frac{U}{\alpha})^2]|\psi|^2 dy + 2k(\frac{\omega}{\alpha^2})\int_a^b U|\psi|^2 dy + \int_a^b \{|\psi'|^2 + [r(k)-(\frac{\omega}{\alpha})^2]|\psi|^2\} dy = 0 . \qquad (2.14)$$

The difference between (2.14) and its conjugate yields

$$(k^2-k^{*2})\int_a^b [1-(\frac{U(y)}{\alpha})^2]|\psi(y)|^2 dy + 2(k-k^*)\int_a^b (\frac{\omega}{\alpha})U(y)|\psi(y)|^2 dy = 0 . \qquad (2.15)$$

The above show that the eigenvalues need not be real, even though $L(k)$ is self-adjoint. Note that (2.14) may be written as

$$b_o k^2 + 2b_1 k + b_2 = 0 , \qquad (2.16)$$

where $b_o$, $b_1$, $b_2$ are the corresponding coefficients in (2.14). Thus, if $r$ is independent of $k$, corresponding to each eigenfunction $\psi_n$, there are two eigenvalues $k_n$ which are roots of (2.16). Let

$$\Delta(\psi) = (b_1^2 - b_o \cdot b_2) \qquad (2.17)$$

be the discriminant for (2.16). Thus, if $\Delta(\psi) \geq 0$, the roots are real. The condition for n-th mode being proper at a fixed $\omega$ is that

$$\Delta(\psi_n) \geq 0 ,$$

otherwise $k_n$ are complex so that this mode either grows or decays in $x$.

As a simple example, consider the BVP

$$L(k)\psi = 0 , \quad -b = a < y < b ,$$
$$\psi(-b) = \psi(b) = 0 , \tag{2.18}$$

where $U$ is constant. Then $L(k)\psi = \psi'' - k^2(1-\mu^2)\psi$, and the eigenfunctions are $\{\mathrm{Sin}\,\frac{n\pi}{2b}(y+b)\}$, corresponding to eigenvalues $k_n$ determined by (2.14)

$$[1 - (\frac{U}{\alpha})^2]k^2 + 2(\frac{U}{\alpha^2})\omega k + [(\frac{n\pi}{2b})^2 - (\frac{\omega}{\alpha})^2] = 0 \tag{2.19}$$

for which

$$\Delta(\psi_n) = (\frac{\omega}{\alpha})^2 - [1 - (\frac{U}{\alpha})^2](\frac{n\pi}{2b})^2 , \quad n = 1,2,\ldots .$$

Thus, for a fixed $\omega$, there are a finite number of proper modes $k_n$ with $n \leq N$, where $N$ is the largest integer less or equal to $\frac{2b\omega}{\alpha\pi(1-m^2)^{\frac{1}{2}}}$ with $m = \frac{U}{\alpha}$. The rest of the modes are improper and may grow or decay.

Next, suppose the velocity profile $U(y)$ is symmetric and varies smoothly over $[-b,b]$. As shown by (2.19), $k = O(n)$ for large $n$, the behavior of higher-order modes may be determined by an asymptotic method. The well-known WKBJ method can be applied. Let

$$\psi \sim A(y)e^{k\theta(y)} \quad \text{as} \quad k \to \infty , \tag{2.20}$$

where $A,\theta$ are determined by the d.e. (2.10). The general solution can then be written as

$$\psi \sim (1-\mu^2)^{-1/4}\{C\,\mathrm{Sin}\,k\,\theta + C_2\,\mathrm{Cos\,h}\,k\theta\} \tag{2.21}$$

where

$$\theta = \int_a^y [1 - \mu^2(s)]^{1/2} ds . \tag{2.22}$$

The approximate solution is valid only when

$$\mu = \alpha^{-1}(u - c) \neq \pm 1 , \quad a < y < b . \tag{2.23}$$

The points $y_o$ at which $\mu(y_o) = \pm 1$ are called turning points. In this case the solution (2.21) is not valid near $y = y_o$, and a different form of solution must be constructed. Introduce an appropriate change of variables [14]

$$\xi = f(y) \ , \quad \zeta = g(y)\,\psi(y) \ , \tag{2.24}$$

the equation (2.3) yields

$$\frac{d^2\zeta}{d\xi^2} - k^2\xi\zeta = \sigma\zeta \ , \tag{2.25}$$

where

$$\xi = f = [\tfrac{2}{3}\int_a^y (1-\mu^2)^{1/2} dy']^{2/3} \ ,$$

$$g = [f'(y)]^{1/2} \ , \tag{2.26}$$

$$\sigma = \frac{1}{2}\frac{f'''}{(f')^3} - \frac{3}{4}\frac{(f'')^2}{(f')^4} - \frac{r}{(f')^2} \ .$$

For large $k$, the equation (2.25) is approximately an Airy equation so that

$$\zeta \sim c\,A_i(\xi) + dB_i(\xi) \quad \text{as} \quad |k| \to \infty \tag{2.27}$$

where $A_i, B_i$ are Airy functions of first and second kind. Hence the solution to (2.3) is given by, referring to (2.26)

$$\psi(y) \sim [f'(y)]^{1/2}\{c\,A_i\,[f(y)] + d\,B_i\,[f(y)]\} \ . \tag{2.28}$$

Suppose that the condition (2.23) holds. Then, in view of (2.21), the eigenvalue problem (2.18) with non-constant $U$ has the asymptotic solution

$$\psi_n \sim (1-\mu_n^2)^{-1/4} \sin k_n \int_{-b}^{y} [\mu_n^2(s)-1]^{1/2} ds, \quad \text{for large } n \ , \tag{2.29}$$

where $\mu_n = [U(y) - \omega/k_n]$, and $k_n$ are solutions of the equation

$$\int_{-b}^{b} \{k^2[1 - \frac{U^2(y)}{\alpha^2}] + 2k(\frac{\omega}{\alpha^2})U(y) - (\frac{\omega}{\alpha})^2\}^{1/2} dy = \pm n\pi i \ . \tag{2.30}$$

We note that (2.30) reduces to (2.19) when $U$ is constant. For large $k$, $|k| >> \omega$, (2.30) has no real solutions. Thus all higher modes are improper for a finite $\omega$. However, if $\omega$ is so large that $\omega >> |k|$, (2.30) gives

$$\int_{-b}^{b} \{\omega - kU(y)\}dy \sim \pm n\pi\alpha$$

or

$$k \sim \{\omega - (\frac{n\pi\alpha}{2b})\} \Big/ \frac{1}{2b}\int_{-b}^{b} U(y)dy \; . \tag{2.31}$$

which is valid if $\omega \sim (\frac{n\pi\alpha}{2b})$ and $n$ is large. This value of $k$ gives rise to

$$p \sim \tilde{p}_n \, e^{-i(\frac{n\pi\alpha}{2bU_o})x + i\frac{\omega}{U_o}(x - U_o t)} \; , \tag{2.32}$$

where $U_o = \frac{1}{2b}\int_{-b}^{b} U(y)dy$ is the mean-velocity of the flow. Here we have tacitly assumed, by continuation, (2.30) holds also for smaller values of $k$. The result (2.32) shows that at high frequency, the disturbance is a spatially modulated wave with the propagation spped $U_o$. On the other hand, if $\omega << |k|$, (2.30) has the approximation

$$k\int_{-b}^{b}\{1 - (\frac{U}{\alpha})^2\}^{1/2}dy + (\frac{\omega}{\alpha^2})\int_{-b}^{b} \frac{U}{\{1-(\frac{U}{\alpha})^2\}^{1/2}} dy \sim \pm n\pi i \; , \tag{2.33}$$

which corresponds to the non-propagating modes.

In the presence of a turning point, we should apply the B.C.'s to (2.28). But the resulting characteristic equation is not readily solvable to get explicit results as before.

We remark that, for simplicity, we impose the B.C. $p = 0$ at $y = \pm b$. More realistically the field in $|y| < b$ must be matched to that in $|y| > b$ at $y = \pm b$. This correction is straight-forward but complicates the characteristic equation whose explicit solution becomes inaccessible.

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Instead of simple modes, consider the superposition of modes excited by various frequencies

$$p(t,x,y) = \sum_{n=1}^{\infty} p_n(t,x,y) , \tag{2.34}$$

where

$$p_n(t,x,y) = \int \mu_n(y,k_n,\omega)\, \psi_n(y,k_n,\omega) q_n(\omega) e^{i\{k_n(\omega)x - \omega t\}} d\omega \tag{2.35}$$

is the n-th mode of $p$, and $k_n(\omega)$ is the n-th eigenvalue of the BVP (2.12) - (2.13) corresponding to the frequency $\omega$. It is easy to verify that $p_n$ given by (2.35) satisfies (2.1) after $u,v$ are eliminated. The solution (2.34) may describe the disturbance due to a source located at $x = 0$.

Since we are interested in the event near the wave front in the large distance down-stream, the asymptotic evaluation of (2.33) will be carried out as $x \to \infty$, $t \to \infty$ with $(x/t)$ fixed. Let

$$t/x = \tau . \tag{2.36}$$

Then (2.33) may be written as

$$p_n = \int \tilde{p}_n(y,k_n,\omega) e^{ix\{k_n(\omega) - \omega\tau\}} d\omega , \tag{2.37}$$

where

$$\tilde{p}_n = \mu(y,k_n)\psi_n q_n . \tag{2.38}$$

Hence, as $x \to \infty$, we may apply the method of stationary phase for asymptotic evaluation. The stationary points of the phase are determined by the equation

$$k_n'(\omega) = \tau . \tag{2.39}$$

Let $\omega^*$ be a solution. Then the contribution to $p_n$ due to this stationary point is [15]

$$P_n \sim \tilde{p}_n(k_n^*,\omega^*)\left[\frac{2\pi}{x\left|k_n''(\omega^*)\right|}\right]^{1/2}\exp i\left\{k_n^* x - \omega^* t - \frac{\pi}{4}\operatorname{sgn} k_n''(\omega^*)\right\} \tag{2.40}$$

where $k_n^* = k_n(\omega^*)$ .

For example, we consider the case of constant $U$ . By (2.19), we have

$$k_n = -M\left(\frac{\omega}{\alpha}\right) \pm \Delta_n^{1/2}, \quad \Delta_n = \left(\frac{\omega}{\alpha}\right)^2 - (1-M^2)\left(\frac{n\pi}{2b}\right)^2 , \tag{2.41}$$

where $M = V/\alpha$ is the Mach number. It follows that

$$k_n'(\omega) = -\left(\frac{M}{\alpha}\right) \pm \frac{\omega}{\alpha^2 \Delta_n^{1/2}} \tag{2.42}$$

$$k_n''(\omega) = \pm \alpha^{-2} \Delta_n^{-1/2}\left(1 - \frac{\omega}{\alpha^2 \Delta_n}\right) . \tag{2.43}$$

Then the equation (2.39) has the solutions

$$\omega_n^* = \pm\alpha(\alpha\tau + M)(1 - M^2)\left(\frac{n\pi}{2b}\right) \quad [1 + (\alpha\tau + M)^2]^{1/2} \tag{2.44}$$

so,

$$k_n = -\omega_n^* \left\{\frac{M}{\alpha} \pm \frac{1}{\alpha(\alpha\tau + M)}\right\} \qquad \text{with} \qquad \tau = t/x \quad . \tag{2.45}$$

Thus we see that, due to boundary absorption, each mode excited by a source at $x = 0$ will evolve into space-time modulated waves which decay like $|x|^{-1/2}$ downstream. Their respective speeds, referring to (2.42) and (2.43) are given by

$$c_n = \left|\omega^*/k_n\right| = \left|M \pm (M + \alpha t/x)^{-1}\right| \quad \alpha . \tag{2.46}$$

For a variable profile $U$ , if the variation is slow, the WKB method can be shown to apply. Thus, if the dispersion equation (2.30) may be solved approximately, a similar procedure can be applied to give an asymptotic evaluation.

## 3. Radiation in an Unbounded Planar Jet.



In the presence of external source, the RHS of the first equation in (2.1) is $q(t,x,y) \neq 0$. Let

$$\tilde{f} = \frac{1}{(2\pi)^2} \iint f(t,x)e^{-i(kx-\omega t)}dxdt . \tag{3.1}$$

Then a Fourier transform of (2.1) with $q \neq 0$ yields (2.4) with the RHS of the first equation equal to $q \neq 0$. The eliminated system corresponding to (2.10) reads

$$L(k)\psi = \psi'' - k^2(1-\mu^2)\psi - r\psi = -ik\tilde{q} , \quad -\infty < y < \infty \tag{3.2}$$

$$R\psi = 0 \quad \text{at} \quad y = \pm \infty \tag{3.3}$$

where the B.C. is introduced by the radiation condition for $p$. Suppose $G(y,y',k,\omega)$ denotes the Green's function for the BVP (3.2), (3.3). Then we have

$$\psi(y,k,\omega) = \int(-ik)G(y,y',k,\omega)\tilde{q}(y',k,\omega)dy' . \tag{3.4}$$

It follows that

$$\begin{aligned} p(t,x,y) = (i\rho_o)^{-1}\iiint k\eta(y,k,\omega)G(y,y',k,\omega) \\ \times \tilde{q}(y',k,\omega)e^{i(kx-\omega t)}dy'dkd\omega . \end{aligned} \tag{3.5}$$

By the theory of contour integration, the major contribution to the integral (3.5) comes from the singularity set of $G$. Let $\psi_1, \psi_2$ be two complementary solutions of (3.2) satisfying the B.C.'s at $y = \infty, -\infty$, respectively. Then we have

$$G = \begin{cases} \Lambda^{-1}(y')\psi_1(y)\psi_2'(y') , & y \geq y' \\ -\Lambda^{-1}(y')\psi_2(y)\psi_1'(y') , & y \leq y' \end{cases} , \tag{3.6}$$

where $\Lambda = (\psi_1\psi_2' - \psi_2\psi_1')$ is the Wronskian which should be a constant.

As an example, suppose $q$ is a point source at $x=0$, $y=0$, so that $q = q_o(t)\delta(x)\delta(y)$. Then (3.5) becomes

$$p(t,x,y) = (i\rho_o)^{-1}\iint k\eta(y,k,\omega)G(y,o,k,\omega)\tilde{q}_o(\omega) \times e^{i(kx-\omega t)}dkd\omega \tag{3.7}$$

For a slowly varying velocity $U$ at low Mach number, the result (2.20) by WKB method gives

$$\begin{aligned} G(y,o,k,\omega) &= G(-y,o,k,\omega) \\ &\sim \frac{1}{2ikg_o}\left[\frac{g_o}{g(y)}\right]^{1/4} e^{ik\int_o^y g(s)ds}, \quad y \geq 0, \end{aligned} \tag{3.8}$$

where $g(y) = [\mu^2(y)-1]^{1/2}$, $g_o = g(o)$. Upon using (3.8) in (3.7), it yields

$$p(t,x,y) \sim (-\rho_o)^{-1}\iint g_o^{-1}(k,\omega)\eta(y,k,\omega)\left[\frac{g_o(k,\omega)}{g(y,k,\omega)}\right]^{1/2} q_o(\omega) \times e^{ik\int_o^y g(s,k,\omega)ds + i(kx-\omega t)} d\omega dk, \quad y \geq 0 .$$

It is difficult to compute the integral in closed form. Sometimes it may be possible to evaluate it approximately.

## 4. Propagation in a Slowly Divergent Planar Jet.

Let the mean-flow be divergent so that, in (1.1), (1.2),

$$\begin{aligned} \bar{U} &= (U,V,0) \ , \\ \bar{u} &= (u,v,o) \ , \end{aligned} \tag{4.1}$$

where $U,V$ are functions of $x,y$ . As a function of $x$ , they are slowly varying. At low Mach number, the mean flow is assumed to be incompressible so that there exists a stream function $\phi(\xi,y)$ , where $\xi = \epsilon x$ is a slow variable with small $\epsilon > 0$ . By definition we have

$$\begin{aligned} U &= \phi_y(\xi,y) \equiv \phi_2(\xi,y) \\ V &= -\phi_x(\xi,y) = -\epsilon\phi_\xi(\xi,y) = -\epsilon\phi_1(\xi,y) \ . \end{aligned} \tag{4.2}$$

In terms of $\phi$ , the governing equations (1.1) - (1.2) can be written in components as follows

$$\begin{aligned} D_\epsilon u + (\epsilon\phi_{12}u + \phi_{22}v) &= -\frac{1}{\rho_o}\frac{\partial p}{\partial x} \ , \\ D_\epsilon v - (\epsilon^2\phi_{11}u + \epsilon\phi_{12}v) &= -\frac{1}{\rho_o}\frac{\partial p}{\partial y} \ , \\ D_\epsilon p + \gamma p_o\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= q \end{aligned} \tag{4.3}$$

where $D_\epsilon = \left(\frac{\partial}{\partial t} + \phi_2\frac{\partial}{\partial x} - \epsilon\phi_1\frac{\partial}{\partial y}\right)$ .

When the skew parameter $\epsilon = 0$ , the system (4.3) reduces to the parallel flow case (2.1). Here we wish to determine the effect of divergence on the propagating waves in the jet. By the two-variable method [16], we write

$$F = (p,u,v) \ .$$

Then (4.3) may be put in the form

$$D_\epsilon F + B_\epsilon F = Q \ , \tag{4.4}$$

where

$$B_\epsilon = \begin{bmatrix} 0 & \gamma P_o \frac{\partial}{\partial x} & \gamma P_o \frac{\partial}{\partial y} \\ \frac{1}{\rho_o}\frac{\partial}{\partial x} & \epsilon\phi_{12} & \phi_{22} \\ \frac{1}{\rho_o}\frac{\partial}{\partial y} & -\epsilon^2\phi_{11} & -\epsilon\phi_{12} \end{bmatrix} , \tag{4.5}$$

$$Q = \begin{bmatrix} q \\ 0 \\ 0 \end{bmatrix} \ . \tag{4.6}$$

First consider the homogeneous B.V.P. in free-space

$$D_\epsilon F + B_\epsilon F = 0 \ , \tag{4.7}$$

on which we impose the radiation conditions at $\infty$ .

Surpressing the dependence on $t$ and $y$ , let

$$F = Z(x,\xi,\epsilon), \qquad \xi = \epsilon x \ , \tag{4.8}$$

so that

$$\frac{\partial F}{\partial x} = \frac{\partial Z}{\partial x} + \epsilon \frac{\partial Z}{\partial \xi} \ . \tag{4.9}$$

Expand $Z$ in a power series in $\epsilon$ to get

$$Z(x,\xi,\epsilon) = Z_o(x,\xi) + \epsilon Z_1(x,\xi) + \ldots \ . \tag{4.10}$$

In view of (4.8) - (4.10), (4.7) yields the successive approximations

$$(D+B)Z_0 = 0 \,, \tag{4.11}$$

$$(D+B)Z_1 = Q_1 \equiv -(D_1+B_1)Z_0 \,, \tag{4.12}$$

$$\cdots\cdots \,,$$

where

$$D = D_0 = \left(\frac{\partial}{\partial t} + \phi \frac{\partial}{\partial x}\right) \,,$$

$$D_1 = \phi_2 \frac{\partial}{\partial \xi} - \phi_1 \frac{\partial}{\partial y} \,,$$

$$B = \begin{bmatrix} 0 & \gamma P_o \frac{\partial}{\partial x} & \gamma P_o \frac{\partial}{\partial y} \\ \frac{1}{\rho_o}\frac{\partial}{\partial x} & 0 & \phi_{22} \\ \frac{1}{\rho_o}\frac{\partial}{\partial y} & 0 & 0 \end{bmatrix} \,, \tag{4.13}$$

$$B_1 = \begin{bmatrix} 0 & \gamma P_o \frac{\partial}{\partial \xi} & 0 \\ \frac{1}{\rho_o}\frac{\partial}{\partial \xi} & \phi_{12} & 0 \\ 0 & 0 & -\phi_{12} \end{bmatrix} .$$

Note that (4.11) is the parallel flow problem (2.1). To proceed we consider the adjoint problem to (4.11)

$$(D^* + B^*)F^* = 0 \tag{4.14}$$

where $D^*$ and $B^*$ are adjoints to $D$ and $B$, respectively. Now, let $F_o(x)$ be a solution of (4.11). Then so is

$$Z_o(x,\xi) = \mathcal{A}(\xi)F_o(x) \,, \tag{4.15}$$

where $\mathcal{A}(\xi)$ is as yet to be determined. A substitution of (4.15) into (4.12) results in

$$(D+B)Z_1 = -(B_1+D_1)\mathcal{a}(\xi)F_o(x) \quad . \tag{4.16}$$

Let $F,G$ be two vector-functions of $t,x,y$ . Then we define

$$\langle F,G\rangle = \iiint (F\cdot G)dtdxdy \quad ,$$

where $F\cdot G$ is their dot product. In view of (4.14), we have

$$\langle F^*,(D+B)Z_1\rangle = \langle (D^*+B^*)F^*,Z_1\rangle = 0 \quad . \tag{4.17}$$

Thus (4.16) implies that

$$\langle F^*,(B_1+D_1)F_o\rangle \mathcal{a}(\xi) = 0 \tag{4.18}$$

which, noting (4.13), may be written in the form

$$\alpha(\xi)\frac{d\mathcal{a}}{d\xi} + \beta(\xi)\mathcal{a} = 0 \quad . \tag{4.19}$$

This is the envelope equation describing the slow modulation of waves due to the divergence effect.

Alternatively the problem can be treated by first applying a Fourier transform to the systems (4.11) and (4.12). The resulting equations may be reduced to scalar equations for the pressure field as done before. This is particularly convenient when dealing with the radiation problem $(q \neq 0)$ .

The evelope equation (4.19) is difficult to solve analytically. But it is amenable to numerical treatment. For example, in the study of the boundary layer stability, Saric and Nayfeh [17] applied two-variable method to account for the slow divergent flow. They derived an evelope equation similar to (4.19). Their numerical procedures for solution is applicable to the present case.

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## 5. Acoustic Waves in an Axisymmetrical Jet.

The analytical techniques for planar jets may be generalized to an axisymmetrical jet. Introduce the cylindrical coordinates $(x,r,\theta)$ , with the corresponding velocity components $(U,V,W)$ and $(u,v,w)$ .

First consider the cylindrical jet, i.e. $U = U(r)$, $V = W \equiv 0$ . In components, the system (1.1) - (1.2) reads

$$\begin{aligned} &Dp + \gamma p_o\{\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{1}{r}\frac{\partial w}{\partial \theta}\} = q \ , \\ &Du + \frac{1}{\rho_o}\frac{\partial p}{\partial x} + \frac{1}{r}\, v\, \frac{\partial}{\partial r}(rU) = 0 \ , \\ &Dv + \frac{1}{\rho_o r}\frac{\partial}{\partial r}(rp) = 0 \ , \\ &Dw + \frac{1}{\rho_o}\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \ . \end{aligned} \tag{5.1}$$

where $D = (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})$ .

For the homogeneous case, $q \equiv 0$ , we set

$$(p,u,v,w) = (\tilde{p},\tilde{u},\tilde{v},\tilde{w})e^{i(kx+m\theta-\omega t)} \ . \tag{5.2}$$

Then (5.1) is reduced to

$$\begin{aligned} &ik\Pi\tilde{p} + \gamma p_o\{ik\tilde{u} + \frac{1}{r}\frac{\partial}{\partial r}(r\tilde{v}) + \frac{im}{r}\tilde{w}\} = 0 \ , \\ &ik\Pi\tilde{u} + (ik/\rho_o)\tilde{p} + \tilde{v}(\frac{1}{r}\frac{\partial}{\partial r}U) = 0 \ , \\ &ik\Pi\tilde{v} + (1/\rho_o)\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{p}) = 0 \ , \\ &ik\Pi\tilde{w} + (im/\rho_o r)\tilde{p} = 0 \ . \end{aligned} \tag{5.3}$$

After eliminating $\tilde{u},\tilde{v},\tilde{w}$, in favor of $\tilde{p}$ , the system (5.3) yields

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$$M_m(k)\tilde{p} = \eta D_r(\eta^{-1}D_r)\tilde{p} - \sigma(r)D_r\tilde{p} - [\frac{m^2}{r^2} + k^2(1-\mu^2)]\tilde{p} = 0 \, , \tag{5.4}$$

where

$$D_r f = \frac{1}{r}\frac{\partial}{\partial r}(rf), \quad \sigma(r) = (D_r\eta)/\eta \, . \tag{5.5}$$

The solutions of (5.4) for $m = 0,1$, known as the axisymmetric and helical modes, are of special interest. In contrast with (2.10), Eq. (5.4) is naturally associated with an equation of Bessel's type. This is true when $U$ is constant. Then (5.4) becomes a Bessel-like equation, but not exactly.

It is possible to generalize the analytical results in §2 to the present case. By (5.5), the equation (5.4) may be written as

$$\tilde{p}'' + (\frac{1}{r} - 2\eta^{-1}\eta')\tilde{p}' - \{\frac{(m^2+1)}{r^2} + k^2(1-\mu^2) + \frac{2\eta'}{r\eta}\}\tilde{p} = 0 \, . \tag{5.6}$$

Set

$$\tilde{p} = \eta\psi \tag{5.7}$$

in (5.6) to get

$$L_m(k)\psi = \psi'' + \frac{1}{r}\psi' - \{k^2(1-\mu^2) + \frac{(m^2+1)}{r^2}\}\psi + g(r)\psi = 0 \, , \tag{5.8}$$

where

$$g(r) = (\frac{\eta''-2\eta'^2}{\eta} - \frac{\eta'}{r}) \, . \tag{5.9}$$

For instance, for large $k$, we can apply the WKB method or a two-variable expansion. The procedure is similar to what was described in §2. One possible approach is to seek a solution of the form

$$\psi = \alpha(r)H_\lambda^{(1)}(r) \quad \text{or} \quad \beta(r)H_\lambda^{(2)}(r) \, , \tag{5.10}$$

where $\lambda = \sqrt{m^2+1}$ is the order of the Henkel's functions of first or second kind. By substituting (5.10) into (5.8), we may derive equations for $\alpha$ or $\beta$, to which we apply the asymptotic method.

For a divergent jet, the mean flow has the velocity components

$$U = \frac{1}{r}\frac{\partial}{\partial r}(r\phi), \quad V = -\frac{\partial}{\partial x}\phi, \quad W \equiv 0 , \tag{5.11}$$

where $\phi(\xi,r)$, $\xi = \epsilon x$, is the slowly-varying stream function. The governing equations (1.1) - (1.2) for this case takes the form

$$\begin{aligned}
&D_\epsilon p + \gamma p_o\left(\frac{\partial u}{\partial x} + D_r v + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) = q , \\
&D_\epsilon u + \frac{1}{\rho_o}\frac{\partial p}{\partial x} + \epsilon\phi_{12} u + \phi_{22} v = 0 , \\
&D_\epsilon v + \frac{1}{\rho_o} D_r p - \epsilon^2\phi_{11} u - \epsilon\phi_{12} v = 0 , \\
&D_\epsilon w + \frac{1}{\rho_o}\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 ,
\end{aligned} \tag{5.12}$$

where

$$D_\epsilon = \left(\frac{\partial}{\partial t} + \phi_2\frac{\partial}{\partial x} - \epsilon\phi_1 D_r\right) , \tag{5.13}$$

and

$$\phi_1 = \frac{\partial\phi}{\partial\xi}, \quad \phi_2 = D_r\phi, \quad \text{and so on.}$$

Again we introduce two variables $x$ and $\xi = \epsilon x$. Then a two-variable expansion procedure as presented in the previous section may be applied to obtain an approximate solution to the homogeneous form of (5.12), $(q \equiv 0)$.

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APPENDIX

"Reconstruction of the Mutual Coherence Function for a Moving Source"
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