Space Oblique Mercator Projection

Mathematical Development

GEOLOGICAL SURVEY BULLETIN 1518

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By JOHN P. SNYDER

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Refined and improved equations that can be applied to Landsat and other near-polar-orbiting satellite parameters to describe a continuous true-to-scale groundtrack and that will contribute to the automated production of image-base maps

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FOREWORD

On July 25, 1972, the National Aeronautics and Space Administration launched an Earth-sensing satellite named ERTS-1. This satellite, which was renamed Landsat-1 early in 1975, circled the Earth in a near-polar orbit and scanned continuous 185-km-wide swaths of the surface. Landsat-1 was followed by Landsats-2 and -3 and thus established a pattern by which the Earth is to be viewed from space.

Conventional map projections are based on a static Earth, but Landsat imagery involves the parameter of time and the need for a dynamic map projection for its cartographic portrayal.

During 1973, U.S. Geological Survey personnel conceived a new dynamic map projection which would, in fact, display Landsat imagery with a minimum of scale distortion. Moreover, the map projection was continuous so that the 185-km swath would involve no zone boundaries regardless of its length. This was accomplished by defining the groundtrack of the satellite as the centerline of the map projection. The projection was made conformal and named the Space Oblique Mercator (SOM). The SOM was relatively easy to define in geometric terms, and NASA began casting Landsat imagery on it early in 1975. The mathematical definition of the SOM is, however, a complex matter which took several years to resolve. In October 1976, at a symposium on "The Changing World of Geodetic Science" sponsored by the Ohio State University, I expressed concern that the equations for the SOM had not been developed in spite of considerable effort on the part of the Geological Survey. I described the task as "a challenge to the academic community." John P. Snyder, a chemical engineer, was among the audience. Map projections had been Snyder's hobby for some 35 years, and since he had found few that he could not mathematically define, the SOM presented a challenge. Moreover, Snyder had recently bought his first programable pocket calculator and was eager to put it to a real test. By May 1977 Snyder had contacted me on this matter, and I was more than willing to provide him all possible assistance, even though we had just established a contract with John L. Junkins of the University of Virginia to perform the same task.

By January 1978 both Junkins and Snyder, acting independently, had developed mathematical solutions to the SOM. Junkins' solution is intended to be more general, for application to non-circular orbits, and is quite complex. His map projection is not conformal, although it is believed this aspect can be rectified. Snyder, on the other hand, developed a relatively simple set of equations which applies to circular orbits (such as Landsat) and provides a rigorous near-conformal map projection.

Snyder's development of an SOM solution is remarkable because he had little formal training in the sciences related to map projections, he developed and tested his equations using only a pocket calculator, and he worked alone and without compensation. He has now developed more complex equations for non-circular orbits.

Snyder's work has been recognized by the U.S. Geological Survey through the John Wesley Powell Award which was presented to him in 1978. As the award indicates, Snyder "provided the sought-after link by which Earth surface data obtained from orbiting spacecraft can now be transformed to any of the common map projections. This is an essential step in automated mapping systems which we see developing."

Alden P. Colvocoresses

Reston, Va.
July 15, 1981

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By JOHN P. SNYDER

INTRODUCTION

In 1974 A. P. Colvocoresses announced a new map projection which would permit continuous mapping of satellite imagery, especially Landsat. He named it the Space Oblique Mercator (SOM) projection (Colvocoresses, 1974).

Until the SOM, no map projection had been devised which showed the groundtrack of an Earth-orbiting satellite continuously and true to scale. In defining the SOM, it was preferred that the areas scanned by the satellite be mapped conformally with a minimum of scale error. Since the satellite track proceeds essentially along an oblique circumference of the Earth, the oblique Mercator was selected as the closest member of the map projection family.

The mapping problem was simplified by the relatively narrow swath scanned by the Landsat (formerly ERTS) satellite at any given time, officially 100 nautical miles (about 185 km), but the mathematical formulas had not been developed when Colvocoresses presented the projection. He offered a geometric analogy using an oscillating cylinder (Colvocoresses, 1975), and made numerous appeals, but adequate formulas were not developed until 1977.

In May 1977 a rigorous derivation was initiated voluntarily by the writer which was to lead to a satisfactory solution of the SOM projection problem by January 1978. The active interest and encouragement shown by A. P. Colvocoresses throughout the work were extremely important factors in its completion.

Coincidentally, in March 1977 a government contract had been issued to John L. Junkins, then of the University of Virginia, to carry out the derivations. His work (Junkins and Turner, 1977) proceeded independently and almost simultaneously with that of the writer. Junkins took a more general, theoretical approach—his formulas were much more complex, but were designed to be more universal, permitting application to any noncircular orbit and finite scanning time. Moreover, Junkins' equations, in present form, do not result in a conformal projection.

During the present work the Landsat orbit was stipulated to be circular. Actually it was planned to have a nominal eccentricity of 0.0001. Its corrected orbit in 1974 had a reported eccentricity of 0.0020, which is one-fortieth the eccentricity of the Earth ellipsoid (Colvocoresses, 1974). The Landsat orbit model has also been stipulated to have a constant altitude, but this is inconsistent with the circular orbit and an ellipsoidal Earth, and would in fact require an orbit with two perigees and two apogees, an unlikely and certainly non-Keplerian situation; therefore, it was agreed that the circular-orbit stipulation be followed.

It should be stressed that the writer does not claim to have found a perfectly conformal projection. It is only a very close approximation, becoming closer as the groundtrack is approached. The groundtrack itself is perfectly true to scale using either set of formulas. Within 1° of the groundtrack (somewhat farther than the 0.83° limit of the Landsat scan), the scale factors using the final formulas below for the sphere are within six-millionths of what they should be for a truly conformal projection. (For the ellipsoid they are within thirty-millionths of being correct for conformality.) They are thus well within USGS mapping requirements.

The following derivations are arranged to provide four successively more complicated levels of equations, including with each level only those features needed. Formulas for the more complex levels may be simplified fairly readily to the formulas for any of the more elementary levels:

1. The first level assumes the Earth to be a sphere and the orbit to be circular. This level is recommended only for understanding the concepts of the Space Oblique Mercator, since it leads to errors of over one-half percent in tropical areas. Formulas are summarized on p. 3.

2. The second level assumes the Earth to be an ellipsoid and the orbit to be circular. This is recommended for normal use with Landsat and normal mapping from satellites. Formulas are summarized on p. 58–61.

3. The third level assumes the Earth to be an ellipsoid and the orbit to be elliptical, but with an eccentricity of 0.05 or less. It is useful for more rigorous plotting of Landsat and similar data, although the satellites normally vary slightly from a perfectly elliptical planar orbit, giving these formulas more theoretical than practical accuracy. Formulas are summarized on p. 81–82.

4. The final level assumes the Earth to be an ellipsoid and the orbit to be elliptical with any eccentricity. As the eccentricity increases, the number of terms for practical computation increases substantially, but the formulas provide a correctly curved groundtrack which is true

to scale. Mapping may be accomplished as accurately as the mapping for levels 2 and 3 above. Formulas are summarized on p. 104–107.

In each case, the ellipsoid is assumed to be centered at the center of the Earth, and an elliptical orbit is assumed to follow Kepler's laws for two bodies. Scanning is assumed to be instantaneous.

SPACE OBLIQUE MERCATOR PROJECTION FOR THE SPHERE

The final forward equations for the Space Oblique Mercator projection for the sphere are as follows:

$$\frac{x}{R} = \int_0^{\lambda'} \frac{H - S^2}{(1 + S^2)^{1/2}}\, d\lambda' - \frac{S}{(1 + S^2)^{1/2}} \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') \tag{1}$$

$$\frac{y}{R} = (H + 1) \int_0^{\lambda'} \frac{S}{(1 + S^2)^{1/2}}\, d\lambda' + \frac{1}{(1 + S^2)^{1/2}} \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') \tag{2}$$

where $S = (P_2/P_1) \sin i \cos \lambda'$ (3)

$H = 1 - (P_2/P_1) \cos i$ (4)

$\tan \lambda' = \cos i \tan \lambda_t + \sin i \tan \phi / \cos \lambda_t$ (5)

$\sin \phi' = \cos i \sin \phi - \sin i \cos \phi \sin \lambda_t$ (6)

$\lambda_t = \lambda + (P_2/P_1)\, \lambda'$. (7)

ϕ = geodetic (or geographic) latitude.

λ = geodetic (or geographic) longitude, measured east from intersection of satellite orbit with Earth's Equator, as satellite crosses in northerly direction (the ascending node).

P_2 = time required for revolution of satellite (103.267 min. for Landsat).

P_1 = length of Earth's rotation with respect to the precessed ascending node. For Landsat, the satellite orbit is Sun-synchronous; that is, it is always the same with respect to the Sun, equating P_1 to the solar day (1,440 min.). (Landsat is orbited to complete exactly 251 revolutions in 18 days.)

i = angle of inclination between the plane of the Earth's Equator and the plane of the satellite orbit, counterclockwise from the Earth's Equator to the orbital plane at the ascending node (99.092° for Landsat).

R = radius of Earth, or of globe at scale of map.

x, y = rectangular coordinates on the map.

The significance of S, H, λ', ϕ', λ_t, as well as calculating techniques for the integrals of equations (1) and (2), iteration and quadrant adjustment for equation (5), inverse equations, and equations for scale factors and distortion will all be developed below. Inverse equations are numbered (80), (87), (19), (92), and (93). Fourier equivalents are equations (73) through (77), (86), and (90).

The regular Mercator projection for a stationary sphere is frequently expressed with the following formulas (Thomas, 1952, p. 2):

$$x = R\lambda \tag{8}$$

$$y = R \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi) \tag{9}$$

where λ is measured in this case east from the origin of the x and y coordinates, and λ and ϕ are given in radians, as are all angles throughout this paper, except where the degree symbol (°) or word is used. Either radians or degrees may be assumed, of course, if used with a trigonometric function such as $\cos \phi$.

The regular Mercator projection consists of meridians projected onto a cylinder tangent to the globe at the equator. The parallels cannot be projected directly from the globe, but must be spaced for conformality according to equation (9). For the transverse Mercator projection, the cylinder is rotated so that it is tangent along a meridian from pole to pole. For the oblique Mercator projection, the cylinder is rotated so that it is tangent along a great circle which crosses the equator at an angle. Direct equations may be provided which show little resemblance to equations (8) and (9) (Thomas, 1952, p. 6). It is preferable, however, for the sake of subsequent derivations to use intermediate transformation equations, substituting λ' and ϕ' from equations below in place of λ and ϕ of (8) and (9).

Referring to figure 1, an orthographic view of the globe centered at the crossing of the transformed equator with the actual equator, the equator of the globe CBDA is AB, the regular north pole is C, and the regular south pole is D. The transformed equator (inclined i to the regular equator) is NG; the transformed poles are H and J. Angle FOB is geographic latitude ϕ, and EF is the projection of the parallel. Arc CMT is a meridian at longitude λ with respect to central meridian CD, so that angle O′C′M′ (polar view) or spherical angle OCM (equatorial view) equals λ. Point M on the transformed globe has transformed latitude ϕ', represented by KL, and transformed longitude λ', represented by arc HMU, with spherical angle OHM equal to λ'.

In the spherical triangle HCM, side HC is an arc of length i, side CM has a length $(90° - \phi)$, and side HM has a length $(90° - \phi')$. Angle C or (HCM) equals $(90° + \lambda)$, and angle H or (CHM) is $(90° - \lambda')$. From the Law of Sines, as applied to this triangle,

$$\sin C/\sin (HM) = \sin H/\sin (CM) \tag{10}$$

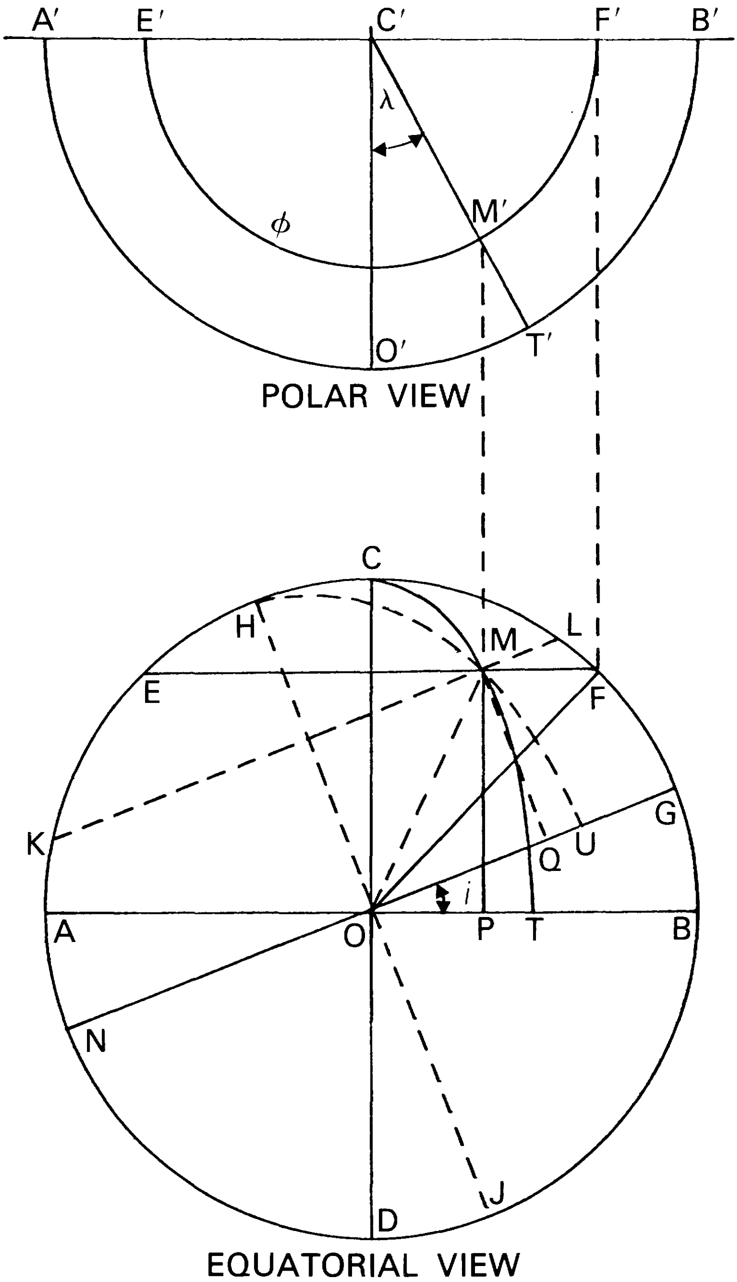

FIGURE 1.—Orthographic view of the Earth.

or $$\sin(90^\circ + \lambda)/\sin(90^\circ - \phi') = \sin(90^\circ - \lambda')/\sin(90^\circ - \phi) \tag{11}$$

or $$\cos\lambda/\cos\phi' = \cos\lambda'/\cos\phi \tag{12}$$

or $$\cos\lambda' = \cos\phi\cos\lambda/\cos\phi' \tag{13}$$

From the Law of Cosines,

$$\cos(\mathrm{HM}) = \cos(\mathrm{HC})\cos(\mathrm{CM}) + \sin(\mathrm{HC})\sin(\mathrm{CM})\cos \mathrm{C} \tag{14}$$

or $$\cos(90^\circ - \phi') = \cos i\cos(90^\circ - \phi) + \sin i\sin(90^\circ - \phi)\cos(90^\circ + \lambda) \tag{15}$$

or $$\sin\phi' = \cos i\sin\phi - \sin i\cos\phi\sin\lambda \tag{16}$$

Using the Law of Cosines for a different side,

$$\cos(\mathrm{CM}) = \cos(\mathrm{HC})\cos(\mathrm{HM}) + \sin(\mathrm{HC})\sin(\mathrm{HM})\cos \mathrm{H} \tag{17}$$

or $$\cos(90^\circ - \phi) = \cos i\cos(90^\circ - \phi') + \sin i\sin(90^\circ - \phi')\cos(90^\circ - \lambda') \tag{18}$$

or $$\sin\phi = \cos i\sin\phi' + \sin i\cos\phi'\sin\lambda' \tag{19}$$

or $$\sin\lambda' = (\sin\phi - \cos i\sin\phi')/\sin i\cos\phi' \tag{20}$$

Dividing (20) by (13), then substituting from (16),

$$\begin{aligned}\tan\lambda' &= (\sin\phi - \cos i\sin\phi')/\sin i\cos\phi\cos\lambda \\ &= [\sin\phi - \cos i(\cos i\sin\phi - \sin i\cos\phi\sin\lambda)]/\sin i\cos\phi\cos\lambda \\ &= [\sin\phi(1 - \cos^2 i) + \sin i\cos i\cos\phi\sin\lambda]/\sin i\cos\phi\cos\lambda \\ &= \sin i\tan\phi/\cos\lambda + \cos i\tan\lambda \end{aligned} \tag{21}$$

Equations (16) and (21) provide the transformation equations in a form suitable for subsequent derivations. This transformed longitude λ' is the angular distance along the transformed equator, measured north from the same intersection of equators from which λ is measured. The transformed latitude ϕ' is the angular distance from the transformed equator, measured positively in the same direction as i, namely counterclockwise from the transformed equator, viewed from the ascending node.

If a satellite is following a uniform circular orbit inclined i to the Earth's Equator and the Earth is assumed to be a stationary sphere, the groundtrack will follow the transformed equator, and λ' will be directly proportional to the time elapsed since the satellite crossed the plane of the Earth's Equator in a northerly direction; that is, the ascending node. (This ascending node for Landsat is on the dark side of the Earth; imaging occurs as the satellite is proceeding southward on the opposite, lighted face.) The angular distance along the scan line will be ϕ', measured from the groundtrack. Scanning by the satellite is assumed to be instantaneous throughout these derivations.

For a stationary spherical Earth with a revolving satellite, these formulas are exact. Let us think, however, of the satellite orbit as fixed in space, and the satellite revolving and the Earth rotating with respect to the orbit. At the time the satellite has reached λ', starting from zero, the parallels will not have changed latitude, but the meridians will have rotated so that, from the viewpoint of the satellite in space, the actual longitude λ will appear along the scanning line at the point where some other longitude λ_t would have appeared if the Earth had remained stationary. The "satellite-apparent" longitude λ_t is thus the geodetic longitude increased by the angle of Earth rotation at the time the satellite has reached λ', or, as listed in the earlier summary (p. 3),

$$\lambda_t = \lambda + (P_2/P_1)\lambda' \tag{7}$$

where P_2 and P_1 are the relative times of orbit and Earth rotation as described after the original listing of equation (7).

The term λ_t should then be substituted for λ in transformation equations (21) and (16) to account for Earth rotation, to give equations (5) and (6) listed earlier:

$$\tan \lambda' = \cos i \tan \lambda_t + \sin i \tan \phi/\cos \lambda_t \tag{5}$$

$$\sin \phi' = \cos i \sin \phi - \sin i \cos \phi \sin \lambda_t \tag{6}$$

Since λ_t is a function of λ', finding λ' for a given ϕ and λ involves trial and error (iteration), and λ' must be calculated before ϕ' is determined so that the proper value of λ_t may be inserted into equation (6), but equations (5), (6), and (7) are exact for the transformation in the case of the rotating sphere. Iteration techniques are discussed on p. 30.

It is not appropriate, however, to substitute λ' and ϕ' from equations (5) and (6) in place of the ϕ and λ of equations (8) and (9) for the rotating sphere and revolving satellite. This substitution is only suitable for the stationary sphere. If this were done for the rotating sphere, the satellite groundtrack would be shown as a straight line and the scan lines as a series of parallel lines perpendicular to the groundtrack. Such a groundtrack would have neither conformality nor true scale.

Due to Earth rotation, the direction of the Landsat groundtrack on the Earth is about 86° from that of the scanlines at the Earth's Equator, even though the orbital track is perpendicular to the scanlines in space. The groundtrack becomes perpendicular to the scanlines when the satellite is closest to the poles (at 180° − 99.092° or about 81° North or South latitude).

To derive the curved satellite groundtrack, the instantaneous slope of the groundtrack relative to the parallels of latitude on the sphere is determined first:

For the groundtrack, ϕ' of equation (6) must be zero, or

$$\cos i \sin \phi_o = \sin i \cos \phi_o \sin \lambda_t \tag{22}$$

$$\sin \lambda_t = \tan \phi_o / \tan i \tag{23}$$

where ϕ_o is the latitude ϕ at the groundtrack. Differentiating,

$$\cos \lambda_t d\lambda_t = (\sec^2\phi_o / \tan i)\, d\phi_o \tag{24}$$

Let i' be the inclination of the groundtrack to a given parallel of latitude if the Earth were stationary. From figure 2, showing an element AB of this Earth-stationary groundtrack near latitude ϕ_o and longitude λ, BE $= d\phi_o$ and AE $= \cos \phi_o d\lambda$ if the sphere is given unit radius. For the stationary situation $\lambda = \lambda_t$, so AE $= \cos \phi_o d\lambda_t$. Then

$$\tan i' = d\phi_o / \cos \phi_o\, d\lambda_t \tag{25}$$

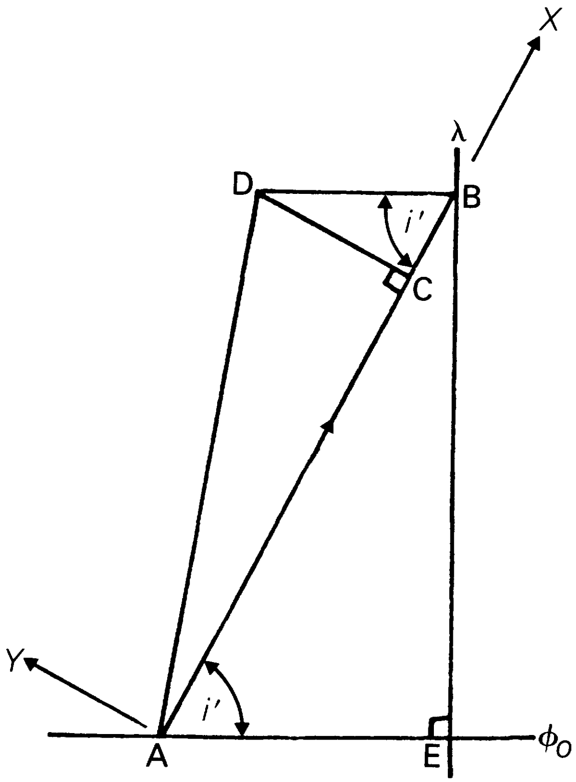

FIGURE 2.—Element (AB) of an Earth-stationary groundtrack near latitude ϕ_o and longitude λ.

Substituting from equation (24),

$$\tan i' = \cos \lambda_t \tan i \cos^2\phi_o \, d\lambda_t / \cos \phi_o \, d\lambda_t$$
$$= \tan i \cos \phi_o \cos \lambda_t \qquad (26)$$

Working further with figure 2, the following elements may be defined:

AB $= d\lambda'$, element of satellite motion along groundtrack, disregarding Earth rotation, during a given time interval dt.

BD $= (P_2/P_1) \cos\phi_o \, d\lambda'$, element of distance resulting from change of longitude due to Earth rotation, along a given parallel of latitude during dt.

AD = projection of line AB onto map to account for change of longitude from B to D during dt.

$\left.\begin{array}{l} \text{AC} = dx \\ \text{CD} = dy \end{array}\right\}$ elements of rectangular coordinates for groundtrack projection AD on SOM map.

Then $$\text{AC} + \text{CB} = \text{AB}$$

or $$dx + \text{CB} = d\lambda'$$

$$\text{CD} = \text{BD} \sin i'$$
$$dy = (P_2/P_1) \cos \phi_o \sin i' d\lambda' \qquad (27)$$
$$\text{CB} = \text{BD} \cos i'$$

Then $$dx = d\lambda'[1 - (P_2/P_1) \cos \phi_o \cos i'] \qquad (28)$$

To eliminate i', rearrange equations (23) and (26) as follows:

$$\tan \phi_o = \tan i \sin \lambda_t \qquad (29)$$
$$\cos \phi_o = 1/(1 + \tan^2\phi_o)^{1/2}$$
$$= \cos i/(1 - \sin^2 i \cos^2\lambda_t)^{1/2} \qquad (30)$$
$$\tan i' = \tan i \cos i \cos \lambda_t/(1 - \sin^2 i \cos^2\lambda_t)^{1/2}$$
$$= \sin i \cos \lambda_t/(1 - \sin^2 i \cos^2\lambda_t)^{1/2} \qquad (31)$$
$$\sin i' = \tan i'/(1 + \tan^2 i')^{1/2}$$
$$= \sin i \cos \lambda_t \qquad (32)$$
$$\cos i' = (1 - \sin^2 i \cos^2\lambda_t)^{1/2} \qquad (33)$$

Substituting from (33) into (28),

$$dx = d\lambda'[1 - (P_2/P_1) \cos \phi_o (1 - \sin^2 i \cos^2\lambda_t)^{1/2}] \qquad (34)$$

Substituting from (30),

$$dx = d\lambda'[1 - (P_2/P_1) \cos i] \qquad (35)$$

Integrating,

$$x = [1 - (P_2/P_1) \cos i] \, \lambda' \qquad (36)$$

Letting $$H=1-(P_2/P_1)\cos i \tag{4}$$

and reentering the radius R of the sphere,

$$x=RH\lambda' \tag{37}$$

Substituting from (32) and (30) into (27),

$$dy=[(P_2/P_1)\cos i\sin i\cos\lambda_t/(1-\sin^2 i\cos^2\lambda_t)^{1/2}]\,d\lambda' \tag{38}$$

Substituting for ϕ from equation (23) into (5),

$$\begin{aligned}\tan\lambda' &= \cos i\tan\lambda_t+\sin i\sin\lambda_t\tan i/\cos\lambda_t\\ &=\tan\lambda_t\,(\cos i+\sin^2 i/\cos i)\\ &=\tan\lambda_t\,(\cos^2 i+\sin^2 i)/\cos i\\ &=\tan\lambda_t/\cos i\end{aligned} \tag{39}$$

$$\tan\lambda_t=\tan\lambda'\cos i \tag{40}$$

$$\cos\lambda_t=1/(1+\tan^2\lambda'\cos^2 i)^{1/2} \tag{41}$$

Substituting from (41) into (38),

$$\begin{aligned}dy&=[(P_2/P_1)\cos i\sin i/(1/\cos^2\lambda_t-\sin^2 i)^{1/2}]\,d\lambda'\\ &=[(P_2/P_1)\cos i\sin i/(1+\tan^2\lambda'\cos^2 i-\sin^2 i)^{1/2}]\,d\lambda'\\ &=[(P_2/P_1)\cos i\sin i/(\cos^2 i\sec^2\lambda')^{1/2}]\,d\lambda'\\ &=(P_2/P_1)\sin i\cos\lambda' d\lambda'\end{aligned} \tag{42}$$

Comparing equation (3), $dy=Sd\lambda'$

Integrating and including R,

$$y=R\,(P_2/P_1)\sin i\sin\lambda' \tag{43}$$

Equations (37) and (43) provide coordinates for the groundtrack with satellite scanlines running perpendicular to the X axis. The transformed parallels of latitude ϕ' are at a distance y in equation (9) from the groundtrack, so equation (9) must be added to equation (43) for the full equation for y. Thus,

$$x=RH\lambda' \tag{37}$$

$$y=R\,[(P_2/P_1)\sin i\sin\lambda'+\ln\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi')] \tag{44}$$

where symbols are as given after equations (1) through (7), and H is found from equation (4).

Equations (37) and (44) provide a groundtrack which is sinusoidal, true to scale, and with conformality (letting $\phi'=0$). It is inclined about 4° to the X axis at the crossing of the equator, as the following calculation shows. Dividing equation (42) by (35),

$$dy/dx=(P_2/P_1)\sin i\cos\lambda'/[1-(P_2/P_1)\cos i]. \tag{45}$$

If $\lambda'=0$, at the equatorial crossing,

$$dy/dx=(P_2/P_1)\sin i/[1-(P_2/P_1)\cos i].$$

If $P_2/P_1 = 18/251$, and $i = 99.092°$, as indicated following equations (1) through (7),

$$dy/dx = 0.07002,$$

of which the arctangent is 4.01°, the angle of slope to the X axis.

Time is proportional to x, but it is not quite proportional to distance along the groundtrack and should not be because of the effect of Earth rotation.

This change from a straight to a curved transformed equator (or groundtrack) is not enough to place scale errors within the satellite swath at less than the desired limit of about one part in 10,000 for mapping accuracy, even though the groundtrack itself is satisfactory. The normal scale factor for a tangent conformal cylindrical projection, whether regular, oblique, or transverse Mercator, is sec ϕ', or 1.000152 at 1° from the transformed equator (somewhat farther than the 0.83° limit of the Landsat scan). Dividing this factor out of the values of h and k in table 1 for various points along one quadrant of the projection based on formulas (37) and (44), and using a scale factor derivation following the same pattern as the differentiation discussed later, it is found that residual errors are over one part in 1,000 at the polar approaches when $\phi' = \pm 1°$. Since the parallels of latitude run parallel to the groundtrack at the polar approach, it may be noted that k is too small at $\phi' = 1°$ (away from the north pole) at $\lambda' = 90°$, and too large at $\phi' = -1°$ (toward the north pole), and by about the same amount: 0.998930/1.000152=0.998778, while 1.001375/1.000152=1.001223, each

TABLE 1.—*Scale factors and angular deviation 1° ϕ' from the Landsat groundtrack given by preliminary formulas for the spherical Space Oblique Mercator projection*

[Scanlines drawn parallel. From equations 37 and 44]

λ'	1°			−1°		
	h	k	ω	h	k	ω
0°	1.000154	1.000151	0.0006°	1.000154	1.000151	0.0006°
15°	.999847	1.000142	.0181	1.000462	1.000159	.0181
30°	.999566	1.000128	.0350	1.000747	1.000169	.0350
45°	.999339	1.000102	.0495	1.000983	1.000187	.0495
60°	.999212	1.000035	.0606	1.001131	1.000233	.0607
75°	.999350	.999776	.0676	1.001059	1.000428	.0677
90°	1.000152	.998930	.0700	1.000152	1.001375	.0700

Note: h = scale factor along meridian of longitude
k = scale factor along parallel of latitude
ω = maximum angular deviation.
λ' = angular distance along groundtrack from ascending node.
ϕ' = angular distance from groundtrack (positive in direction away from north pole).
On groundtrack $\phi' = 0$, $h = k = 1.0$ and $\omega = 0.0°$.

quotient being about 0.001223 from unity. The maximum angular deformation is consequently almost identical each place: 0.0700°. At the equatorial crossing ($\lambda' = 0°$), this discrepancy disappears, so that k is less than h by 0.000003, whether ϕ' is 1° or $-1°$.

Therefore, after much conceptualizing, the retrospectively simple answer became apparent: the groundtrack should be bent more sharply on the projection in the polar areas but not in the equatorial areas, and the scanlines should continue to intersect the groundtrack at the same angles as before to prevent distortion along the groundtrack. The scanlines would thus be skewed with respect to the Y axis.

By observing the magnitude of the residual scale error at various values of λ', it was decided to double the slope of the groundtrack and to rotate the scanlines so that they still intersected the groundtrack at the proper angle. This greatly reduced the residual errors, but it was found that doubling the angle of the slope instead of doubling the slope itself resulted in both simpler formulas and a slightly improved scale factor. This form had scale-error residuals (after dividing out sec λ') of up to 0.000020. The last slight modification brought the maximum residual down to about 0.000006, without complicating the formulas. The approach was strictly empirical, with the preceding formulas and the desire to retain the exact scale and conformality of the groundtrack already achieved.

Following is the derivation of equations for x and y retaining the proper groundtrack, but altering its slope to the final form.

By substituting equations (3) and (4) into equation (45),

$$dy/dx = S/H \tag{46}$$

for the groundtrack of equations (37) and (44). Calling the angle of this slope $\theta = \arctan(S/H)$, this angle is to be increased by angle $\theta_1 = \arctan S$. Temporarily calling the revised coordinates x' and y', to contrast with the x and y of (37) and (44), the new angle of slope,

$$\theta' = \arctan(dy'/dx')$$
$$= \theta + \theta_1.$$

Since $$\tan\theta' = (\tan\theta + \tan\theta_1)/(1 - \tan\theta\tan\theta_1)$$

then $$dy'/dx' = (S/H + S)/(1 - S^2/H)$$
$$= (S + HS)/(H - S^2) \tag{47}$$

To maintain the same scale along the groundtrack,

$$(dx)^2 + (dy)^2 = (dx')^2 + (dy')^2$$

Rearranging,

$$[1 + (dy/dx)^2]\,(dx)^2 = [1 + (dy'/dx')^2]\,(dx')^2$$
$$(dx')^2 = \left\{[1 + (dy/dx)^2]/[1 + (dy'/dx')^2]\right\}(dx)^2$$
$$= \left\{[1 + (S/H)^2]/[1 + ((S + HS)/(H - S^2))^2]\right\}(dx)^2$$

This simplifies to

$$dx' = [(H - S^2)/H(1+S^2)^{1/2}]dx \tag{48}$$

Substituting from equation (35) into (48),

$$\begin{aligned} dx' &= [(H-S^2)\ H/H\ (1+S^2)^{1/2}]d\lambda' \\ &= [(H-S^2)/(1+S^2)^{1/2}]d\lambda' \end{aligned} \tag{49}$$

Integrating,

$$x' = \int_0^{\lambda'} [(H-S^2)/(1+S^2)^{1/2}]d\lambda' \tag{50}$$

Substituting equation (49) into (47),

$$\begin{aligned} dy' &= [(S+HS)/(H-S^2)]dx' \\ &= [S\ (1+H)\ (H-S^2)/(H-S^2)\ (1+S^2)^{1/2}]d\lambda' \\ &= [S\ (H+1)/(1+S^2)^{1/2}]d\lambda' \end{aligned} \tag{51}$$

Integrating,

$$y' = (H+1) \int_0^{\lambda'} [S/(1+S^2)^{1/2}]d\lambda' \tag{52}$$

For points away from the satellite path, the second term of equation (44) must be used, but corrected for inclination at angle θ_1 with respect to vertical. Since $\tan \theta_1 = S$, as above,

$$\sin \theta_1 = \tan \theta_1/(1+\tan^2\theta_1)^{1/2} = S/(1+S^2)^{1/2}$$
$$\cos \theta_1 = 1/(1+\tan^2\theta_1)^{1/2} = 1/(1+S^2)^{1/2}$$

Thus, from equation (50) must be subtracted $\sin \theta_1$ times $\ln \tan (\frac{1}{4}\pi + \frac{1}{2}\phi')$, and to equation (52) must be added $\cos \theta_1$ times $\ln \tan (\frac{1}{4}\pi + \frac{1}{2}\phi')$. Then the final equations for the new x and y, eliminating the prime marks and including the radius R of the sphere, become

$$x/R = \int_0^{\lambda'} \frac{H-S^2}{\sqrt{1+S^2}}\, d\lambda' - \frac{S}{\sqrt{1+S^2}} \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') \tag{1}$$

$$y/R = (H+1) \int_0^{\lambda'} \frac{S}{\sqrt{1+S^2}}\, d\lambda' + \frac{1}{\sqrt{1+S^2}} \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') \tag{2}$$

The resulting scale factors, after dividing by $\sec \phi'$, as calculated below, are within 0.000006 of correct for a conformal projection, in a zone 1° on either side of the satellite groundtrack. This is still not perfectly conformal, but for the sphere the discrepancy within the required scanning range is negligible. The X axis is not now proportional to time, but λ' remains so. For Landsat, the poles are near to, but not on, the new X axis. The scanlines vary between $\pm 4.05°$ from vertical, at counterclockwise angle arctan S, and $\pm 4.01°$ (at clockwise angle arctan (S/H)) from normal to the satellite groundtrack, which now is inclined 8.06° to the X axis at the Equator. The 4.01° is as calculated following equation (45), while 4.05° is the arctan of $S = (18/251) \times \sin 99.092° = 0.07081$. Figure 3 shows a 30° graticule extended to most of

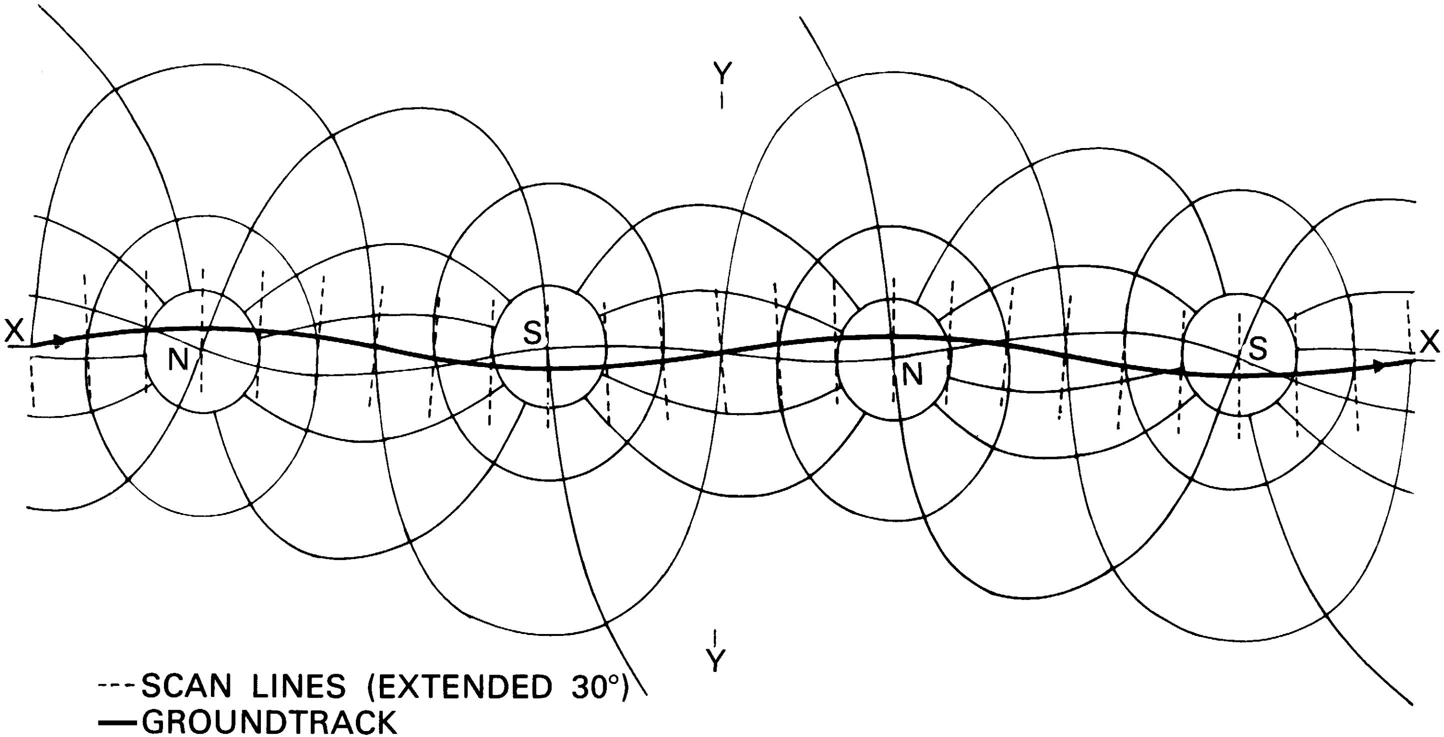

Figure 3.—Space Oblique Mercator projection applied to the sphere for two orbits. Heavy line indicates satellite groundtrack; dashed lines, scan; orbit inclination, 99°; period, 103 minutes.

the globe for two orbits. The progressive rotation of meridians may be observed.

Numerical integration is now required, but this can be reduced to a one-time calculation of constants for a Fourier series for any given satellite orbit.

FOURIER SERIES

The Fourier series for a repeating function may be found thus:

$$F(x)=\tfrac{1}{2}a_0+a_1\cos\frac{2\pi x}{T}+a_2\cos\frac{4\pi x}{T}+\ldots+a_m\cos\frac{2\pi mx}{T}+\ldots$$
$$+b_1\sin\frac{2\pi x}{T}+\ldots+b_n\sin\frac{2\pi nx}{T}+\ldots \quad (53)$$

where

$$a_0=\frac{2}{T}\int_C^{C+T} F(x)\,dx \quad (54)$$

$$a_m=\frac{2}{T}\int_C^{C+T} F(x)\cos\frac{2\pi mx}{T}\,dx \quad (55)$$

$$b_n=\frac{2}{T}\int_C^{C+T} F(x)\sin\frac{2\pi nx}{T}\,dx \quad (56)$$

and $F(x+T)=F(x)$, with C and T constants. (57)

Taking the first term of equation (1),

$$x_a/R=\int_0^{\lambda'}[(H-S^2)/(1+S^2)^{1/2}]d\lambda' \quad (58)$$

Let

$$F_x(\lambda')=(H-S^2)/(1+S^2)^{1/2} \quad (59)$$

The repeating cycle for $F(\lambda')$, for use in equation (57), is $T=2\pi$. Let $C=0$. Letting $\tfrac{1}{2}a_0=B$, and because of symmetry integrating only to $\pi/2$ and multiplying by 4, equation (54) becomes

$$B=(2/\pi)\int_0^{\pi/2} F_x(\lambda')\,d\lambda' \quad (60)$$

The term $F(x)$ in equations (55) through (57) should in theory be $[F(\lambda')-B]$ to remove the noncyclical portion of x. After first publication of the formulas (Snyder, 1978b), it was realized that B in these cases cancels out and may be omitted from equations for a_m, etc. Substituting in equations (53) through (56),

$$F_x(\lambda')=B+\sum_{m=1}^{m} a_m\cos\frac{2\pi m\lambda'}{2\pi}+\sum_{n=1}^{n} b_n\sin\frac{2\pi n\lambda'}{2\pi} \quad (61)$$

where

$$a_m=\frac{1}{\pi}\int_0^{2\pi}F_x(\lambda')\cos m\lambda' d\lambda' \quad (62)$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} F_x(\lambda')\sin n\lambda' d\lambda' \tag{63}$$

In calculating these constants, it is found that b_n, and a_1, a_3, a_5, etc., are all zero. Therefore, eliminating the unnecessary terms from (61),

$$F_x(\lambda')=B+\sum_{m=2}^{m} a_m \cos m\lambda' \tag{64}$$

where m is even. Equation (58) requires the integration of $F_x(\lambda')$, or of (64). Since

$$\int \cos m\lambda'\, d\lambda'=\frac{1}{m}\sin m\lambda' \tag{64a}$$

we may combine equations (58), (59), (64) and (64a) for the following:

$$\frac{x_a}{R}=B\lambda'+\sum_{m=2}^{m}\frac{a_m}{m}\sin m\lambda' \tag{65}$$

where a_m is found from equation (62), m is an even number, and B is found from (60). It is found that three terms of the series give 10-place accuracy for Landsat and other orbits tested. As a multiplier of B, λ' must be in radians, or the calculated value of B must be multiplied by $\pi/180$ to use λ' in degrees.

For similar handling of the first term of (2), inserting $(H+1)$ into the integral (it is a constant for the sphere, but a variable for the ellipsoid):

$$y_a/R=\int_0^{\lambda'}[(H+1)S/(1+S^2)^{1/2}]d\lambda' \tag{66}$$

let $$F_y(\lambda')=(H+1)S/(1+S^2)^{1/2} \tag{67}$$

As in equations (53) and (61),

$$F_y(\lambda')=\tfrac{1}{2}a_0+\sum_{m=1}^{m} a_m\cos m\lambda'+\sum_{n=1}^{n} b_n \sin n\lambda' \tag{68}$$

where $$a_0=\frac{1}{\pi}\int_0^{2\pi} F_y(\lambda')d\lambda' \tag{69}$$

$$a_m=\frac{1}{\pi}\int_0^{2\pi} F_y(\lambda')\cos m\lambda' d\lambda' \tag{70}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} F_y(\lambda')\sin n\lambda' d\lambda' \tag{71}$$

This time, it is found that a_0, b_n and a_2, a_4, a_6, etc., are all zero. With integration as before, combining equations (66), (67), (68) and (64a),

$$\frac{y_a}{R}=\sum_{m=1}^{m}\frac{a_m}{m}\sin m\lambda' \tag{72}$$

where a_m is found from equation (70) this time, and m is an odd number. It is found that three terms give 10-place accuracy for this orbit.

To recapitulate, the Fourier equations as applied to equations (1) and (2) give the following (incorporating (59), (60), (62), (65), (67), (70) and (72), and consolidating constants):

$$\frac{x}{R}=B\lambda'+A_2 \sin 2\lambda'+A_4 \sin 4\lambda'+\ldots$$

$$-\frac{S}{\sqrt{1+S^2}} \ln \tan \left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right) \tag{73}$$

$$\frac{y}{R}=C_1 \sin \lambda'+C_3 \sin 3\lambda'+\ldots$$

$$+\frac{1}{\sqrt{1+S^2}} \ln \tan \left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right) \tag{74}$$

where

$$B=\frac{2}{\pi}\int_0^{\pi/2} \frac{H-S^2}{\sqrt{1+S^2}}\, d\lambda' \tag{75}$$

$$A_n=\frac{1}{\pi n}\int_0^{2\pi} \frac{H-S^2}{\sqrt{1+S^2}} \cos n\lambda' d\lambda' \tag{76}$$

$$C_n=\frac{H+1}{\pi n}\int_0^{2\pi} \frac{S}{\sqrt{1+S^2}} \cos n\lambda' d\lambda' \tag{77}$$

It is recommended that Simpson's rule be used for calculating any of the numerical integrations encountered in the equations in this study. In the general forms, $y=\int_a^b F(x)dx$, the interval from $x=a$ to $x=b$ must be divided into an even number n of equal intervals, each Δx wide, and the value of $F(x)$ is calculated for each value of x. Then

$$y=\frac{\Delta x}{3}\left[F(x_a)+4F(x_a+\Delta x)+2F(x_a+2\Delta x)+4F(x_a+3\Delta x)\right.$$
$$\left.+2F(x_a+4\Delta x)+\ldots+4F(x_b-\Delta x)+F(x_b)\right] \tag{78}$$

This is increasingly accurate as Δx is reduced, but is far more accurate than trapezoidal integration. In the case of the Landsat orbit and probably most others, a sufficiently small interval for sphere or ellipsoid is 9° in λ' for 10-place accuracy in determining all Fourier coefficients, while 15° is a satisfactory interval for 7- to 8-place accuracy, which is sufficient for most purposes.

Another interesting aspect of the Fourier constants for the spherical or ellipsoidal SOM formulas is that while the 2π or 360° calculation is necessary to determine whether the constant is zero or not, once this is determined the needed constant may be calculated by integrating from 0° to 90° or $\pi/2$, instead of to 360°, and then multiplying the answer by 4. This results from symmetry of the quadrants.

Thus the computer time for calculating the Fourier constants for the SOM needs to be only about 1/36, using a $\Delta\lambda'$ of 9° and one quadrant, instead of say 1° intervals for the entire circumference.

For the Landsat on the sphere, using $P_2/P_1=18/251$ and $i=99.092°$, the Fourier constants are found to be as follows (showing decimal places in excess of those needed):

$$
\begin{aligned}
B &= 1.0075654142 \quad &&\text{for } \lambda' \text{ in radians}\\
&= 0.0175853339 \quad &&\text{for } \lambda' \text{ in degrees}\\
A_2 &= -0.0018819808\\
A_4 &= 0.0000006868\\
A_6 &= -0.0000000004\\
C_1 &= 0.1421597410\\
C_3 &= -0.0000296182\\
C_5 &= 0.0000000167
\end{aligned}
$$

INVERSE EQUATIONS

Equations (1) through (7) give x and y in terms of ϕ' and λ', and ϕ' and λ' in terms of ϕ and λ. It is also desirable to have the inverse. The equations for ϕ' and λ' in terms of x and y will be derived first. The equations are easily inverted, although Fourier series equivalents are the most practical means of calculating with the resulting forms.

Multiplying equation (2) by S, we obtain

$$yS/R=S(H+1)\int_0^{\lambda'} \frac{S}{\sqrt{1+S^2}}\,d\lambda'+\frac{S}{\sqrt{1+S^2}}\ln\tan\left(\frac{\pi}{4}+\frac{\phi'}{2}\right) \tag{79}$$

Adding this equation to equation (1), the last terms, and thus ϕ', cancel out;

$$\frac{x+yS}{R}=\int_0^{\lambda'} \frac{H-S^2}{\sqrt{1+S^2}}\,d\lambda'+S(H+1)\int_0^{\lambda'} \frac{S}{\sqrt{1+S^2}}\,d\lambda' \tag{80}$$

Since ϕ' is eliminated, λ' may be found in terms of x and y from this equation, but both numerical integration and iteration are involved, resulting in excessive computer time. Thus equation (80) is an almost academic equation of little practical use. By changing it to Fourier series, however, repeated numerical integration is eliminated, and while trial and error (or iteration) is required for the remaining equation, convergence is rapid. To convert to the Fourier series, substitute the first portions of equations (73) and (74) in place of the respective integrals of (80), multiplying the series of (74) by S:

$$\frac{x+yS}{R}=B\,\lambda'+A_2\sin 2\lambda'+A_4\sin 4\lambda'+\ldots+S(C_1\sin\lambda'+C_3\sin 3\lambda'+C_5\sin 5\lambda'+\ldots) \tag{81}$$

Since $S = D \cos \lambda'$, where $D = (P_2/P_1) \sin i$, from equation (3),

$$\frac{x+yS}{R} = B\lambda' + A_2 \sin 2\lambda' + A_4 \sin 4\lambda' + \ldots$$
$$+ D \cos \lambda' (C_1 \sin \lambda' + C_3 \sin 3\lambda' + C_5 \sin 5\lambda' + \ldots) \quad (82)$$

Taking a standard trigonometric identity,

$$\sin a + \sin b = 2 \sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b) \quad (83)$$

let $\quad a = (m+n)\lambda'$ and $b = (m-n)\lambda'$

Substituting, $\sin (m+n)\lambda' + \sin (m-n)\lambda' = 2 \sin \tfrac{1}{2}(2m)\lambda' \cos \tfrac{1}{2}(2n)\lambda'$

or

$$\sin m\lambda' \cos n\lambda' = \tfrac{1}{2} \sin (m+n)\lambda' + \tfrac{1}{2} \sin (m-n)\lambda' \quad (84)$$

Applying (84) to (82),

$$\frac{x+yS}{R} = B\lambda' + A_2 \sin 2\lambda' + A_4 \sin 4\lambda' + A_6 \sin 6\lambda' + \ldots$$
$$+ \tfrac{1}{2}DC_1 \sin 2\lambda' + \tfrac{1}{2}DC_3 (\sin 4\lambda' + \sin 2\lambda')$$
$$+ \tfrac{1}{2}DC_5 (\sin 6\lambda' + \sin 4\lambda') + \ldots$$
$$= B\lambda' + E_2 \sin 2\lambda' + E_4 \sin 4\lambda' + E_6 \sin 6\lambda' + \ldots \quad (85)$$

where

$$E_2 = A_2 + \tfrac{1}{2}D(C_1 + C_3)$$
$$E_4 = A_4 + \tfrac{1}{2}D(C_3 + C_5)$$
$$E_6 = A_6 + \tfrac{1}{2}D(C_5 + C_7)$$

For the same values of P_2/P_1, i, and the Fourier constants as before,

$D =$ 0.0708121348
$E_2 =$ 0.003150288
$E_4 = -0.0000003619$
$E_6 =$ 0.0000000002 (negligible).

Solution of (85) may then be accomplished by transposing as follows:

$$B\lambda' = (x/R) + (y/R)\, D \cos \lambda' - E_2 \sin 2\lambda' - E_4 \sin 4\lambda' - \ldots \quad (86)$$

After substituting the known values of x, y, and R, almost any value of λ' may be tried in the right side of the equation, followed by solving for λ' on the left side and using the latter λ' for the next trial, etc., until there is no significant change between successive trial λ''s.

To find ϕ' in terms of x and y, it is most convenient to find λ' first from equation (86) and then to substitute it into a rearranged equation (2):

$$\ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') = \sqrt{1+S^2} \left[\frac{y}{R} - (H+1) \int_0^{\lambda'} \frac{S}{\sqrt{1+S^2}}\, d\lambda' \right] \quad (87)$$

While no iteration is involved in (87), it is preferable to convert to Fourier series to eliminate the numerical integration, first converting $\sqrt{1+S^2}$ to combine with the integral. Using binomial expansion,

$$
\begin{aligned}
(1+S^2)^{1/2} &= [1+(D\cos\lambda')^2]^{1/2} &&(88)\\
&= 1+\tfrac{1}{2}D^2\cos^2\lambda' - \tfrac{1}{8}D^4\cos^4\lambda' + \tfrac{1}{16}D^6\cos^6\lambda' + \ldots\\
&= 1+\tfrac{1}{2}D^2(\tfrac{1}{2})(1+\cos 2\lambda') - \tfrac{1}{8}D^4(\tfrac{1}{8})(3+4\cos 2\lambda'\\
&\quad +\cos 4\lambda') + \tfrac{1}{16}D^6(\tfrac{1}{32})(10+15\cos 2\lambda' + 6\cos 4\lambda'\\
&\quad +\cos 6\lambda') + \ldots\\
&= (1+D^4/4 - 3D^4/64 + 5D^6/256) + (D^2/4 - D^4/16\\
&\quad +15D^6/512)\cos 2\lambda' + (-D^4/64 + 3D^6/256)\cos 4\lambda'\\
&\quad +(D^6/512)\cos 6\lambda' + \ldots\\
&= G_0 + G_2\cos 2\lambda' + G_4\cos 4\lambda' + G_6\cos 6\lambda' + \ldots &&(89)
\end{aligned}
$$

where

$$
\begin{aligned}
G_0 &= 1+D^2/4 - 3D^4/64 + (5D^6/256)\\
G_2 &= D^2/4 - D^4/16 + (15D^6/512)\\
G_4 &= -D^4/64 + (3D^6/256)\\
G_6 &= (D^6/512)
\end{aligned}
$$

Substituting from equations (88), (89), and (74) into (87), and dropping G_6, which is only 0.0000000002:

$$
\begin{aligned}
\ln\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') = (y/R)\sqrt{1+D^2\cos^2\lambda'} - (G_0 + G_2\cos 2\lambda'\\
+G_4\cos 4\lambda' + \ldots)\cdot(C_1\sin\lambda'\\
+C_3\sin 3\lambda' + C_5\sin 5\lambda' + \ldots)
\end{aligned}
$$

Using equation (84) to combine the last part,

$$
\begin{aligned}
\ln\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') &= (y/R)\sqrt{1+D^2\cos^2\lambda'} - [C_1G_0\sin\lambda'\\
&\quad +C_3G_0\sin 3\lambda' + C_5G_0\sin 5\lambda' + \tfrac{1}{2}C_1G_2\sin 3\lambda'\\
&\quad +\tfrac{1}{2}C_1G_2\sin(-\lambda') + \tfrac{1}{2}C_3G_2\sin 5\lambda'\\
&\quad +\tfrac{1}{2}C_3G_2\sin\lambda' + \tfrac{1}{2}C_5G_2\sin 7\lambda'\\
&\quad +\tfrac{1}{2}C_5G_2\sin 3\lambda' + \tfrac{1}{2}C_1G_4\sin 5\lambda'\\
&\quad +\tfrac{1}{2}C_1G_4\sin(-3\lambda') + \tfrac{1}{2}C_3G_4\sin 7\lambda'\\
&\quad +\tfrac{1}{2}C_3G_4\sin(-\lambda') + \tfrac{1}{2}C_5G_4\sin 9\lambda'\\
&\quad +\tfrac{1}{2}C_5G_4\sin\lambda' + \ldots]\\
&= (y/\mathrm{R})\sqrt{1+D^2\cos^2\lambda'} - L_1\sin\lambda'\\
&\quad -L_3\sin 3\lambda' - L_5\sin 5\lambda' - \ldots &&(90)
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 &= C_1G_0 - \tfrac{1}{2}C_1G_2 + \tfrac{1}{2}C_3G_2 - (\tfrac{1}{2}C_3G_4) + (\tfrac{1}{2}C_5G_4)\\
L_3 &= C_3G_0 + \tfrac{1}{2}C_1G_2 + (\tfrac{1}{2}C_5G_2) - \tfrac{1}{2}C_1G_4\\
L_5 &= (C_5G_0 + \tfrac{1}{2}C_3G_2 + \tfrac{1}{2}C_1G_4)
\end{aligned}
$$

The following values of constants G_n and L_n for Landsat have been calculated. The parenthetical products in the formulas for G_n and L_n, and thus G_6 and L_5, may be ignored for 7-place accuracy:

$$
\begin{aligned}
G_0 &= 1.001252413\\
G_2 &= 0.0012520218\\
G_4 &= -\,0.0000003914
\end{aligned}
$$

$G_6 = \ \ 0.0000000002$ (negligible)
$L_1 = \ \ 0.1422487716$
$L_3 = \ \ 0.0000593661$
$L_5 = -\ 0.0000000296$ (negligible).

The inverses of equations (5), (6), and (7) are also desirable so that ϕ and λ may be calculated for a given ϕ' and λ'.

Equation (19) already provides ϕ in terms of ϕ' and λ':

$$\sin \phi = \cos i \sin \phi' + \sin i \cos \phi' \sin \lambda' \tag{19}$$

Rearranging equation (13),

$$\cos \lambda = \cos \phi' \cos \lambda' / \cos \phi \tag{91}$$

Rearranging equation (16),

$$\sin \lambda = (\cos i \sin \phi - \sin \phi') / \sin i \cos \phi \tag{91a}$$

Dividing (91a) by (91), then substituting from (19),

$$\begin{aligned} \tan \lambda &= (\cos i \sin \phi - \sin \phi') / \sin i \cos \phi' \cos \lambda' \\ &= [\cos i\ (\cos i \sin \phi' + \sin i \cos \phi' \sin \lambda') \\ &\quad - \sin \phi'] / \sin i \cos \phi' \cos \lambda' \\ &= \cos i \tan \lambda' - \sin i \tan \phi' / \cos \lambda' \end{aligned} \tag{91b}$$

This is the inverse for the stationary globe. As in equation (5), λ_t must be substituted for λ, so that

$$\tan \lambda_t = \tan \lambda' \cos i - \tan \phi' \sin i / \cos \lambda', \tag{92}$$

and

$$\lambda = \lambda_t - (P_2/P_1)\, \lambda' \tag{93}$$

provides the geodetic longitude from a transposition of (7). Equations (92) and (93) require adjustment to place λ in the proper quadrant. (See step 1 following equation (119).)

Equations (19), (92), and (93) are the formulas for ϕ and λ in terms of ϕ' and λ'.

DISTORTION ANALYSIS

To indicate the accuracy of map projection formulas it is important that scale factors and angular distortion be determined for representative points throughout the map. This is equally needed on the SOM formulas, where conformality is sought but not entirely achieved. The fundamental formulas for analysis of scale factors on the sphere involve comparing differential lengths between points on the map with the corresponding distance on the globe.

The distance on the map for a differential change of latitude is:

$$(\partial s / \partial \phi)_m = [(\partial x / \partial \phi)^2 + (\partial y / \partial \phi)^2]^{1/2} \tag{94}$$

and for a differential change of longitude,

$$(\partial s/\partial\lambda)_m=[(\partial x/\partial\lambda)^2+(\partial y/\partial\lambda)^2]^{1/2} \tag{95}$$

The corresponding distance on the globe,

$$(\partial s/\partial\phi)_g=R \tag{96}$$

$$(\partial s/\partial\lambda)_g=R\cos\phi \tag{97}$$

Dividing (94) by (96) gives the scale factor h along a meridian (constant longitude):

$$h=[(\partial x/\partial\phi)^2+(\partial y/\partial\phi)^2]^{1/2}/R \tag{98}$$

The corresponding scale factor k along a parallel (constant latitude) is found by dividing (95) by (97):

$$k=[(\partial x/\partial\lambda)^2+(\partial y/\partial\lambda)^2]^{1/2}/R\cos\phi \tag{99}$$

A more general scale factor m is found by combining (98) and (99), but using finite increments,

$$m=\frac{1}{R}\left[\frac{(\Delta x)^2+(\Delta y)^2}{(\Delta\phi)^2+(\cos\phi\ \Delta\lambda)^2}\right]^{1/2} \tag{100}$$

This gives the scale factor in any direction, allowing variation in both ϕ and λ. It is useful if small finite increments are given for a map projection that is complex to differentiate (especially if applied with proper changes to the ellipsoid, as described later). If desired in the differential form, the Δ's can be changed to ∂'s and then divided through by $\partial\phi$ or $\partial\lambda$ to give calculable expressions.

In addition to h and k, a third function, the maximum angular deformation ω first described by Tissot in 1881, should be included. To obtain a composite equation for the sphere, we begin with certain formulas in Maling (1973), corrected for two errors.

$$\sin \tfrac{1}{2}\omega=(a-b)/(a+b) \tag{101}$$

$$\cos\theta'=\left(\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\lambda}+\frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\lambda}\right)\Big/h\,k\cos\phi \tag{102}$$

$$a+b=(h^2+k^2+2hk\sin\theta')^{1/2} \tag{103}$$

$$a-b=(h^2+k^2-2hk\sin\theta')^{1/2} \tag{104}$$

where θ' is the angle at which the meridian and parallel intersect at a given latitude and longitude, and h and k are determined from equations (98) and (99). Solving for $2hk\sin\theta'$ in equation (104) by substituting differentials, letting $R=1$, since it cancels out,

$$2hk \sin \theta' = 2hk \sqrt{1-\cos^2 \theta'}$$

$$= 2hk \left[1 - \left(\frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \lambda} + \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \lambda}\right)^2 \Big/ h^2k^2 \cos^2 \phi\right]^{1/2}$$

$$= \frac{2}{\cos \phi} \left\{\left[\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2\right]\left[\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2\right] - \left[\frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \lambda} + \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \lambda}\right]^2\right\}^{1/2}$$

This simplifies to

$$2hk \sin \theta' = \frac{2}{\cos \phi}\left(\frac{\partial y}{\partial \phi}\frac{\partial x}{\partial \lambda} - \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \lambda}\right)$$

Substituting into equation (104),

$$(a-b)^2 = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \frac{1}{\cos^2 \phi}\left[\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2\right] - \frac{2}{\cos \phi}\left(\frac{\partial y}{\partial \phi}\frac{\partial x}{\partial \lambda} - \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \lambda}\right) = \left[\frac{\partial x}{\partial \phi} + \frac{1}{\cos \phi}\left(\frac{\partial y}{\partial \lambda}\right)\right]^2 + \left[\frac{\partial y}{\partial \phi} - \frac{1}{\cos \phi}\left(\frac{\partial x}{\partial \lambda}\right)\right]^2 \tag{105}$$

Similarly, equation (103) becomes

$$(a+b)^2 = \left[\frac{\partial x}{\partial \phi} - \frac{1}{\cos \phi}\left(\frac{\partial y}{\partial \lambda}\right)\right]^2 + \left[\frac{\partial y}{\partial \phi} + \frac{1}{\cos \phi}\left(\frac{\partial x}{\partial \lambda}\right)\right]^2 \tag{106}$$

Substituting from (105) and (106) into (101) and multiplying through by $\cos^2 \phi$,

$$\sin \tfrac{1}{2}\omega = \left[\frac{\left(\cos \phi \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \lambda}\right)^2 + \left(\cos \phi \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \lambda}\right)^2}{\left(\cos \phi \frac{\partial x}{\partial \phi} - \frac{\partial y}{\partial \lambda}\right)^2 + \left(\cos \phi \frac{\partial y}{\partial \phi} + \frac{\partial x}{\partial \lambda}\right)^2}\right]^{1/2} \tag{107}$$

The tangent form is slightly simpler, using the sine-to-tangent identity:

$$\tan \tfrac{1}{2}\omega = \sin \tfrac{1}{2}\omega / (1 - \sin^2 \tfrac{1}{2}\omega)^{1/2}$$

$$= \tfrac{1}{2}\left[\frac{\left(\cos \phi \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \lambda}\right)^2 + \left(\cos \phi \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \lambda}\right)^2}{\cos \phi \left(\frac{\partial y}{\partial \phi}\frac{\partial x}{\partial \lambda} - \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \lambda}\right)}\right]^{1/2} \tag{108}$$

With equations (98), (99), and (108), a fairly complete picture of the distortion at any point may be obtained. For a truly conformal projection, h, k, (and m) are equal, and ω is zero.

To evaluate these for the final spherical SOM formulas (1) to (7), and inverses (19), (73) through (77), (80), (86), (87), (92), and (93), differentiation is carried out so that h, k, and ω may be calculated preferably for any values of ϕ' and λ', since these variables enable us to pinpoint the relation of a point to the groundtrack of zero distortion. Distortion as a function of ϕ and λ may be determined with little additional derivation, but it is omitted here as superfluous.

Differentiating equations (3) and (7),

$$dS=-(P_2/P_1)\ \sin i\ \sin\lambda'\ d\lambda'=-S\ \tan\lambda' d\lambda' \tag{109}$$

$$d\lambda_t=d\lambda+(P_2/P_1)\ d\lambda' \tag{110}$$

From equation (5),

$$\begin{aligned}\sec^2\lambda' d\lambda'=\cos i\ \sec^2\lambda_t d\lambda_t+(\sin i\ \sec^2\phi/\cos\lambda_t)\ d\phi\\ +\sin i\ \tan\phi\ \sec\lambda_t\ \tan\lambda_t d\lambda_t\end{aligned} \tag{111}$$

Substituting (110) into (111),

$$\begin{aligned}\sec^2\lambda' d\lambda=\cos i\ \sec^2\lambda_t\ [d\lambda+(P_2/P_1)\ d\lambda']+(\sin i\ \sec^2\phi/\cos\lambda_t)d\phi\\ +\sin i\ \tan\phi\ \sec\lambda_t\tan\lambda_t\ [d\lambda+(P_2/P_1)\ d\lambda']\end{aligned}$$

Separating by differentials,

$$\begin{aligned}&[\sec^2\lambda'-(P_2/P_1)\ (\cos i/\cos^2\lambda_t+\sin i\ \tan\phi\ \sin\lambda_t/\cos^2\lambda_t)]\ d\lambda'\\ &\quad=(\cos i/\cos^2\lambda_t+\sin i\ \tan\phi\ \sin\lambda_t/\cos^2\lambda_t)\ d\lambda\\ &\quad\quad+(\sin i/\cos^2\phi\ \cos\lambda_t)\ d\phi\end{aligned} \tag{112}$$

Taking partials,

$$\frac{\partial\lambda'}{\partial\phi}=\frac{\sin i}{\cos^2\phi\left[\dfrac{\cos\lambda_t}{\cos^2\lambda'}-\dfrac{P_2}{P_1}\dfrac{1}{\cos\lambda_t}\ (\cos i+\sin i\ \tan\phi\ \sin\lambda_t)\right]} \tag{113}$$

$$\frac{\partial\lambda'}{\partial\lambda}=\frac{\cos i+\sin i\ \tan\phi\ \sin\lambda_t}{\dfrac{\cos^2\lambda_t}{\cos^2\lambda'}-\dfrac{P_2}{P_1}\ (\cos i+\sin i\ \tan\phi\ \sin\lambda_t)} \tag{114}$$

Differentiating equation (6), and then substituting from (110),

$$\begin{aligned}&\cos\phi' d\phi'=\cos i\ \cos\phi\ d\phi-\sin i\ \cos\phi\ \cos\lambda_t d\lambda_t+\sin i\ \sin\phi\ \sin\lambda_t d\phi\\ &\quad=(-\sin i\ \cos\phi\ \cos\lambda_t)\ [d\lambda+(P_2/P_1)\ d\lambda']+(\cos i\ \cos\phi\\ &\quad\quad+\sin i\ \sin\phi\ \sin\lambda_t)\ d\phi\end{aligned} \tag{115}$$

Taking partials,

$$\begin{aligned}\frac{\partial\phi'}{\partial\phi}=(\cos\phi/\cos\phi')\Big[&\cos i+\sin i\ \tan\phi\ \sin\lambda_t\\ &-\frac{P_2}{P_1}\sin i\ \cos\lambda_t\frac{\partial\lambda'}{\partial\phi}\Big]\end{aligned} \tag{116}$$

$$\frac{\partial\phi'}{\partial\lambda}=-\frac{\sin i\ \cos\phi\ \cos\lambda_t}{\cos\phi'}\left(1+\frac{P_2}{P_1}\frac{\partial\lambda'}{\partial\lambda}\right) \tag{117}$$

Differentiating equation (1),

$$dx/R = \frac{H-S^2}{(1+S^2)^{1/2}}\,d\lambda' - \ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right)\left[\frac{-S^2}{(1+S^2)^{3/2}} + \frac{1}{(1+S^2)^{1/2}}\right]dS - \frac{S}{(1+S^2)^{1/2}}\sec\phi' d\phi'$$

Taking partials,

$$\frac{\partial x}{R\partial\phi} = \frac{1}{(1+S^2)^{1/2}}\left[(H-S^2)\frac{\partial\lambda'}{\partial\phi} - S\sec\phi'\frac{\partial\phi'}{\partial\phi} + \frac{S}{1+S^2}\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right)\tan\lambda'\frac{\partial\lambda'}{\partial\phi}\right] \tag{118}$$

The partial $\partial x/R\partial\lambda$ is the same equation with λ in place of ϕ (at four places).

Differentiating equation (2),

$$dy/R = \frac{S(H+1)}{(1+S^2)^{1/2}}\,d\lambda' - \frac{S}{(1+S^2)^{3/2}}\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right)dS + \frac{1}{(1+S^2)^{1/2}}\sec\phi' d\phi'$$

$$\frac{\partial y}{R\partial\phi} = \frac{1}{(1+S^2)^{1/2}}\left[S(H+1)\frac{\partial\lambda'}{\partial\phi} + \sec\phi'\frac{\partial\phi'}{\partial\phi} + \frac{S^2}{1+S^2}\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right)\tan\lambda'\frac{\partial\lambda'}{\partial\phi}\right] \tag{119}$$

The partial $\partial y/R\partial\lambda$ is the same equation with λ in place of ϕ (at four places).

These equations permit calculation of distortion for the spherical form of the SOM. Assembling the steps,

1. For a given ϕ' and λ', calculate ϕ, λ_t, and λ from equations (19), (92), and (93), making sure that the λ_t calculated from the arctan in equation (92) is placed in the proper quadrant. To do this, subtract from the λ_t initially computed (falling between $-90°$ and $+90°$) this factor: 90° times (1 ± 1, taking the opposite sign of $\cos\lambda'$) times ±1, taking the sign of the initial λ_t (assume minus if λ_t is zero). If $\cos\lambda'$ is zero, add 10^{-8} degree to λ' for the calculation. Thus the subtrahend is 0, $\pm180°$, $\pm180°$, 0, 0, etc., in successive quadrants of the orbit, depending on the sign of the initial λ_t.
2. Calculate S and H from equations (3) and (4).
3. Calculate $(\partial\lambda'/\partial\phi)$, $(\partial\lambda'/\partial\lambda)$, $(\partial\phi'/\partial\phi)$, and $(\partial\phi'/\partial\lambda)$ from equations (113), (114), (116), and (117), respectively.
4. Calculate $(\partial x/\partial\phi)$, $(\partial x/\partial\lambda)$, $(\partial y/\partial\phi)$, and $(\partial y/\partial\lambda)$ from equations (118) and (119).
5. Calculate h, k and ω from equations (98), (99), and (108).

Table 2 shows these values for every 5° of a Landsat quadrant at 1° from the groundtrack, along the groundtrack, and each 15° of a Landsat quadrant at greater distances from the groundtrack.

TABLE 2.—*Scale factors and angular deviation away from Landsat groundtrack given by final formulas for the spherical Space Oblique Mercator projection*

[See notes at end of table]

At 1° from groundtrack

λ'	$\phi'=1°$				$\phi'=-1°$			
	h	k	ω	$m_{\phi'}$	h	k	ω	$m_{\phi'}$
0°	1.000154	1.000151	0.0006°	1.000152	1.000154	1.000151	0.0006°	1.000152
5	1.000153	1.000151	0.0006	1.000151	1.000154	1.000151	0.0006	1.000152
10	1.000153	1.000151	0.0006	1.000151	1.000155	1.000151	0.0006	1.000153
15	1.000153	1.000151	0.0006	1.000150	1.000155	1.000151	0.0006	1.000153
20	1.000152	1.000151	0.0006	1.000150	1.000156	1.000151	0.0006	1.000154
25	1.000152	1.000151	0.0006	1.000150	1.000156	1.000151	0.0006	1.000154
30	1.000152	1.000151	0.0005	1.000149	1.000156	1.000151	0.0005	1.000154
35	1.000152	1.000150	0.0005	1.000149	1.000156	1.000151	0.0005	1.000154
40	1.000152	1.000150	0.0005	1.000150	1.000156	1.000151	0.0005	1.000154
45	1.000152	1.000150	0.0004	1.000150	1.000156	1.000151	0.0005	1.000154
50	1.000152	1.000150	0.0004	1.000150	1.000156	1.000151	0.0004	1.000154
55	1.000152	1.000150	0.0004	1.000150	1.000155	1.000151	0.0004	1.000154
60	1.000153	1.000151	0.0003	1.000151	1.000155	1.000151	0.0003	1.000154
65	1.000153	1.000151	0.0003	1.000151	1.000155	1.000151	0.0003	1.000153
70	1.000153	1.000151	0.0002	1.000152	1.000154	1.000151	0.0002	1.000153
75	1.000153	1.000151	0.0002	1.000152	1.000154	1.000151	0.0002	1.000153
80	1.000153	1.000151	0.0001	1.000152	1.000153	1.000152	0.0001	1.000153
85	1.000153	1.000152	0.0001	1.000152	1.000153	1.000152	0.0001	1.000152
90	1.000152	1.000151	0.0001	1.000152	1.000152	1.000152	0.0000	1.000152

Beyond 1° from groundtrack

λ'	ϕ'	h	k	ω	λ'	ϕ'	h	k	ω
0°	15°	1.03567	1.03489	0.142°	45°	15°	1.03582	1.03455	0.099°
	10	1.01560	1.01526	0.062		10	1.01563	1.01515	0.044
	5	1.00386	1.00378	0.015		5	1.00386	1.00376	0.011
	−5	1.00386	1.00378	0.015		−5	1.00386	1.00379	0.011
	−10	1.01560	1.01526	0.062		−10	1.01553	1.01539	0.044
	−15	1.03567	1.03489	0.142		−15	1.03540	1.03535	0.102
15	15	1.03574	1.03474	0.136	60	15	1.03575	1.03459	0.071
	10	1.01561	1.01521	0.060		10	1.01562	1.01517	0.031
	5	1.00386	1.00377	0.015		5	1.00386	1.00377	0.008
	−5	1.00387	1.00378	0.015		−5	1.00386	1.00380	0.008
	−10	1.01558	1.01530	0.060		−10	1.01550	1.01542	0.032
	−15	1.03558	1.03505	0.137		−15	1.03531	1.03546	0.073
30	15	1.03579	1.03461	0.122	75	15	1.03551	1.03483	0.038
	10	1.01562	1.01518	0.053		10	1.01554	1.01525	0.016
	5	1.00386	1.00377	0.013		5	1.00385	1.00378	0.004
	−5	1.00387	1.00379	0.013		−5	1.00385	1.00380	0.004
	−10	1.01556	1.01535	0.054		−10	1.01548	1.01544	0.017
	−15	1.03549	1.03521	0.124		−15	1.03526	1.03552	0.039

Table 2.—*Continued*

Beyond 1° from groundtrack									
λ'	ϕ'	h	k	ω	λ'	ϕ'	h	k	ω
90°	15°	1.03528	1.03505	0.012°	90°	−5°	1.00382	1.00383	0.000°
	10	1.01543	1.01536	0.004		−10	1.01543	1.01549	0.004
	5	1.00382	1.00381	0.000		−15	1.03528	1.03551	0.013

Notes: λ' = angular position along groundtrack, from ascending node.
ϕ' = angular distance away from groundtrack, positive in direction away from north pole.
h = scale factor along meridian of longitude.
k = scale factor along parallel of latitude.
ω = maximum angular deformation.
$m_{\phi'}$ = scale factor along line of constant ϕ'.
$m_{\lambda'}$ = scale factor along line of constant λ'.
= sec ϕ', or 1.000152 at $\phi'=1°$.
If ϕ' = 0°, h, k, and $m_{\phi'}=1.0$, while $\omega=0$.

It is also of interest to calculate the scale factor on the SOM at constant ϕ' or at constant λ', especially since the equations simplify somewhat in the case of the sphere.

At constant ϕ', we return to equation (100), letting $R=1$, and using differentials with respect to λ':

$$m=\left[\frac{\left(\frac{\partial x}{\partial\lambda'}\right)^2+\left(\frac{\partial y}{\partial\lambda'}\right)^2}{\left(\frac{\partial\phi}{\partial\lambda'}\right)^2+\cos^2\phi\left(\frac{\partial\lambda}{\partial\lambda'}\right)^2}\right]^{1/2} \tag{120}$$

For the numerator of this, we may refer to equations (118) and (119), but replacing ϕ in each with λ'. Also let

$$V=[S/(1+S^2)]\tan\lambda'\ \ln\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi')+H \tag{121}$$

Then from equations (118) and (119), with $R=1$, and since $\partial\phi'$ is zero,

$$\partial x/\partial\lambda'=(1+S^2)^{-1/2}(V-S^2) \tag{122}$$

$$\partial y/\partial\lambda'=(1+S^2)^{-1/2}S(V+1) \tag{123}$$

Combining for the numerator of equation (120),

$$\begin{aligned}(\partial x/\partial\lambda')^2+(\partial y/\partial\lambda')^2 &= (1+S^2)^{-1}[(V-S^2)^2+S^2(V+1)^2]\\ &= (1+S^2)^{-1}[V^2-2VS^2+S^4+S^2V^2+2VS^2+S^2]\\ &= (1+S^2)^{-1}[V^2(1+S^2)+S^2(1+S^2)]\\ &= V^2+S^2\end{aligned} \tag{124}$$

For the denominator of equation (120), it is not correct to use reciprocals of previously derived partial derivatives, since ϕ' is not held constant in those derivations. Instead, it is necessary to differentiate inverse equations (19) and (92). For (92),

$$\sec^2\lambda_t d\lambda_t = \sec^2\lambda' \cos i\, d\lambda' - (\sec^2\phi' \sin i/\cos\lambda')\, d\phi'$$
$$-\tan\phi' \sin i \sec\lambda' \tan\lambda' d\lambda'$$

Substituting from (110) for $d\lambda_t$,

$$\sec^2\lambda_t\, [d\lambda + (P_2/P_1)\, d\lambda'] = \sec^2\lambda' \cos i\, d\lambda' - (\sec^2\phi' \sin i/\cos\lambda')\, d\phi'$$
$$-\tan\phi' \sin i \sec\lambda' \tan\lambda' d\lambda'$$

Separating differentials and rearranging,

$$d\lambda/\cos^2\lambda_t = [\cos i/\cos^2\lambda' - \tan\phi' \sin i \sin\lambda'/\cos^2\lambda'$$
$$-(P_2/P_1)/\cos^2\lambda_t]\, d\lambda' - (\sin i/\cos\lambda' \cos^2\phi')\, d\phi'$$

Then, using partial differentials,

$$\frac{\partial\lambda}{\partial\lambda'} = \frac{\cos^2\lambda_t}{\cos^2\lambda'}(\cos^2 i - \tan\phi' \sin i \sin\lambda') - \frac{P_2}{P_1} \tag{125}$$

$$\frac{\partial\lambda}{\partial\phi'} = -\sin i \cos^2\lambda_t/\cos\lambda' \cos^2\phi' \tag{126}$$

Differentiating equation (19),

$$\cos\phi\, d\phi = \cos\phi' \cos i\, d\phi' + \cos\lambda' \cos\phi' \sin i\, d\lambda'$$
$$-\sin\lambda' \sin\phi' \sin i\, d\phi'$$

Rearranging and using partials,

$$\frac{\partial\phi}{\partial\lambda'} = \cos\phi' \cos\lambda' \sin i/\cos\phi \tag{127}$$

$$\frac{\partial\phi}{\partial\phi'} = (\cos\phi' \cos i - \sin\phi' \sin i \sin\lambda')/\cos\phi \tag{128}$$

While the denominator of equation (120) cannot be readily consolidated, it is found from equations (127) and (125):

$$\left(\frac{\partial\phi}{\partial\lambda'}\right)^2 + \cos^2\phi\left(\frac{\partial\lambda}{\partial\lambda'}\right)^2 = \left(\frac{\cos\phi' \cos\lambda' \sin i}{\cos\phi}\right)^2 + \cos^2\phi\left[\frac{\cos^2\lambda_t}{\cos^2\lambda'}(\cos i - \tan\phi' \sin i \sin\lambda') - \frac{P_2}{P_1}\right]^2 \tag{129}$$

Equations (124) and (129) may be used in equation (120) to calculate m for lines of constant ϕ', after calculating steps 1 and 2 listed after equation (119). Table 2 shows the results for Landsat, using the SOM spherical formulas.

At constant λ', substituting ϕ' in place of λ' in equation (120),

$$m_{\lambda'} = \left[\frac{\left(\frac{\partial x}{\partial\phi'}\right)^2 + \left(\frac{\partial y}{\partial\phi'}\right)^2}{\left(\frac{\partial\phi}{\partial\phi'}\right)^2 + \cos^2\phi\left(\frac{\partial\lambda}{\partial\phi'}\right)^2}\right]^{1/2}$$

For the numerator, replacing ϕ with ϕ' in equations (118) and (119), and since $d\lambda'$ is zero, and again letting $R=1$,

$$\begin{aligned}(\partial x/\partial\phi')^2+(\partial y/\partial\phi')^2 &=(1+S^2)^{-1}\,(S^2\sec^2\phi'+\sec^2\phi')\\ &=(1+S^2)^{-1}\,(1+S^2)\sec^2\phi'\\ &=\sec^2\phi'\end{aligned}$$

For the denominator, equations (128) and (126) may be used. Actually the denominator equals 1, although it takes considerable work on the formulas to simplify them. It is more easily verified by making several representative calculations on a programable calculator. Thus,

$$m_{\lambda'}=\sec\phi'$$

in accordance with the actual design of the scanlines, which are lines of constant λ', since the incorporation of equation (9) into (44) and then into (1) and (2) changed the direction but not the scale along any given scanline.

FORMULAS FOR THE SATELLITE GROUNDTRACK

If $\phi'=0°$, previous equations may be readily reduced for calculations along the groundtrack. The symbols ϕ_0 and λ_0 will be used for values of ϕ and λ, respectively, along the groundtrack.

Equations (1) and (2) reduce to the portion determined as equations (50) and (52), namely,

$$\frac{x_0}{R}=\int_0^{\lambda'}\frac{H-S^2}{\sqrt{1+S^2}}\,d\lambda' \tag{130}$$

$$\frac{y_0}{R}=(H+1)\int_0^{\lambda'}\frac{S}{\sqrt{1+S^2}}\,d\lambda' \tag{131}$$

with H and S unchanged from equations (3) and (4).

Equations (19), (92), and (93) become

$$\sin\phi_0=\sin\lambda'\sin i \tag{132}$$

$$\lambda_0=\lambda_{t_0}-(P_2/P_1)\,\lambda' \tag{93}$$

where $$\tan\lambda_{t_0}=\tan\lambda'\cos i \tag{133}$$

The latter three equations may be easily inverted to give λ' for a given ϕ_0 or λ_0:

$$\sin\lambda'=\sin\phi_0/\sin i \tag{134}$$

$$\tan\lambda'=\tan\lambda_{t_0}/\cos i \tag{135}$$

where $$\lambda_{t_0}=\lambda_0+(P_2/P_1)\,\lambda' \tag{136}$$

Equations (135) and (136) require combined iteration, since λ' cannot be isolated, just as in equations (5) and (7). This is discussed below. To find ϕ_0 for a given λ_0, use equations (135) and (136), then (132). To find λ_0 for a given ϕ_0, use equations (134), (133), and (93) in order.

If λ' should be desired for a given x or y along the groundtrack, iteration is also required, although the flatness of the groundtrack curve and its periodicity make the calculation from y undesirable. For λ' from x, transpose the series part of equation (73):

$$B\lambda'=(x_0/R)-A_2 \sin 2\lambda'-A_4 \sin 4\lambda'-\ldots \qquad (137)$$

Iteration may proceed in the manner described after equation (86).

ITERATION PROCEDURES

There are two types of iterations in the equations so far:

1. Equations (86) and (137) follow the very simple procedure given just after equation (86).

2. Equations (5) with (7) and (135) with (136) may be rapidly converged with the procedure below. Note that unless a procedure such as this is used, attempts to calculate λ' automatically will result in quadrant problems and discontinuities near λ' of 90°, 270°, etc., in the arctan calculation.

The equations converge rapidly if, after selecting the desired ϕ and λ, the λ' of the nearest polar approach, λ'_p, is used as the first trial λ' on the right side of the equation, solving for the corresponding λ' on the left side, adding a factor (see below), and using that result as the next trial λ'. The λ'_p value may be calculated as $\lambda'_p=90°\times(4N+2\pm1)$, where N is the number of orbits completed at the last ascending node before the satellite passes the nearest pole, and the $\pm$ is minus in the northern hemisphere and plus in the southern (either for the equator). (There is actually substantial overlap, so that the minus may be extended almost, but not quite, to the south pole, and the plus almost to the north pole; the arbitrary switch of signs at the equator is convenient and covers all situations.) Since the computer normally calculates the arctan as an angle between −90° and 90°, it is necessary to add the proper factor. Each λ' given on the left side by the computer must be increased by λ'_p minus the following: 90° times sin λ'_p times ±1 (the $\pm$taking the sign of cos λ_{t_p}, where $\lambda_{t_p}=\lambda+(P_2/P_1)\,\lambda'_p$). If cos λ'_p is zero, the final λ' is λ'_p. Thus λ'_p is 90°, 90°, 270°, 270°, 450°, etc., and the adder to arctangent is 0°, 180°, 180°, 360°, 360°, etc., for each successive quadrant beginning at the origin (ϕ, λ, ϕ', and $\lambda'=0$). These quadrants automatically change along the equator, rather than along the scanline which crosses the equator at the node.

RADIUS OF CURVATURE ANALYSIS

Junkins and Turner (1977, p. 18) applied two basic constraints to their development of the SOM formulas. First, they maintained the groundtrack perfectly true to scale and conformal, as did the present writer. Secondly, they made the curvature of the groundtrack on the map the same as its curvature on the Earth. The writer did not attempt to do this until much later and in fact did not think of this constraint until Junkins mentioned it. The writer's groundtrack very closely approaches this concept of curvature, judging from the very small differences in coordinates between the two derivations, but it is as a result of empirical reduction of scale error, as previously described.

Junkins and Turner's report shows larger errors than the calculations based on the present formulas, even in the ellipsoidal forms (below), but this appears to be due to errors in some of the concepts for developing the formulas for distance from the groundtrack (Junkins and Turner, 1977, p. 115–118). The writer's own formulas embodying the curvature restraint are derived at the end of this work. Because they are much lengthier than the formulas using an empirical curvature, even when simplified for a circular orbit, they are recommended only for a non-circular orbit. The increased scale accuracy for a circular orbit appears to be negligible.

Following is derived the equation for the curvature of the groundtrack on the sphere. It will be rederived for the ellipsoid later and incorporated into mapping formulas. The standard formula for radius of curvature in plane coordinates is

$$r_c=[1+(dy/dx)^2]^{3/2}/(d^2y/dx^2) \tag{138}$$

To convert this to spherical coordinates, it was simplest to use an oblique orthographic projection, the azimuthal with the simplest formulas, tangent to the globe at the point where the groundtrack curvature is to be calculated. To follow conventional practice, x and y are used in equations (138) through (145), but they are not the SOM coordinates derived earlier. These formulas for a sphere of radius 1.0 are as follows:

$$x=\cos\phi\sin\lambda \tag{139}$$

$$y=\cos\alpha\sin\phi-\sin\alpha\cos\phi\cos\lambda \tag{140}$$

where α is the latitude at the tangent point, and λ is zero.

Since

$$dy/dx = (dy/d\phi)/(dx/d\phi)$$

differentiating each with respect to ϕ,

$$dx/d\phi=-\sin\phi\sin\lambda+\cos\phi\cos\lambda\, d\lambda/d\phi \tag{141}$$

$$\begin{aligned} dy/d\phi &= \cos\alpha\cos\phi+\sin\alpha\sin\phi\cos\lambda \\ &\quad +\sin\alpha\cos\phi\sin\lambda\, d\lambda/d\phi \end{aligned} \tag{142}$$

Dividing,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos\alpha\cos\phi+\sin\alpha\sin\phi\cos\lambda+\sin\alpha\cos\phi\sin\lambda\,(d\lambda/d\phi)}{-\sin\phi\sin\lambda+\cos\phi\cos\lambda\,(d\lambda/d\phi)} \\ &= \frac{\cos\alpha\sec\lambda+\sin\alpha\tan\phi+\sin\alpha\tan\lambda\,(d\lambda/d\phi)}{-\tan\phi\tan\lambda+(d\lambda/d\phi)} \end{aligned} \tag{143}$$

If $\alpha=\phi$ and $\lambda=0$ at the point of tangency, where the slope on the map is the same as the slope on the sphere itself,

$$\frac{dy}{dx}=\frac{\cos\phi+\sin^2\phi/\cos\phi}{d\lambda/d\phi}$$
$$=\sec\phi/(d\lambda/d\phi) \tag{144}$$

For the second derivative, we differentiate equation (143) with respect to ϕ, and divide by $(dx/d\phi)$ from equation (141):

$$\frac{d^2y}{dx^2}=\frac{d\ (dy/dx)}{d\phi}\Big/\frac{dx}{d\phi}$$
$$=\frac{\begin{array}{l}[-\tan\phi\tan\lambda+(d\lambda/d\phi)]\,[\cos\alpha\sec\lambda\tan\lambda\ (d\lambda/d\phi)\\ +\sin\alpha\sec^2\phi+\sin\alpha\sec^2\lambda\ (d\lambda/d\phi)^2+\sin\alpha\tan\lambda\ (d^2\lambda/d\phi^2)]\\ -[\cos\alpha\sec\lambda+\sin\alpha\tan\phi+\sin\alpha\tan\lambda\ (d\lambda/d\phi)]\\ [-\sec^2\phi\tan\lambda-\tan\phi\sec^2\lambda\ (d\lambda/d\phi)+(d^2\lambda/d\phi^2)]\end{array}}{[-\tan\phi\tan\lambda+(d\lambda/d\phi)]^2\,[-\sin\phi\sin\lambda+\cos\phi\cos\lambda\ (d\lambda/d\phi)]}$$

If $\alpha=\phi$ and $\lambda=0$,

$$\frac{d^2y}{dx^2}=\frac{\begin{array}{l}(d\lambda/d\phi)\,[\sin\phi/\cos^2\phi+\sin\phi\ (d\lambda/d\phi)^2]-[\cos\phi+\\ \sin^2\phi/\cos\phi]\,[-\tan\phi\ (d\lambda/d\phi)+(d^2\lambda/d\phi^2)]\end{array}}{(d\lambda/d\phi)^2\cos\phi\ (d\lambda/d\phi)}$$

Multiplying numerator and denominator by $\cos^2\phi/\cos^2\phi$,

$$\frac{d^2y}{dx^2}=\frac{\begin{array}{l}(d\lambda/d\phi)\ (\sin\phi+\sin\phi\cos^2\phi+\sin^3\phi)-(d^2\lambda/d\phi^2)\\ (\cos^3\phi+\sin^2\phi\cos\phi)+\sin\phi\cos^2\phi\ (d\lambda/d\phi)^3\end{array}}{\cos^3\phi\ (d\lambda/d\phi)^3}$$
$$=\frac{2\sin\phi\ (d\lambda/d\phi)-\cos\phi\ (d^2\lambda/d\phi^2)+\sin\phi\cos^2\phi\ (d\lambda/d\phi)^3}{\cos^3\phi\ (d\lambda/d\phi)^3} \tag{145}$$

Substituting equations (145) and (144) into (138), and inserting subzeros, since only the groundtrack is being considered:

$$r_c=\frac{[1+\sec^2\phi_o/(d\lambda_o/d\phi_o)^2]^{3/2}\cos^3\phi_o\ (d\lambda_o/d\phi_o)^3}{2\sin\phi_o\ (d\lambda_o/d\phi_o)-\cos\phi_o\ (d^2\lambda_o/d\phi_o{}^2)+\sin\phi_o\cos^2\phi_o\ (d\lambda_o/d\phi_o)^3} \tag{146}$$

For differentials in (146), we go to groundtrack formulas. Combining (133) and (93),

$$\lambda_o=\arctan\ (\cos i\tan\lambda')-(P_2/P_1)\lambda' \tag{147}$$

Differentiating,

$$d\lambda_o/d\lambda'=\cos i\sec^2\lambda'/(1+\cos^2 i\tan^2\lambda')-P_2/P_1 \tag{148}$$

Differentiating equation (134),

$$\cos\lambda'(d\lambda'/d\phi_o)=\cos\phi_o/\sin i \tag{149}$$

Multiplying (148) by transposed (149),

$$(d\lambda_o/d\phi_o)=[\cos i \sec^2\lambda'/(1+\cos^2 i \tan^2\lambda')-P_2/P_1] \cos \phi_o/\sin i \cos \lambda'$$
$$=[\cos i/(\cos^2\lambda'+\cos^2 i \sin^2\lambda')-P_2/P_1] \cos \phi_o/\sin i \cos \lambda'$$
$$=[\cos i/(1-\sin^2 i \sin^2\lambda')-P_2/P_1] \cos \phi_o/\sin i \cos \lambda'$$

Substituting from equation (134) for λ',

$$(d\lambda_o/d\phi_o)=(\cos i/\cos^2 \phi_o-P_2/P_1) \cos \phi_o/(\sin^2 i-\sin^2 \phi_o)^{1/2} \qquad (150)$$

Differentiating this,

$$d^2\lambda_o/d\phi_o^2=\frac{(\cos i/\cos^2\phi_o-P_2/P_1)\ [(\sin^2 i-\sin^2\phi_o)^{1/2}\ (-\sin \phi_o)-\frac{1}{2}\cos\phi_o\ (\sin^2 i-\sin^2\phi_o)^{-1/2}\ (-2 \sin \phi_o \cos \phi_o)]}{\sin^2 i-\sin^2 \phi_o}$$
$$+\frac{\cos \phi_o}{(\sin^2 i-\sin^2 \phi_o)^{1/2}}\ (2 \cos i \sec \phi_o \sec \phi_o \tan \phi_o)$$
$$=\frac{1}{(\sin^2 i-\sin^2\phi_o)^{3/2}}\ \{[-\sin \phi_o\ (\sin^2 i-\sin^2 \phi_o)$$
$$+\sin \phi_o \cos^2 \phi_o]\ (\cos i/\cos^2\phi_o-P_2/P_1)$$
$$+2 \cos i \sin \phi_o \sec^2\phi_o\ (\sin^2 i-\sin^2 \phi_o)\}$$
$$=\frac{\sin \phi_o \cos i}{(\sin^2 i-\sin^2 \phi_o)^{3/2}}\ [\cos^2 i/\cos^2 \phi_o-(P_2/P_1) \cos i$$
$$+2 \sin^2 i/\cos^2 \phi_o-2 \tan^2 \phi_o]$$
$$=\frac{\sin \phi_o \cos i}{(\sin^2 i-\sin^2 \phi_o)^{3/2}}\ [2-\cos^2 i/\cos^2 \phi_o-(P_2/P_1) \cos i] \qquad (151)$$

Equations (150) and (151) may be substituted in equation (146) to give the radius of curvature r_c in terms of groundtrack latitude ϕ_o. Without giving the intermediate steps, the combination can be somewhat simplified to the following:

$$r_c=\frac{\cos^2 \phi_o\ [1-2\ (P_2/P_1) \cos i+(\cos \phi_o\ P_2/P_1)^2]^{3/2}}{\sin \phi_o\ \{(\cos i/\cos^2 \phi_o-P_2/P_1)\ [2(\cos^2 \phi_o-\cos^2 i)+(\cos i-\cos^2 \phi_o\ P_2/P_1)^2]-\cos i\ (2-\cos^2 i/\cos^2 \phi_o-\cos i\ P_2/P_1)\}} \qquad (152)$$

Actually, $r_c=\tan z_o$, (153)

where z_o is the polar distance of the small circle with radius of curvature r_c; that is, the pole of the groundtrack curve is z_o away from it. It is also desirable to know the latitude and longitude of this pole. This involves determining the direction of the groundtrack at a given point and moving a distance z_o perpendicular to it along a great circle path.

For the direction, or azimuth, the direction Z_g east of north of the groundtrack is determined from the fundamental relationship,

$$\tan Z_g=\cos \phi_o\ d\lambda_o/d\phi_o$$

The azimuth of the momentary pole of the circle of curvature from the groundtrack is the perpendicular Z, east of north:

$$Z=\arctan\,[\cos\phi_o\,(d\lambda_o/d\phi_o)]+90°$$

From equation (150),

$$Z=\arctan\,[(\cos i/\cos^2\phi_o-P_2/P_1)\cos^2\phi_o/(\sin^2 i-\sin^2\phi_o)^{1/2}]+90°$$
$$=\arctan\,\{[\cos i-(P_2/P_1)\cos^2\phi_o]/(\cos^2\phi_o-\cos^2 i)^{1/2}\quad+90° \tag{154}$$

If (ϕ_1, λ_1) are coordinates of the pole, the standard formula for the great circle distance of the groundtrack is written

$$\cos z_o=\sin\phi_o\sin\phi_1+\cos\phi_o\cos\phi_1\cos(\lambda_o-\lambda_1) \tag{155}$$

If a spherical triangle is drawn with the geographic pole, the pole of the circle of curvature, and the point on the groundtrack as vertices, the Law of Sines provides an azimuth formula as follows:

$$\sin(\lambda_o-\lambda_1)/\sin z_o=\sin Z/\cos\phi_1 \tag{156}$$

Since ϕ_o, λ_o, Z, and z_o are known or determined from equations previously given, we have two equations (155) and (156) in two unknowns.

To eliminate $(\lambda_o-\lambda_1)$, rearrange (155) and (156) as follows:

$$\cos(\lambda_o-\lambda_1)=\frac{\cos z_o-\sin\phi_o\sin\phi_1}{\cos\phi_o\cos\phi_1} \tag{157}$$

$$\sin(\lambda_o-\lambda_1)=\sin Z\sin z_o/\cos\phi_1 \tag{158}$$

Squaring each and adding the two equations gives 1.0 for the left side:

$$1=(\cos z_o-\sin\phi_o\sin\phi_1)^2/\cos^2\phi_o\cos^2\phi_1+\sin^2 Z\sin^2 z_o/\cos^2\phi_1$$

Solving for ϕ_1 leads to the following:

$$\sin\phi_1=\sin\phi_o\cos z_o-\cos\phi_o\sin z_o\cos Z \tag{159}$$

Equation (158) may be used to find λ_1, after determining ϕ_1 from (159). The $(\lambda_o-\lambda_1)$ must fall between 90° and 180° for a first-quadrant portion of the groundtrack. Table 3 shows the interesting pattern of these points for the Landsat groundtrack, with $i=99.092°$ and $P_2/P_1=18/251$. Each quadrant is symmetrical.

TABLE 3.—*Location of pole of curvature for Landsat groundtrack relative to sphere*
[$i=99.092°$; $P_2/P_1=18/251$. Each quadrant symmetrical]

ϕ_o	λ_o	λ'	z_o	Z	ϕ_1	λ_1
0°	0.0°	0.0°	−90.0°	76.90276°	13.09724°	90.0°
10	− 2.34332	10.12857	−88.60379	76.82351	13.21777	88.57343
20	− 4.79247	20.26557	−87.25038	76.56102	13.56506	87.23070
30	− 7.48298	30.42181	−85.98117	76.03133	14.09788	86.04627
40	−10.62969	40.61451	−84.83482	75.05302	14.75274	85.07670
50	−14.64319	50.87686	−83.84607	73.23903	15.45126	84.35388
60	−20.48721	61.28790	−83.04467	69.64980	16.10931	83.88063

TABLE 3.—*CONTINUED*

ϕ_0	λ_0	λ'	z_0	Z	ϕ_1	λ_1
70°	−31.25479°	72.11023°	−82.45460°	61.25029°	16.64701°	83.62960°
80	−71.32738	85.81878	−82.09342	24.20186	16.99861	83.54691
*	−96.45418	90.0	−82.07238	.00000	17.01962	83.54582

* $\phi_0 = 180° - i = 80.908°$.

Note: ϕ_0=geodetic latitude along groundtrack.

λ_0=geodetic longitude along groundtrack, relative to longitude of ascending node.

λ'=transformed longitude along groundtrack.

z_0=polar distance of small circle with radius of curvature of groundtrack.

Z=azimuth of radius of curvature of groundtrack, east of north.

ϕ_1=geodetic latitude of pole of radius of curvature of groundtrack.

λ_1=geodetic longitude of pole of radius of curvature of groundtrack, relative to longitude of ascending node.

The Space Oblique Mercator projection may be more correctly considered equivalent to an oblique Lambert Conformal Conic projection with one constantly changing standard transformed parallel of distance z_0 from a constantly shifting transformed pole with coordinates (ϕ_1, λ_1). This concept results in a substantial reduction (by a factor of 10 to 500 in various tests) in non-conformality for the sphere. With the shifting of the transformed pole, true conformality is still not achieved. Since the scanlines are skewed with respect to the groundtrack, they are not radii of the conic projection, nor are they truly straight using this concept. The formulas involve a lengthy iteration to find the radius of curvature and related parameters for each latitude and longitude, in addition to using Fourier constants for calculations of x and y. The considerably expanded complexity merely for the sphere does not appear justified, since the empirical approach provides such an accurate projection. Therefore, the rest of the formulas for the conic interpretation of the SOM for the sphere are not given in this work, and the concept has not been applied to the ellipsoid.

SPACE OBLIQUE MERCATOR PROJECTION FOR THE ELLIPSOID

Accurate though the above formulas may be for the sphere, there are errors of over one-half of a percent in using these (or any accurate spherical formulas) in place of formulas based on the ellipsoid, especially in tropical latitudes (Snyder, 1978a). In maps of very large areas, errors fundamental to plane projection far outweigh the effect of the ellipsoid, but for topographic mapping of small areas or strips, the use of the ellipsoid is essential for high-quality mapping. Therefore, as indicated earlier, the development of SOM formulas for the sphere served largely to avoid even more complex derivations and calculations in determining the feasibility of various concepts.

At first, the writer assumed that Hotine's classic work on the ellipsoidal oblique Mercator would be the logical link in changing from sphere to ellipsoid (Hotine, 1946, 1947). Indeed, John B. Rowland (written commun., 1977) of the Geological Survey has applied it in five stationary zones to approximate each north-to-south pass of the satellite, with consequent discontinuities. Hotine pointed out that an ellipsoidal version of the oblique Mercator cannot be exactly developed (both the regular and transverse Mercator projections may be derived exactly for the ellipsoid), but he prepared an approximation by using an "aposphere" tangent to the ellipsoid at a chosen point. The scale of his central line is true only at the point of tangency. Within several degrees of the tangent point, the central line is essentially true to scale, but it gradually deviates. The projection is exactly conformal, however. For normal usage of the ellipsoidal oblique Mercator, this is completely satisfactory. In the case of a revolving satellite, the increasing error of the central line soon becomes unacceptable. The curve presented by the plotting of the groundtrack presents an additional serious problem in attempting to adapt the Hotine to the SOM for the ellipsoid.

Thus it was soon evident that it was better to disregard the Hotine derivations in developing the ellipsoidal SOM formulas, and to work instead from the basic geometry of the ellipse and ellipsoid, with assistance from Thomas (1952). It will be assumed that the center of the ellipsoid coincides with the center of the Earth's mass, slightly incorrect for most official ellipsoids as used with local datums, but accurate for satellite-determined ellipsoids such as GRS 1980.

ELEMENTS OF THE ELLIPSE

In the ellipse of figure 4, a is the major semi-axis AC or CE, b is the minor semi-axis BC or PC, and the eccentricity e,

$$e=(1-b^2/a^2)^{1/2} \tag{160}$$

The geocentric latitude for point L is ϕ_g, with the north pole at P and the equator along AE.

The geographic latitude ϕ_n is the inclination to the equator of a perpendicular to the surface at point L. Radii shown to point L are ρ_g or CL, the radius to the center; ρ_m, the radius of curvature of meridian PE at point L; and ρ_p, the radius of curvature of the surface in a plane perpendicular to the plane of the meridian PE, and also perpendicular to a plane tangent to the Earth at L. Both ρ_m and ρ_p lie along LD, although they are unequal. In much of the literature, ρ_m and ρ_p are given the symbols R and N, respectively, but the latter are not used here, to avoid confusion with other symbols.

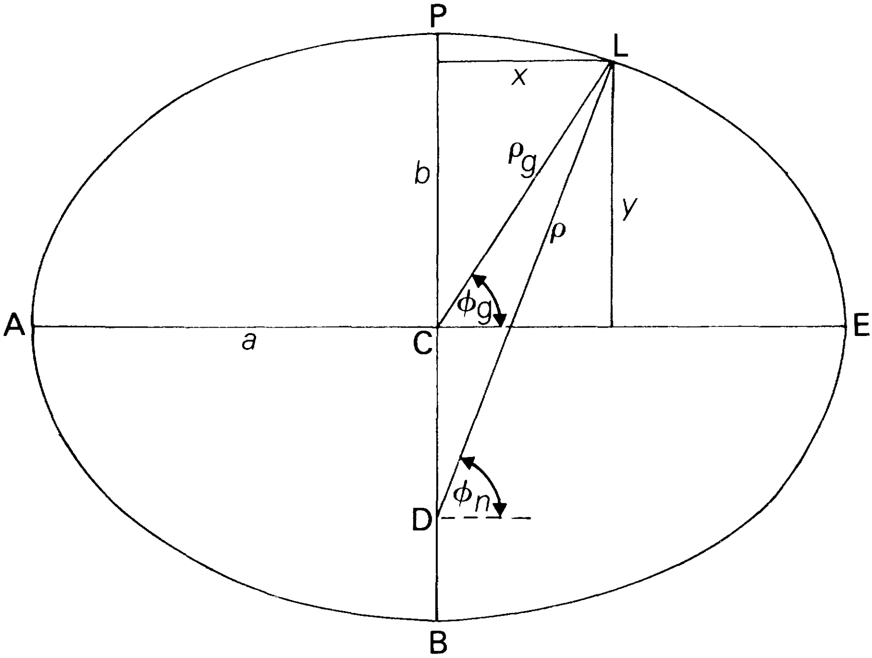

FIGURE 4.—Exaggerated elliptical cross section of the Earth.

The basic equation for the ellipse (using conventional x and y coordinates, which are not those for the SOM map, in equations (161) through (167))

$$x^2/a^2+y^2/b^2=1 \tag{161}$$

is differentiated:

$$(2x/a^2)dx+(2y/b^2)dy=0$$
$$dy/dx=-xb^2/ya^2.$$

Since, from (161),

$$y=b(a^2-x^2)^{1/2}/a \tag{162}$$

then $$dy/dx=-xb/a(a^2-x^2)^{1/2}$$

This is the slope of the ellipse, and is therefore the negative reciprocal of the slope of the perpendicular DL, or

$$dy/dx=-1/\tan \phi_n$$

Therefore,

$$\tan \phi_n=a(a^2-x^2)^{1/2}/bx \tag{163}$$

For geocentric coordinates:

$$\tan \phi_g=y/x=b(a^2-x^2)^{1/2}/ax$$

Substituting from (163) and (160),

$$\tan \phi_g = b^2 \tan \phi_n / a^2$$
$$= (1-e^2) \tan \phi_n \qquad (164)$$

Solving for x from equation (163),

$$b^2 x^2 \tan^2 \phi_n = a^2(a^2 - x^2)$$
$$x^2(b^2 \tan^2 \phi_n + a^2) = a^4$$
$$x = a/(b^2 \tan^2 \phi_n / a^2 + 1)^{1/2}$$
$$= a/[(1-e^2) \tan^2 \phi_n + 1]^{1/2}$$
$$= a \cos \phi_n / (1 - e^2 \sin^2 \phi_n)^{1/2} \qquad (165)$$

Solving for y from equations (162) and (165),

$$y = b[a^2 - a^2 \cos^2 \phi_n / (1 - e^2 \sin^2 \phi_n)]^{1/2} / a$$
$$= b[1 - \cos^2 \phi_n / (1 - e^2 \sin^2 \phi_n)]^{1/2}$$
$$= b \sin \phi_n \, [(1-e^2)/(1 - e^2 \sin^2 \phi_n)]^{1/2}$$
$$= a(1-e^2) \sin \phi_n / (1 - e^2 \sin^2 \phi_n)^{1/2} \qquad (166)$$

For geocentric radius ρ_g, using the Pythagorean theorem and equations (165) and (166),

$$\rho_g = (x^2 + y^2)^{1/2}$$
$$= a \sqrt{\frac{\cos^2 \phi_n + (1-e^2)^2 \sin^2 \phi_n}{1 - e^2 \sin^2 \phi_n}}$$
$$= a \sqrt{\frac{1 - e^2 (2-e^2) \sin^2 \phi_n}{1 - e^2 \sin^2 \phi_n}} \qquad (167)$$

It is also useful to have ρ_g in terms of ϕ_g.

Multiplying numerator and denominator of (167) by $\sec \phi_n$,

$$\rho_g = a \sqrt{\frac{\sec^2 \phi_n - e^2 (2-e^2) \tan^2 \phi_n}{\sec^2 \phi_n - e^2 \tan^2 \phi_n}}$$
$$= a \sqrt{\frac{1 + \tan^2 \phi_n \, (1-e^2)^2}{1 + \tan^2 \phi_n \, (1-e^2)}}$$

Substituting from (164),

$$\rho_g = a \sqrt{\frac{1 + \tan^2 \phi_g}{1 + \tan^2 \phi_g / (1-e^2)}}$$
$$= a[(1-e^2)/(1 - e^2 \cos^2 \phi_g)]^{1/2} \qquad (168)$$

For radii ρ_m and ρ_p the derivations are given in several standard works and will be omitted here (see Thomas, 1952, p. 59, eqn. 152, 154).

$$\rho_m = a(1-e^2)/(1 - e^2 \sin^2 \phi_n)^{3/2} \qquad (169)$$

$$\rho_p = a/(1 - e^2 \sin^2 \phi_n)^{1/2} \qquad (170)$$

ORBITAL PLANE ON THE ELLIPSOID

Referring to figure 5, the satellite orbit cuts the Earth with an inclination i (99.092° for Landsat) at the equatorial crossing. The cross section in the orbital plane is also an ellipse having the same major semi-axis a, but a different minor semi-axis r (fig. 6). If the satellite orbit is assumed to be circular, the geocentric angle λ', as measured from the equatorial crossing, will be changed at a uniform rate with respect to time. Therefore, the elements of the orbital plane are as follows:

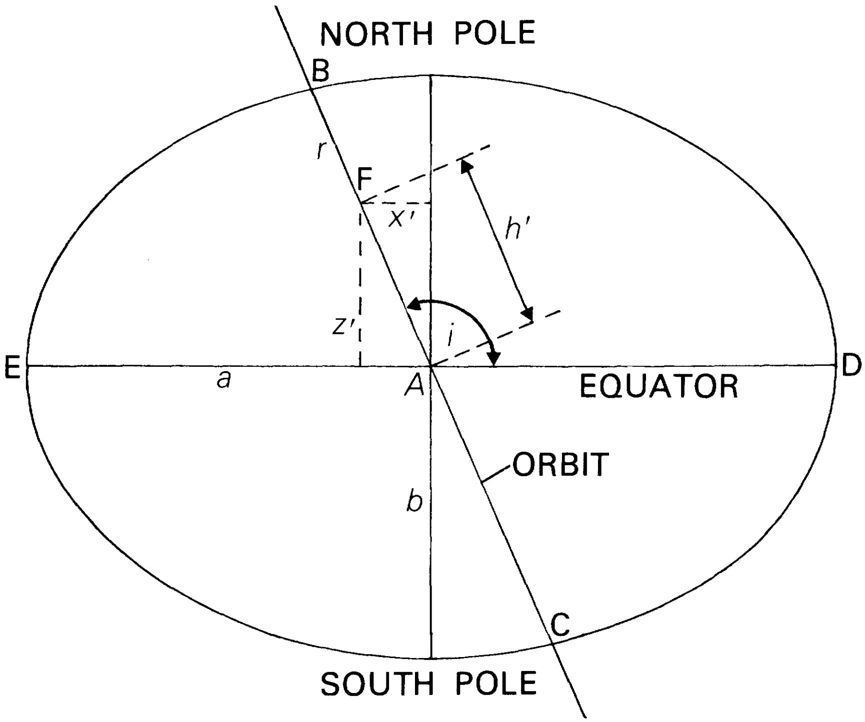

FIGURE 5.—Equatorial view of ellipsoid showing intersection with orbital plane.

The minor semi-axis r in figure 6 or figure 5 (where r=AB or AC) is the same as ρ_g of figure 4, if ϕ_g is made equal to i. Therefore, using equation (168),

$$r = a[(1-e^2)/(1-e^2 \cos^2 i)]^{1/2}. \tag{171}$$

Again using equation (168), but calculating the radius ρ_o in figure 6 to the surface of the Earth in terms of λ' and the ellipse of dimensions a and r, we may use $(1-r^2/a^2)$ in place of e^2 (see equation (160)):

$$\rho_0 = a(r/a)/[1-(1-r^2/a^2)\cos^2\lambda']^{1/2} \tag{172}$$

Then
$$h' = \rho_0 \sin\lambda' \tag{172a}$$
$$= a(r/a)\sin\lambda'/[1-(1-r^2/a^2)\cos^2\lambda']^{1/2} \tag{173}$$

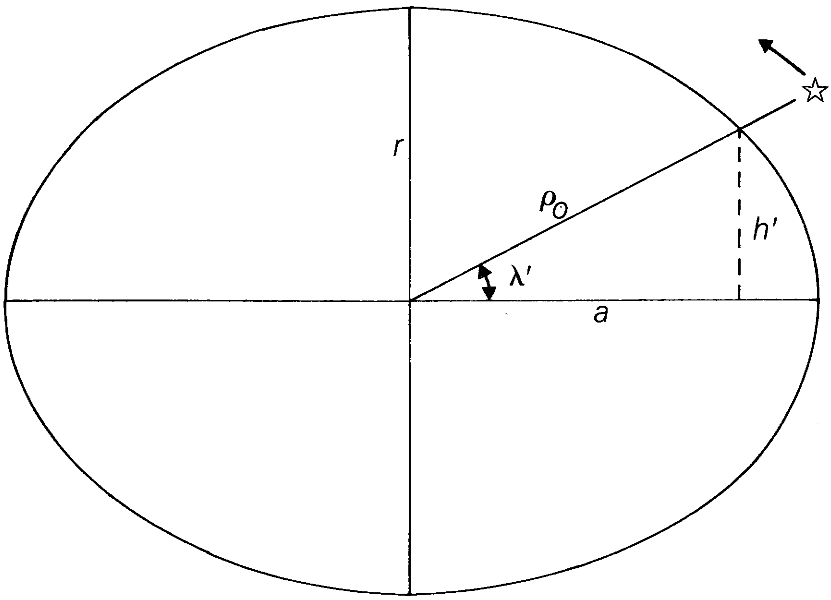

FIGURE 6.—Plane of satellite orbit.

Relating h' to rectangular coordinates x' and z' in the plane of figure 5, following the usual convention for the vertical axis of the ellipsoid, for points on the satellite groundtrack BC at a given time angle λ',

$$x' = h'\cos i \tag{173a}$$
$$= a(r/a)\sin\lambda'\cos i/[1-(1-r^2/a^2)\cos^2\lambda']^{1/2} \tag{174}$$
$$z' = h'\sin i$$
$$= a(r/a)\sin\lambda'\cos i/[1-(1-r^2/a^2)\cos\lambda']^{1/2} \tag{175}$$

Taking the same octant (upper left) as that used in figure 5, let us plot geodetic (or geographic) latitude ϕ (the same as ϕ_n) and satellite-apparent longitude λ_t, using equations (165) and (166) for the outer meridian, but multiplying x' by $\sin\lambda_t$ to give coordinates for any intermediate point (fig. 7). Longitude λ_t is measured from the central meridian, and is calculated from geodetic longitude from equation (7), just as for the sphere.

$$x' = a\cos\phi\sin\lambda_t/(1-e^2\sin^2\phi)^{1/2} \tag{176}$$

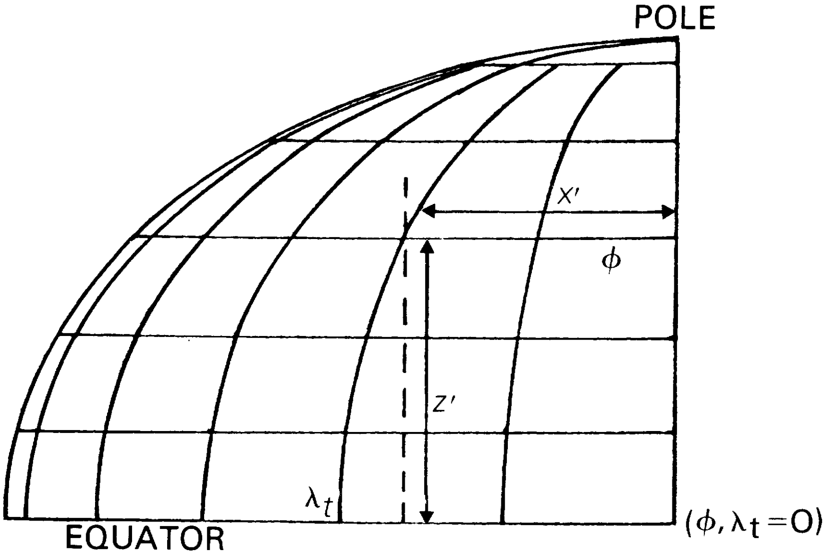

FIGURE 7.—Octant of ellipsoid with geographic latitude ϕ and satellite-apparent longitude λ, plotted.

$$z'=a(1-e^2)\sin\phi/(1-e^2\sin^2\phi)^{1/2} \tag{177}$$

(Note that x' is negative, but so is λ_t)

By equating (174) to (176) and (175) to (177) and solving simultaneously, ϕ and λ may be determined at various places along the groundtrack. This will be done later. To determine the true distance along the satellite groundtrack on the ellipsoid as a function of λ', but for a stationary Earth (to be corrected shortly for rotation), we return to figure 6 and apply equation (169) using $(1-r^2/a^2)$ for e^2 and ϕ'_n as a fictitious perpendicular equivalent of λ', just as ϕ_n relates to ϕ_g. Let us call the elemental distance along the groundtrack ds.

Then $ds/d\phi'_n=\rho_m$ adapted to the orbital plane, or

$$ds/d\phi'_n=a[1-(1-r^2/a^2)]/[1-(1-r^2/a^2)\sin^2\phi'_n]^{3/2}$$

$$=a(r^2/a^2)/[(1-(1-r^2/a^2)\sin^2\phi'_n]^{3/2} \tag{178}$$

Applying (164) to figures 4 and 6,

$$\tan\lambda'=(r^2/a^2)\tan\phi'_n \tag{179}$$

Equation (179) may be rearranged to give

$$\sin\phi'_n=\sin\lambda'/[1-(1-r^4/a^4)\cos^2\lambda']^{1/2} \tag{180}$$

Differentiating (179),

$$\sec^2\lambda'\,d\lambda'=(r^2/a^2)\sec^2\phi'_n\,d\phi'_n$$

$$d\phi'_n/d\lambda'=\cos^2\phi'_n/(r^2/a^2)\cos^2\lambda'$$

Substituting from (180) for ϕ'_n,

$$d\phi'_n/d\lambda'=(r^2/a^2)/[1-(1-r^4/a^4)\cos^2\lambda']$$

Multiplying this equation by (178), and substituting from (180) into (178),

$$ds/d\lambda'=a(r^4/a^4)/\left[1-\frac{(1-r^2/a^2)\sin^2\lambda'}{1-(1-r^4/a^4)\cos^2\lambda'}\right]^{3/2}[1-(1-r^4/a^4)\cos^2\lambda'] \tag{181}$$

PLOTTED SATELLITE GROUNDTRACK ON THE PROJECTION

Referring again to figure 2, but defining the elements with respect to the ellipsoid,

AB$=ds$, element of satellite motion during a given time interval proportional to angular satellite motion $d\lambda'$. Earth rotation is disregarded along line AB.

BD$=(P_2P_2)\ a\cos\phi_0 d\lambda'/(1-e\sin^2\phi_0)^{1/2}$, element of distance resulting from change of longitude along a given parallel of latitude ϕ_0 during $d\lambda'$, using equation (170) multiplied by $\cos\phi_0$ to give distance along the parallel, and by $(P_2/P_1)\ d\lambda'$ to give distance during satellite motion $d\lambda'$. P_2 and P_1 are defined after equations (1) through (7).

AD=projection of line AB onto map to account for change of longitude from point B to D during time of satellite travel through $d\lambda'$.

$\left.\begin{array}{l}\text{AC}=dx\\ \text{CD}=dy\end{array}\right\}$ elements of rectangular coordinates for groundtrack projection AD on the SOM map.

Following a derivation similar to that for equations (27) through (45), but applicable to the ellipsoid,

$$\text{AC}+\text{CB}=\text{AB}$$

$$dx+\text{CB}=ds$$

$$\text{CD}=\text{BD}\sin i'$$

where i' is, as for figure 2, the inclination of the groundtrack to a given parallel of latitude if the Earth were stationary.

$$dy=(P_2/P_1)\ a\cos\phi_0\sin i'\ d\lambda'/(1-e^2\sin^2\phi_0)^{1/2}. \tag{182}$$

$$\text{CB}=\text{BD}\cos i'$$

$$dx=ds-(P_2/P_1)\ a\cos\phi_0\cos i'\ d\lambda'/(1-e^2\sin^2\phi_0)^{1/2}. \tag{183}$$

To determine i', using equations (169) and (170),

$$\begin{aligned}\tan i'&=\text{BE}/\text{AE}\\ &=\rho_m\, d\phi_0/\rho_p\cos\phi_0\, d\lambda_t\\ &=(1-e^2)\, d\phi_0/(1-e^2\sin^2\phi_0)\cos\phi_0\, d\lambda_t.\end{aligned} \tag{184}$$

Equating (175) to (177),

$$z'/a=(r/a)\sin\lambda'\sin i/[1-(1-r^2/a^2)\cos^2\lambda']^{1/2}, \text{ and also}$$
$$=(1-e^2)\sin\phi_o/(1-e^2\sin^2\phi_o)^{1/2}. \qquad (185)$$

Equating (174) to (176),

$$x'/a=(r/a)\sin\lambda'\cos i/[1-(1-r^2/a^2)\cos^2\lambda']^{1/2}, \text{ and also}$$
$$=\cos\phi_o\sin\lambda_t/(1-e^2\sin^2\phi_o)^{1/2}. \qquad (186)$$

Rearranging (185) and also (186), several common terms may be moved to the left side of each:

$$(r/a)\sin\lambda'(1-e^2\sin^2\phi_o)^{1/2}/[1-(1-r^2/a^2)\cos^2\lambda']^{1/2}$$
$$=(1-e^2)\sin\phi_o/\sin i$$

and also

$$=\cos\phi_o\sin\lambda_t/\cos i \qquad (186a)$$

Equating the two right sides of (186a),

$$\sin\lambda_t=\tan\phi_o(1-e^2)/\tan i \qquad (187)$$

Differentiating (187),

$$\cos\lambda_t\,d\lambda_t=(1-e^2)\sec^2\phi_o\,d\phi_o/\tan i$$

and substituting into (184),

$$\tan i'=\cos\lambda_t\tan i\cos\phi_o/(1-e^2\sin^2\phi_o) \qquad (188)$$

Converting to the sine function,

$$\sin i'=\cos\lambda_t\tan i\cos\phi_o/[(1-e^2\sin^2\phi_o)^2+\cos^2\lambda_t\tan^2 i\cos\phi_o]^{1/2}$$

Substituting for λ_t from (187), after a few steps we obtain

$$\sin i'=\sqrt{\frac{\sin^2 i-\cos^2 i\tan^2\phi_o(1-e^2)^2}{1-e^4\sin^2\phi_o\cos^2 i}} \qquad (189)$$

Substituting into equation (182),

$$\frac{dy}{d\lambda'}=a\frac{P_2}{P_1}\sqrt{\frac{\sin^2 i-\sin^2\phi_o[1-e^2\cos^2 i\,(2-e^2)]}{(1-e^2\sin^2\phi_o)(1-e^4\sin^2\phi_o\cos^2 i)}} \qquad (190)$$

Reverting to (185) to solve for ϕ_o in terms of λ', first let

$$\sqrt{A}=(r/a)\sin\lambda'/[1-(1-r^2/a^2)\cos^2\lambda']^{1/2} \qquad (191)$$

Then, from (185),

$$\frac{(1-e^2)\sin\phi_o}{\sin i(1-e^2\sin^2\phi_o)^{1/2}}=\sqrt{A}$$

or

$$\sin\phi_o=\sqrt{A}\sin i/[(1-e^2)^2+Ae^2\sin^2 i]^{1/2} \qquad (192)$$

Substituting from (171) to simplify A,

$$A=\frac{1-e^2}{1-e^2\cos^2 i}\sin^2\lambda'/\left[1-\left(1-\frac{1-e^2}{1-e^2\cos^2 i}\right)\cos^2\lambda'\right]$$
$$=(1-e^2)\sin^2\lambda'/[1-e^2(1-\sin^2 i\sin^2\lambda')] \qquad (192a)$$

Substituting into (192), this reduces to

$$\sin \phi_0 = \sin i \sin \lambda'/[(1-e^2)^2 + e^2 \sin^2 i \sin^2\lambda'(2-e^2)]^{1/2} \tag{193}$$

The tangent form is simpler:

$$\tan \phi_0 = \sin i \sin \lambda'/(1-e^2)(1-\sin^2 i \sin^2\lambda')^{1/2} \tag{194}$$

Using an auxiliary angle, such that

$$\sin \psi = \sin i \sin \lambda', \tag{194a}$$

this simplifies still further:

$$\tan \phi_0 = \sin \psi/(1-e^2)(1-\sin^2 \psi)^{1/2} = \tan \psi/(1-e^2) \tag{194b}$$

Comparing this to (164), it is apparent that ψ is the same as ϕ_g, the geocentric latitude. Therefore,

$$\tan \phi_0 = \tan \phi_g/(1-e^2) \tag{195}$$

where

$$\sin \phi_g = \sin i \sin \lambda' \tag{196}$$

Substituting (193) into (190) and carrying out several laborious steps of algebra and trigonometry, a harmonious exact relationship may be developed, using certain operators which are constants for a given orbit:

$$\frac{dy}{a d\lambda'} = \frac{P_2}{P_1} \sin i \cos \lambda'[(1+T \sin^2\lambda')/(1+W \sin^2 \lambda')(1+Q \sin^2 \lambda')]^{1/2}, \tag{197}$$

where $T = e^2 \sin^2 i\ (2-e^2)/(1-e^2)^2$ (198)

$$Q = e^2 \sin^2 i/(1-e^2) \tag{199}$$

$$W = [(1-e^2 \cos^2 i)/(1-e^2)]^2 - 1 \tag{200}$$

This is the ellipsoidal version of equation (42) for the sphere, and may be given the symbol S, corresponding to equation (3). That is,

$$S = dy/a d\lambda' \tag{201}$$

Similarly equation (189) may be converted to the cosine form,

$$\cos i' = \frac{\cos i\ (1-e^2 \sin^2 \phi_0)}{\cos \phi_0(1-e^4 \sin^2 \phi_0 \cos^2 i)^{1/2}} \tag{202}$$

and substituted into equation (183) to give

$$\frac{dx}{d\lambda'} = \frac{ds}{d\lambda'} - a\frac{P_2}{P_1} \frac{\cos i(1-e^2 \sin^2 \phi_0)^{1/2}}{(1-e^4 \sin^2 \phi_0 \cos^2 i)^{1/2}} \tag{203}$$

Substituting from equation (193) for $\sin \phi_0$ and (181) for $ds/d\lambda'$, equation (203) also may be harmonically simplified after many laborious steps to

$$\frac{dx}{a d\lambda'} = \sqrt{\frac{1+Q \sin^2 \lambda'}{1+W \sin^2 \lambda'}} \left[\frac{1+W \sin^2 \lambda'}{(1+Q \sin^2 \lambda')^2} - \frac{P_2}{P_1} \cos i\right] \tag{204}$$

where W and Q are as in equations (199) and (200). This corresponds to equation (38) for the sphere, and may be given the symbol H, corresponding to equation (4), or

$$H = dx/ad\lambda'. \tag{205}$$

Unlike the spherical H, which is a constant for a given satellite orbit, the ellipsoidal H varies slightly with position along the groundtrack. This gives an ellipsoidal SOM groundtrack corresponding to the spherical groundtrack plotting of equations (39) and (43). Like them, equations (197) and (204) give a conformal groundtrack which is true to scale, but this is again not sufficient to reduce the scale error to the proper range away from the groundtrack. The slope must be changed, rotating scanlines at the same time, as for the sphere, but the optimum correction is slightly more complicated than before. Although the calculation of scale errors requires formulas for ϕ' and λ' in terms of ϕ and λ, which are derived subsequently, the rotation formulas are included at this point for convenience.

Combining equations (201) and (205), equation (46) is again obtained:

$$dy/dx = S/H. \tag{46}$$

After changing the slope with the same formula used for the sphere, and calculating scale factors at constant ϕ' and λ' along the sides of quadrilaterals 0.01° on a side, the slope was further revised so that the new angle θ' of the slope, temporarily using x' and y' again for revised coordinates, is found as follows:

$$\begin{aligned}\theta' &= \arctan\,(dy'/dx')\\ &= \theta + \theta_1\end{aligned}$$

where $\theta = \arctan\,(S/H)$ or the angle of the original slope

$$\theta_1 = \arctan\,(S/J) \tag{206}$$

and

$$J = (1-e^2)^3 \tag{207}$$

J is an empirical factor giving almost correct scale factors for the ellipsoid, but reducing as desired to $J=1$ for the sphere, since e is then zero.

Since

$$\tan\theta' = (\tan\theta + \tan\theta_1)/(1 - \tan\theta\,\tan\theta_1)$$

then

$$\begin{aligned}dy'/dx' &= (S/H + S/J)/(1 - S^2/HJ)\\ &= (JS + HS)/(HJ - S^2)\end{aligned} \tag{208}$$

Further derivation is pursued exactly as in the case of equations (47) through (52), but using equation (208) instead of (47). The following equivalent to (48) is obtained:

$$dx' = [(HJ - S^2)/H(J^2 + S^2)^{1/2}]\,dx \tag{209}$$

Substituting from (205),

$$dx'=a[(HJ-S^2)/(J^2+S^2)^{1/2}]\,d\lambda' \tag{210}$$

Integrating,

$$x'=a\int_0^{\lambda'}[(HJ-S^2)/(J^2+S^2)^{1/2}]\,d\lambda' \tag{211}$$

Substituting from (210) into (208), and reducing,

$$dy'=a[S(H+J)/(J^2+S^2)^{1/2}]\,d\lambda' \tag{212}$$

or

$$y'=a\int_0^{\lambda'}[S(H+J)/(J^2+S^2)^{1/2}]\,d\lambda' \tag{213}$$

Equations (211) and (213), if primes are dropped from x' and y', apply only to the satellite groundtrack. The factors for the scanlines are derived below.

SCANLINE PROJECTIONS

The normal (or perpendicular) radius of curvature ρ_n at a given point and in a given direction on the ellipsoid, using the nomenclature of equations (169) and (170), is as follows (Thomas, 1952, p. 60, eqn. 157):

$$\rho_n=\rho_m\,\rho_p/(\rho_m\sin^2\gamma+\rho_p\cos^2\gamma) \tag{214}$$

where γ is the azimuth east or west from north.

The azimuth of the scanlines is perpendicular to the satellite groundtrack in space (but not perpendicular on the rotating globe or projected map). The space-fixed azimuth of the groundtrack north from east is i' (equations (182) ff). The azimuth of the scanline west from north (space-fixed or Earth-fixed) is also i'. Therefore,

$$\rho_n=\rho_m\,\rho_p/(\rho_m\sin^2 i'+\rho_p\cos^2 i') \tag{215}$$

The scale in the scanline direction is based on this radius. The differential of the map distance S_s along the scanline, as a function of the angular distance ϕ' from the satellite groundtrack, positive (as for the sphere) in the direction of increasing i, is

$$dS_s=\rho_n\,d\phi'/\cos\phi' \tag{216}$$

for a conformal cylindrical projection. Actually, ρ_n should vary with angle ϕ', but for simplicity it is taken as the radius at the satellite groundtrack, in the proper direction, however. This error practically cancels out (less than one-millionth) in final equations.

Substituting into (216) from (215),

$$dS_s=\frac{1}{\cos\phi'}\,\frac{\rho_m\,\rho_p\,d\phi'}{\rho_m\,(1-\cos^2 i')+\rho_p\cos^2 i'}$$

$$=\frac{1}{\cos\phi'}\,\frac{\rho_m\,\rho_p\,d\phi'}{\rho_m-(\rho_m-\rho_p)\cos^2 i'}$$

Substituting for ρ_m, ρ_p, and cos i' from (169), (170), and (202), and combining,

$$\frac{dS_s}{d\phi'}=\frac{a}{\cos\phi'}\ \frac{1}{(1-e^2\sin^2\phi_o)^{1/2}\left[1+\dfrac{e^2\cos^2 i\ (1-e^2\sin^2\phi_o)^2}{(1-e^2)\ (1-e^4\sin^2\phi_o\cos^2 i)}\right]}$$

Since all these values are constants for a given scanline, except for ϕ',

$$S_s=\frac{a\ \ln\tan\left(\frac{1}{4}\pi+\frac{1}{2}\phi'\right)}{(1-e^2\sin^2\phi_o)^{1/2}\left[1+\dfrac{e^2\cos^2 i\ (1-e^2\sin^2\phi_o)^2}{(1-e^2)\ (1-e^4\sin^2\phi_o\cos^2 i)}\right]} \tag{217}$$

Another symbol is used to simplify this and future expressions. Let

$$F=(1-e^2\sin^2\phi_o)^{1/2}\left[1+\frac{e^2\cos^2 i\ (1-e^2\sin\phi_o)^2}{(1-e^2)\ (1-e^4\sin^2\phi_o\cos^2 i)}\right]$$

By substituting for sin ϕ_o from equation (193), and using several steps to simplify, it is found that

$$F=\sqrt{\frac{1+Q\sin^2\lambda'}{1+T\sin^2\lambda'}}\left[1+\frac{U(1+Q\sin^2\lambda')^2}{(1+W\sin^2\lambda')\ (1+T\sin^2\lambda')}\right] \tag{218}$$

where Q, T, and W are as in equations (198) - (200) and

$$U=e^2\cos^2 i/(1-e^2) \tag{219}$$

Thus equation (217) becomes

$$S_s=(a/F)\ \ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right) \tag{220}$$

Geometrically, (a/F) is ρ_n or the radius of curvature of the ellipsoid at the satellite groundtrack in the direction of the scan line.

The scan lines must be rotated θ_1 counterclockwise from vertical, where

$$\theta_1=\arctan\ (S/J). \tag{206}$$

As in the discussion following equation (52), sin θ_1 times S_s must be subtracted from equation (211), and cos θ_1 times S_s must be added to equation (213).

$$\begin{aligned}\sin\theta_1&=\tan\theta_1/(1+\tan^2\theta_1)^{1/2}\\&=(S/J)/(1+S^2/J^2)^{1/2}\\&=S/(J^2+S^2)^{1/2}\\\cos\theta_1&=1/(1+\tan^2\theta_1)^{1/2}\\&=J/(J^2+S^2)^{1/2}\end{aligned}$$

Dropping the primes on x' and y' for the final rectangular coordinates,

$$x/a=\int_0^{\lambda'}\frac{HJ-S^2}{(J^2+S^2)^{1/2}}\,d\lambda'-\frac{S}{F\,(J^2+S^2)^{1/2}}\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right) \tag{221}$$

$$y/a=\int_0^{\lambda'}\frac{S\,(H+J)}{(J^2+S^2)^{1/2}}\,d\lambda'+\frac{J}{F\,(J^2+S^2)^{1/2}}\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right) \tag{222}$$

TRANSFORMATION OF GEODETIC COORDINATES

In order to determine the geodetic latitude and longitude at a given ϕ' and λ', figure 8 may be used. This figure shows an octant of the equatorial aspect of an orthographic view of the ellipsoid, with ϕ and λ_t as in figure 7, but with the groundtrack, a transformed latitude ϕ' and transformed longitude λ'. A given scanline is a portion of the transformed meridian for λ'.

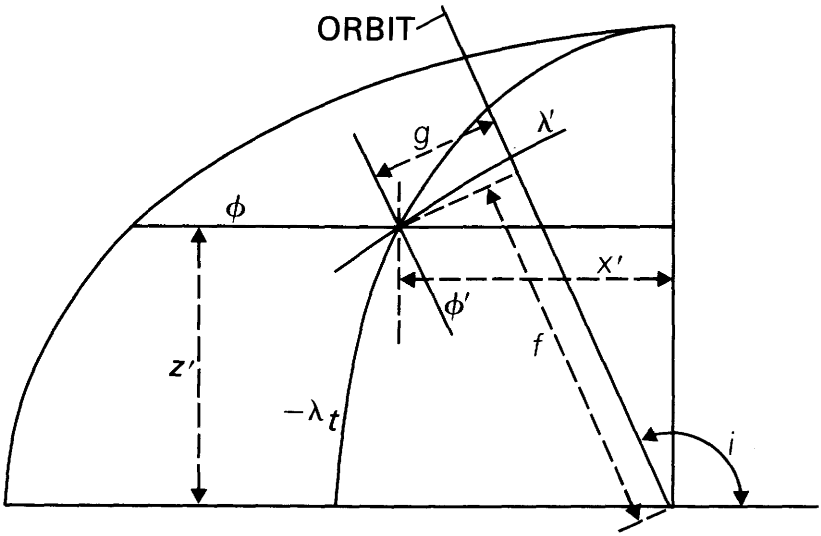

FIGURE 8.—Octant of ellipsoid with orbit.

The scanline is perpendicular to the groundtrack in the plane tangent to the ellipsoid, but in figure 8 it is only exactly perpendicular at the Equator. At the closest polar approach it is slightly skewed in order to remain tangent to the surface of the ellipsoid. At intermediate latitudes, its skewness will be less. For the point (ϕ, λ_t), equations for x' and z' have already been given:

$$x'=a \cos \phi \sin \lambda_t/(1-e^2 \sin^2 \phi)^{1/2} \tag{176}$$

$$z'=a (1-e^2) \sin \phi/(1-e^2 \sin^2 \phi)^{1/2} \tag{177}$$

The equation for y', in a direction perpendicular to the plane of figure 8, and increasing toward the reader, is found by multiplying x in equation (165) by $\cos \lambda_t$, or

$$y'=a \cos \phi \cos \lambda_t/(1-e^2 \sin^2 \phi)^{1/2} \tag{223}$$

Since y' is the same length when viewed in the plane of the orbit (fig. 9), the distance f from the ascending node to the projection of (ϕ, λ_t) in figure 8 onto the groundtrack is

$$f=y' \tan \lambda' \tag{224}$$

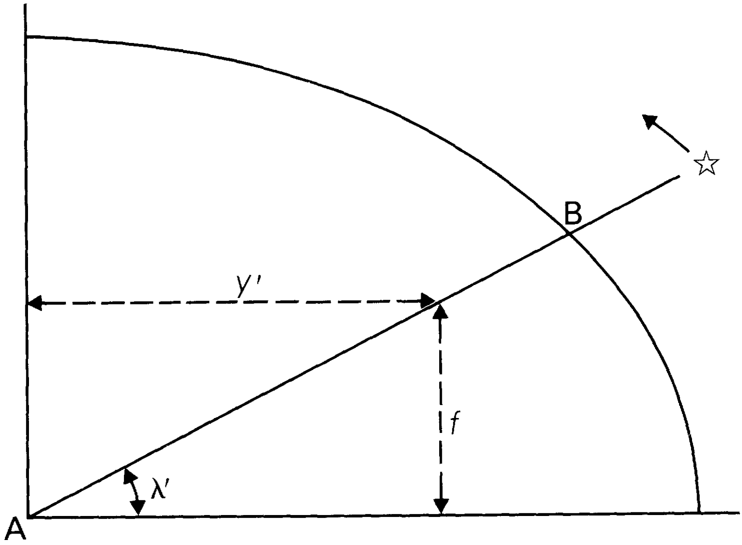

FIGURE 9.—Quadrant of plane of satellite orbit for ellipsoid.

The distance g as projected onto figure 8, from the discussion for equations (216) through (220) is nearly, but not exactly,

$$g=(a/F)\sin\phi' \tag{225}$$

with ϕ' positive to the left of the groundtrack in this view. Subsequent scale analysis shows that this approximation gives scale factors along the scanlines that are in error by no more than one part in a million at the groundtrack. Greater errors away from the groundtrack are due to other approximations as well.

By simple trigonometric relationships in figure 8, remembering that x' is shown as a negative value,

$$x'=f\cos i-g\sin i \tag{226}$$

$$z'=f\sin i+g\cos i \tag{227}$$

Solving for f and g by multiplying (226) by sin i, (227) by cos i, and subtracting for g, then multiplying (226) by cos i, (227) by sin i, and adding for f, and substituting from (224) and (225) for f and g,

$$(a/F)\sin\phi'=z'\cos i-x'\sin i \tag{228}$$

$$y'\tan\lambda'=x'\cos i+z'\sin i \tag{229}$$

Substituting for x' and z' from (176) and (177) into (228), and solving for ϕ':

$$\sin\phi'=\frac{F}{(1-e^2\sin^2\phi)^{1/2}}[(1-e^2)\sin\phi\cos i-\cos\phi\sin i\sin\lambda_t] \tag{230}$$

Substituting into (229), and solving for λ':

$$\tan \lambda' = \tan \lambda_t \cos i + (1-e^2) \tan \phi \sin i / \cos \lambda_t \tag{235}$$

with equation (7) as before to relate λ_t to λ and λ'. For iteration, follow type (2) in the section "Iteration procedures."

For formulas for λ' along the groundtrack in terms of ϕ_0 or λ_0, with ϕ' of course equal to zero, we may easily invert (195) and (196),

$$\sin \lambda' = \sin \phi_g / \sin i \tag{236}$$

where

$$\tan \phi_g = (1-e^2) \tan \phi_0 \tag{237}$$

to get λ' in terms of ϕ_0. For λ' in terms of λ_0, we may equate (176) with (173a), substituting (172a) for h':

$$\frac{a \cos \phi \sin \lambda_{t_0}}{(1-e^2 \sin^2 \phi)^{1/2}} = \rho_0 \sin \lambda' \cos i \tag{238}$$

Since in figure 9, when measured to B at the groundtrack,

$$\begin{aligned} y' &= \mathrm{AB} \cos \lambda' \\ &= \rho_0 \cos \lambda' \end{aligned} \tag{238a}$$

Equating (223) with (238a),

$$\frac{a \cos \phi \cos \lambda_{t_0}}{(1-e^2 \sin^2 \phi)^{1/2}} = \rho_0 \cos \lambda' \tag{239}$$

Dividing (238) by (239), and transposing, we get a simple rigorous equation which is identical with (135), the equivalent formula for the sphere:

$$\tan \lambda' = \tan \lambda_{t_0} / \cos i, \tag{135}$$

where

$$\lambda_{t_0} = \lambda_0 + (P_2/P_1)\, \lambda' \tag{136}$$

as before.

INVERSE EQUATIONS

Equations (230) and (235) may be inverted as follows to find ϕ and λ from ϕ' and λ': Equation (235) is transposed and rearranged to solve for $\tan \phi$:

$$(1-e^2) \tan \phi \sin i / \cos \lambda_t = \tan \lambda' - \tan \lambda_t \cos i$$

$$\begin{aligned} \tan \phi &= \cos \lambda_t\, (\tan \lambda' - \cos i \tan \lambda_t)/(1-e^2) \sin i \\ &= (\tan \lambda' \cos \lambda_t - \cos i \sin \lambda_t)/(1-e^2) \sin i \end{aligned} \tag{240}$$

This is a satisfactory form in which to leave this equation (except for the case $\cos \lambda' = 0$, discussed later) but it requires the calculation of λ_t from ϕ' and λ' first. To invert equation (230) for this purpose requires many more steps. Transposing (230), we isolate $\sin \lambda_t$:

$$\cos\phi \sin i \sin\lambda_t=(1-e^2)\sin\phi\cos i-\sin\phi'\,(1-e^2\sin^2\phi)^{\frac{1}{2}}/F$$
$$\begin{aligned}\sin\lambda_t&=(1-e^2)\tan\phi/\tan i-(\sin\phi'/F\sin i)\,(1-e^2\sin^2\phi)^{\frac{1}{2}}/\cos\phi\\&=(1-e^2)\tan\phi/\tan i-(\sin\phi'/F\sin i)\,(\sec^2\phi-e^2\tan^2\phi)^{\frac{1}{2}}\\&=(1-e^2)\tan\phi/\tan i-(\sin\phi'/F\sin i)\,[1+(1-e^2)\tan^2\phi]^{\frac{1}{2}}\end{aligned}$$

Substituting from (240) for $\tan\phi$,

$$\sin\lambda_t=\frac{\tan\lambda'\cos\lambda_t-\cos i\sin\lambda_t}{\sin i\tan i}-\frac{\sin\phi'}{F\sin i}\left[1+\frac{(\tan\lambda'\cos\lambda_t-\cos i\sin\lambda_t)^2}{(1-e^2)\sin^2 i}\right]^{\frac{1}{2}}$$

Multiplying through by $\sin^2 i$,

$$\sin^2 i\sin\lambda_t=\tan\lambda'\cos\lambda_t\cos i-\cos^2 i\sin\lambda_t-\frac{\sin\phi'}{F}\left[\sin^2 i+\frac{(\tan\lambda'\cos\lambda_t-\cos i\sin\lambda_t)^2}{(1-e^2)}\right]^{\frac{1}{2}}$$

Combining the left and the second right terms, then dividing by $\cos\lambda_t$,

$$\tan\lambda_t=\tan\lambda'\cos i-\frac{\sin\phi'}{F}\left[\frac{\sin^2 i}{\cos^2\lambda_t}+\frac{(\tan\lambda'-\cos i\tan\lambda_t)^2}{(1-e^2)}\right]^{\frac{1}{2}}$$

Transposing and squaring,

$$\tan^2\lambda_t-2\tan\lambda_t\tan\lambda'\cos i+\tan^2\lambda'\cos^2 i=\frac{\sin^2\phi'}{F^2}\left[\sin^2 i\,(1+\tan^2\lambda_t)+\frac{(\tan\lambda'-\cos i\tan\lambda_t)^2}{(1-e^2)}\right]$$

Expanding and collecting, terms

$$\tan^2\lambda_t\left[1-\frac{\sin^2\phi'}{F^2}\left(\sin^2 i+\frac{\cos^2 i}{1-e^2}\right)\right]-\tan\lambda_t\left(1-\frac{\sin^2\phi'}{F^2(1-e^2)}\right)(2\tan\lambda'\cos i)-\frac{\sin^2\phi'}{F^2}\left(\sin^2 i+\frac{\tan^2\lambda'}{1-e^2}\right)+\tan^2\lambda'\cos^2 i=0$$

Solving for $\tan\lambda_t$ as a quadratic equation,

$$\tan\lambda_t=[-B\pm(B^2-4AC)^{\frac{1}{2}}]/2A, \tag{241}$$

where

$$B=-\left(1-\frac{\sin^2\phi'}{F^2(1-e^2)}\right)(2\tan\lambda'\cos i)$$
$$A=1-\frac{\sin^2\phi'}{F^2}\left(\sin^2 i+\frac{\cos^2 i}{1-e^2}\right)$$
$$C=-\frac{\sin^2\phi'}{F^2}\left(\sin^2 i+\frac{\tan^2\lambda'}{1-e^2}\right)+\tan^2\lambda'\cos^2 i$$

To simplify these terms before combining them, first let

$$K=\sin\phi'/F. \tag{242}$$

Then the term $-B$ becomes,

$$-B=2[1-K^2/(1-e^2)]\tan\lambda'\cos i. \tag{243}$$

while

$$2A = 2\left[1-K^2\left(1-\cos^2 i+\frac{\cos^2 i}{1-e^2}\right)\right]$$
$$=2\left[1-K^2\left(1+\frac{\cos^2 i-\cos^2 i+e^2\cos^2 i}{1-e^2}\right)\right]$$
$$=2\left[1-K^2\left(1+\frac{e^2\cos^2 i}{1-e^2}\right)\right]$$
$$=2\left[1-K^2(1+U)\right] \tag{244}$$

using the term U from equation (219).

The radical $(B^2-4AC)^{\frac{1}{2}}$ is more complicated:

$$B^2-4AC=4[1-K^2/(1-e^2)]^2\tan^2\lambda'\cos^2 i$$
$$-4[1-K^2(1+U)][-K^2$$
$$\left(\sin^2 i+\frac{\tan^2\lambda'}{1-e^2}\right)+\tan^2\lambda'\cos^2 i]$$

$$\tfrac{1}{4}(B^2-4AC)=\tan^2\lambda'\cos^2 i-2K^2\tan^2\lambda'\cos^2 i/(1-e^2)$$
$$+K^4\tan^2\lambda'\cos^2 i/(1-e^2)^2+K^2\sin^2 i$$
$$+K^2\tan^2\lambda'/(1-e^2)-\tan^2\lambda'\cos^2 i$$
$$-K^4(1+U)[\sin^2 i+\tan^2\lambda'/(1-e^2)]$$
$$+K^2(1+U)\tan^2\lambda'\cos^2 i$$

Cancelling the first and sixth terms on the right, multiplying by $(1-e^2)/K^2$, and subsequently resubstituting its trigonometric form for U,

$$(B^2-4AC)(1-e^2)/4K^2=-2\tan^2\lambda'\cos^2 i+K^2\tan^2\lambda'\cos^2 i/(1-e^2)$$
$$+\sin^2 i\,(1-e^2)+\tan^2\lambda'-K^2(1+U)$$
$$[\sin^2 i\,(1-e^2)+\tan^2\lambda']+(1+U)\tan^2\lambda'$$
$$\cos^2 i\,(1-e^2)$$
$$=\tan^2\lambda'[-2\cos^2 i+1+\cos^2 i\,(1-e^2)$$
$$+e^2\cos^4 i]+\sin^2 i\,(1-e^2)+K^2$$
$$[\tan^2\lambda'\cos^2 i/(1-e^2)-\sin^2 i\,(1-e^2)-\tan^2\lambda'$$
$$-e^2\sin^2 i\cos^2 i-e^2\cos^2 i\tan^2\lambda'/(1-e^2)]$$
$$=\tan^2\lambda'[-2(1-\sin^2 i)+1+(1-\sin^2 i)$$
$$(1-e^2)+e^2(1-\sin^2 i)^2]+\sin^2 i\,(1-e^2)$$
$$+K^2\ \{\tan^2\lambda'[\cos^2 i/(1-e^2)-1-e^2\cos^2 i/$$
$$(1-e^2)]-\sin^2 i\,(1-e^2)-e^2\sin^2 i\,(1-\sin^2 i)\}$$
$$=\tan^2\lambda'\sin^2 i\,(1-e^2+e^2\sin^2 i)+K^2$$
$$[\tan^2\lambda'(\cos^2 i-1)-\sin^2 i\,(1-e^2\sin^2 i)]$$
$$+\sin^2 i\,(1-e^2)$$
$$=\tan^2\lambda'\sin^2 i\,(1-e^2\cos^2 i)-K^2\sin^2 i$$
$$(\tan^2\lambda'+1-e^2\sin^2 i)+\sin^2 i\,(1-e^2)$$

Multiplying by $\cos^2\lambda'/\sin^2 i$,

$$\begin{aligned}(B^2-4AC)(1-e^2)\cos^2\lambda'/4K^2\sin^2 i &= \sin^2\lambda'(1-e^2\cos^2 i)-K^2\\&\quad(\sin^2\lambda'+\cos^2\lambda'-e^2\sin^2 i\\&\quad\cos^2\lambda')+\cos^2\lambda'(1-e^2)\\&=\sin^2\lambda'-e^2\sin^2\lambda'(1-\sin^2 i)\\&\quad+(1-\sin^2\lambda')(1-e^2)-K^2\\&\quad[1-e^2\sin^2 i(1-\sin^2\lambda')]\\&=1-e^2+e^2\sin^2 i\sin^2\lambda'-K^2\\&\quad(1-e^2\sin^2 i+e^2\sin^2 i\sin^2\lambda')\\&=(1-e^2)+Q\sin^2\lambda'(1-e^2)-K^2\\&\quad[1-e^2+e^2\cos^2 i+Q\sin^2\lambda'\\&\quad(1-e^2)],\end{aligned}$$

making use of Q from equation (199). We may again divide out $(1-e^2)$:

$$(B^2-4AC)\cos^2\lambda'/4K^2\sin^2 i=(1+Q\sin^2\lambda')(1-K^2)-K^2U.$$

The radical of equation (241) is then

$$(B^2-4AC)^{\frac{1}{2}}=(2K\sin i/\cos\lambda')[(1+Q\sin^2\lambda')(1-K^2)-K^2U]^{\frac{1}{2}} \tag{245}$$

Combining equations (241) through (245) and dividing out the 2 (calculating experience shows that the $\pm$ needs to be a $-$):

$$\tan\lambda_t=\frac{[1-K^2/(1-e^2)]\tan\lambda'\cos i-(K\sin i/\cos\lambda')[(1+Q\sin^2\lambda')(1-K^2)-K^2U]^{\frac{1}{2}}}{1-K^2(1+U)} \tag{246}$$

This is the final formula for λ_t in terms of λ' and ϕ', where K is found from (242).

Then
$$\lambda=\lambda_t-(P_2/P_1)\lambda' \tag{93}$$

If $\cos\lambda'=0$ in equation (240) the equation is indeterminate, since $\cos\lambda_t$ is also zero. From equation (235), if λ_t is 90°, λ' is $-90°$, and vice versa. This applies to each previous or subsequent cycle of 360° in λ'. Thus, in (230), if $\cos\lambda'=0$, $\sin\lambda_t=\pm1$, taking minus the sign of $\sin\lambda'$, or, for $\cos\lambda'=0$,

$$\sin\phi'=[F/(1-e^2\sin^2\phi)^{\frac{1}{2}}][(1-e^2)\sin\phi\cos i\pm\cos\phi\sin i]$$

taking the sign of $\sin\lambda'$. To find ϕ in terms of ϕ', i, and F, first let $(\sin\phi'/F)=K$. Transposing and squaring,

$$\begin{aligned}K^2-K^2e^2\sin^2\phi &= (1-e^2)^2\sin^2\phi\cos^2 i\ \pm2\sin\phi\cos\phi\sin i\\&\quad\cos i(1-e^2)+\cos^2\phi\sin^2 i\\&=(1-e^2)^2\sin^2\phi\cos^2 i+\sin^2 i-\sin^2\phi\sin^2 i\\&\quad\pm2\sin\phi(1-\sin^2\phi)^{\frac{1}{2}}\sin i\cos i(1-e^2)\end{aligned}$$

$$\begin{aligned}\sin^2\phi[(1-e^2)^2\cos^2 i-\sin^2 i+K^2e^2]+[\sin^2 i-K^2]&=\\ \pm2\sin\phi(1-\sin^2\phi)^{\frac{1}{2}}\sin i\cos i(1-e^2)\end{aligned}$$

Let
$$A=(1-e^2)^2\cos^2 i-\sin^2 i+K^2e^2 \tag{246a}$$

and
$$B=\sin^2 i-K^2$$

Substituting and squaring,

$$A^2 \sin^4 \phi + 2AB \sin^2 \phi + B^2 = 4 \sin^2 \phi\,(1-\sin^2 \phi) \sin^2 i \cos^2 i\,(1-e^2)^2$$

$$[A^2 + 4(1-e^2)^2 \sin^2 i \cos^2 i] \sin^4 \phi + [2AB - 4(1-e^2)^2 \sin^2 i \cos^2 i] \sin^2 \phi + B^2 = 0$$

Solving for $\sin^2 \phi$ with the binomial theorem,

$$\begin{aligned} \sin^2 \phi = {} & \{-2AB + 4(1-e^2)^2 \sin^2 i \cos^2 i \pm [4A^2B^2 - 16AB \\ & (1-e^2)^2 \sin^2 i \cos^2 i + 16(1-e^2)^4 \sin^4 i \cos^4 i - 4A^2B^2 \\ & -4B^2 (1-e^2)^2 \sin^2 i \cos^2 i]^{\frac{1}{2}}\} / [2A^2 + 8(1-e^2)^2 \sin^2 i \cos^2 i] \\ = {} & \{-AB + 2(1-e^2)^2 \sin^2 i \cos^2 i \pm 2(1-e^2) \sin i \cos i \\ & [(1-e^2)^2 \sin^2 i \cos^2 i - B(A+B)]^{\frac{1}{2}}\} / \\ & [A^2 + 4(1-e^2)^2 \sin^2 i \cos^2 i] \end{aligned} \tag{246b}$$

Resubstituting for A and B in the radical,

$$\begin{aligned} (1-e^2)^2 \sin^2 i \cos^2 i - B(A+B) = {} & (1-e^2)^2 \sin^2 i \cos^2 i - (1-e^2)^2 \\ & \cos^2 i \sin^2 i + (1-e^2)^2 K^2 \cos^2 i \\ & -K^4 (1-e^2) + K^2 (1-e^2) \sin^2 i \\ = {} & K^2 (1-e^2) [(1-e^2) \cos^2 i + \sin^2 i - K^2] \\ = {} & K^2 (1-e^2)(1-e^2 \cos^2 i - K^2) \end{aligned}$$

Letting

$$A' = (1-e^2) \sin i \cos i \tag{246c}$$

and returning to equation (246b), the formula for ϕ in terms of ϕ' when $\cos \lambda' = 0$ is as follows:

$$\begin{aligned} \sin \phi = \pm \{ & [-A(\sin^2 i - K^2) \\ & + 2A' [A \pm K (1-e^2)^{\frac{1}{2}} (1-e^2 \cos^2 i - K^2)^{\frac{1}{2}}]] / (A^2 + 4A'^2)\}^{\frac{1}{2}} \end{aligned} \tag{246d}$$

Experience with the formula indicates that each of the two "±" signs should take the sign of $\sin \lambda''$.

In the unlikely event that the inclination i of the orbit is 0°—an equatorial orbit—equation (240) will be indeterminate. In that case, equation (230) may be rewritten for $i=0$ as follows:

$$\sin \phi' = \frac{F}{(1-e^2 \sin^2 \phi)^{1/2}} (1-e^2) \sin \phi \tag{247}$$

By considering the geometry, it becomes evident that ϕ and ϕ' are independent of λ in this case. Solving (247) for ϕ,

$$\frac{\sin \phi}{(1-e^2 \sin^2 \phi)^{1/2}} = \frac{\sin \phi'}{F(1-e^2)}$$

Substituting K from equation (242),

$$\frac{\sin \phi}{(1-e^2 \sin^2 \phi)^{1/2}} = \frac{K}{1-e^2}$$

$$(1-e^2)^2 \sin^2 \phi = K^2 (1-e^2 \sin^2 \phi)$$

$$\sin^2 \phi [(1-e^2)^2 + e^2 K^2] = K^2$$

$$\sin \phi = K / [(1-e^2)^2 + e^2 K^2]^{\frac{1}{2}} \tag{248}$$

only if $i=0$. Under these conditions, equation (246) may still be used, but it simplifies merely to $\lambda_t=\lambda'$, after substituting equation (219) for U.

In a previous paper (Snyder, 1978b) inverse formulas different from (240) and (246) are shown as equations (34) through (36) and (38). These are partially empirical equations, not exact inverses of present equations (230) and (235) although they are satisfactory within the Landsat scanning range. Equations (240) and (246), derived after the earlier publication, are exact inverses (even though the projection itself involves some empirical assumptions) and are slightly simpler to compute. Therefore, they replace the earlier formulas, for which the derivation is omitted here.

Inverse equations for ϕ' and λ' in terms of x and y are derived from ellipsoidal formulas (221) and (222) in a manner completely analogous to and as simply as the spherical equivalents, although terms F and J are in one set and not the other. Comparable to steps for equations (80) and (87), equation (222) is multiplied by (S/J) and added to (221), cancelling out the right term of each:

$$\frac{x+(S/J)y}{a}=\int_0^{\lambda'}\frac{HJ-S^2}{(J^2+S^2)^{1/2}}\,d\lambda'+\frac{S}{J}\int_0^{\lambda'}\frac{S(H+J)}{(J^2+S^2)^{1/2}}\,d\lambda' \tag{249}$$

This allows solution for λ' in terms of x and y, but like equation (80), it is impractical as it stands, because it requires both numerical integration and iteration unless Fourier series are used, in which case the iteration converges rapidly, as for (80). These series are developed below. For ϕ' in terms of x and y, equation (222) is transposed:

$$\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'\right)=\frac{F(J^2+S^2)^{\frac{1}{2}}}{J}\left[\frac{y}{a}-\int_0^{\lambda'}\frac{S(H+J)}{(J^2+S^2)^{1/2}}\,d\lambda'\right] \tag{250}$$

Like equation (87), this first requires calculation of λ' from (249), or its Fourier equivalent, but another Fourier series is recommended to eliminate numerical integration in (250), even though iteration is not required.

CORRECTION FOR VERTICAL SCANNING

The above equations were derived for Landsat scanning in which the sensor would follow a groundtrack directly between the satellite and the center of the ellipsoid, or geocentric scanning. Actually, the sensor follows a groundtrack which is vertical, or normal to the plane at the surface of the ellipsoid. This is a small but significant correction. (In the case of the sphere, geocentric and vertical groundtracks are identical.) As with other groundtrack calculations, the positions for the vertical groundtrack may be rigorously determined.

In figure 10, the satellite is at S. (SA is not a projection of the orbital plane, but merely a vector in the orbital plane at some position on the

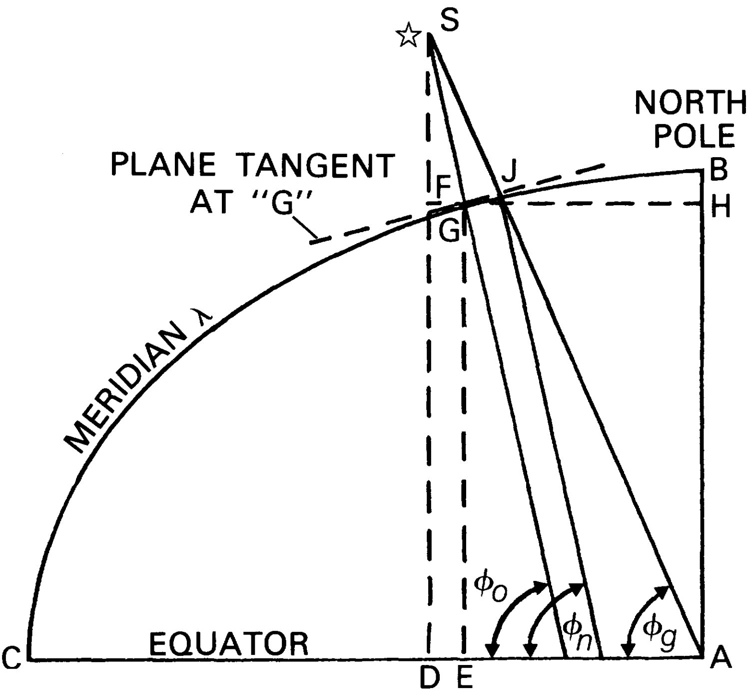

FIGURE 10.—Correction for vertical scanning on an ellipsoid.

orbit crossing meridian CB.) The radius of the circular orbit is a constant R_o=SA. The satellite scans vertically with true groundtrack at G, at which geodetic latitude is ϕ_o. The geocentric groundtrack is at J, at geodetic latitude ϕ_n and geocentric latitude ϕ_g. Geometrically,

$$SD=SF+GE=R_o \sin \phi_g \tag{251}$$

$$FH=FG+EA=R_o \cos \phi_g \tag{252}$$

$$SF/FG=\tan \phi_o \tag{253}$$

From equations (165) and (166), for rectangular coordinates of point G,

$$EA=x=a \cos \phi_o/(1-e^2 \sin^2 \phi_o)^{\frac{1}{2}} \tag{254}$$

$$GE=y=a\ (1-e^2) \sin \phi_o/(1-e^2 \sin^2 \phi_o)^{\frac{1}{2}} \tag{255}$$

Combining (251) through (255),

$$\tan \phi_o=[R_o \sin \phi_g-a\ (1-e^2) \sin \phi_o/(1-e^2 \sin^2 \phi_o)^{\frac{1}{2}}]/ [R_o \cos \phi_g-a \cos \phi_o/(1-e^2 \sin^2 \phi_o)^{\frac{1}{2}}]$$

Transposing and multiplying by $\cos \phi_o$,

$$R_o \sin \phi_o \cos \phi_g-a \sin \phi_o \cos \phi_o/(1-e^2 \sin^2 \phi_o)^{\frac{1}{2}}=R_o \cos \phi_o \sin \phi_g -a \sin \phi_o \cos \phi_o/(1-e^2 \sin^2 \phi_o)^{\frac{1}{2}}+ae^2 \sin \phi_o \cos \phi_o/(1-e^2 \sin^2 \phi_o)^{\frac{1}{2}}$$

Combining terms,

$$\sin(\phi_0-\phi_g)=ae^2\sin\phi_0\cos\phi_0/R_0(1-e^2\sin^2\phi_0)^{\frac{1}{2}}$$

or
$$\phi_g=\phi_0-\arcsin\left[\frac{ae^2\sin 2\phi_0}{2R_0(1-e^2\sin^2\phi_0)^{1/2}}\right] \tag{256}$$

Since we have previously obtained (236),

$$\sin\lambda'=\sin\phi_g/\sin i, \tag{236}$$

we can now calculate λ' for a given ϕ_0 along a vertical groundtrack, using (236) and (256). For the inverse, ϕ_0 in terms of λ', it was found that the direct inverse of (236) and (256), which requires iteration, converges rapidly enough to be more satisfactory than an approximation. Thus:

$$\phi_0=\arcsin(\sin\lambda'\sin i)+\arcsin\left[\frac{ae^2\sin 2\phi_0}{2R_0(1-e^2\sin^2\phi_0)^{1/2}}\right] \tag{257}$$

For λ' in terms of λ_0 and vice versa, there is no change from the geocentric formulas to correct for vertical scanning, since the groundtrack merely changes latitude along the same meridian when it drops from a geocentric to vertical position under the satellite. Thus (135) and (136) apply, with iteration required in the direction shown, and no iteration for the inverses:

$$\tan\lambda'=\tan\lambda_t/\cos i \tag{135}$$

where
$$\lambda_{t_0}=\lambda_0+(P_2/P_1)\,\lambda' \tag{136}$$

Inversely,

$$\lambda_0=\lambda_{t_0}-(P_2/P_1)\,\lambda' \tag{93}$$

$$\tan\lambda_{t_0}=\tan\lambda'\cos i \tag{133}$$

The best means of correcting the foregoing formulas for the geocentric groundtrack with respect to the vertical groundtrack was a matter of some concern. It was apparent from scale factor calculations that the earlier equations are quite accurate (though not perfectly, of course), as far as plotting meridians and parallels onto the grid of the map projection, and they are based on a groundtrack which cuts the ellipsoid with a plane, while the "plane" is "warped" for the vertical groundtrack. After various approaches were tried, it was decided to retain the geocentric formulas exactly as derived, but to change ϕ' and λ' in them to ϕ'' and λ'', defined as pseudotransformed latitude and longitude, respectively, relative to a geocentric groundtrack. Transformed coordinates ϕ' and λ' relative to the vertical groundtrack are then converted to the equivalent ϕ'' and λ'' as follows.

Equations (257), (93), and (133) are used to calculate ϕ_0 and λ_0 for a given λ'. Then equations (235) and (7) are used to calculate λ''_0 (instead of λ') for the same ϕ_o and λ_o, where λ_o'' is the adjusted position of λ' in a direction parallel to the groundtrack:

$$\tan \lambda_o'' = \tan \lambda_{t_o} \cos i + (1-e^2) \tan \phi_o \sin i / \cos \lambda_{t_o} \tag{258}$$

$$\lambda_{t_o} = \lambda_o + (P_2/P_1)\, \lambda'' \tag{259}$$

Equation (230) is similarly used to find ϕ_o'' (instead of ϕ') for this ϕ_o and λ_o, where ϕ_o'' is the angular displacement of the true groundtrack from the geocentric groundtrack:

$$\sin \phi_o'' = \frac{F}{(1-e^2 \sin^2 \phi_o)^{1/2}} [(1-e^2) \sin \phi_o \cos i - \cos \phi_o \sin i \sin \lambda_{t_o}] \tag{260}$$

Thus ϕ'' and λ'' for the groundtrack can be determined as functions of λ'. Then ϕ'' for any given angular displacement ϕ' in the satellite scanning is

$$\phi'' = \phi' + \phi_o'' \tag{261}$$

and λ'' for a given time-angle λ' in the satellite orbit is

$$\lambda'' = \lambda' + (\lambda_o'' - \lambda'). \tag{262}$$

These corrections from ϕ' to ϕ'' and λ' to λ'' are small and cyclical. Therefore, it is better to eliminate the repetitive lengthy calculations for each value by using Fourier series as derived below.

SUMMARY OF CLOSED EQUATIONS

At this point it is well to bring together the various closed equations derived above for the ellipsoidal SOM before giving the Fourier equivalents, and at the same time changing ϕ' and λ' to ϕ'' and λ'' so that they can be used with the vertical groundtrack corrections.

1. For x and y in terms of ϕ'' and λ'' (from equations (221) and (222)):

$$\frac{x}{a} = \int_0^{\lambda''} \frac{HJ - S^2}{(J^2+S^2)^{1/2}}\, d\lambda'' - \frac{S}{F(J^2+S^2)^{1/2}} \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'') \tag{263}$$

$$\frac{y}{a} = \int_0^{\lambda''} \frac{S(H+J)}{(J^2+S^2)^{1/2}}\, d\lambda'' + \frac{J}{F(J^2+S^2)^{1/2}} \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'') \tag{264}$$

where, from equations (201) and (197),

$$S = (P_2/P_1) \sin i \cos \lambda'' [(1+T \sin^2 \lambda'')/(1+W \sin^2 \lambda'')(1+Q \sin^2 \lambda'')]^{\frac{1}{2}} \tag{265}$$

from equations (205) and (204)

$$H = \sqrt{\frac{1+Q \sin^2 \lambda''}{1+W \sin^2 \lambda''}} \left[\frac{1+W \sin^2 \lambda''}{(1+Q \sin^2 \lambda'')^2} - (P_2/P_1) \cos i\right] \tag{266}$$

from equation (218),

$$F = \sqrt{\frac{1+Q \sin^2 \lambda''}{1+T \sin^2 \lambda''}} \left[1 + \frac{U(1+Q \sin^2 \lambda'')^2}{(1+W \sin^2 \lambda'')(1+T \sin^2 \lambda'')}\right] \tag{267}$$

and from equations (207), (198) through (200), and (219),

$$J=(1-e^2)^3 \tag{207}$$

$$Q=e^2\sin^2 i/(1-e^2) \tag{199}$$

$$T=e^2\sin^2 i\ (2-e^2)/(1-e^2)^2 \tag{198}$$

$$W=[(1-e^2\cos^2 i)/(1-e^2)]^2-1 \tag{200}$$

$$U=e^2\cos^2 i/(1-e^2) \tag{219}$$

2. For ϕ and λ in terms of ϕ'' and λ'', from equations (240), (242), (246), (248), and (93),

$$\lambda=\lambda_t-(P_2/P_1)\,\lambda'' \tag{268}$$

$$\tan\lambda_t=\frac{\left(1-\dfrac{K^2}{1-e^2}\right)\tan\lambda''\cos i-\dfrac{K\sin i}{\cos\lambda''}[(1+Q\sin^2\lambda'')\,(1-K^2)-K^2U]^{\frac{1}{2}}}{1-K^2\,(1+U)} \tag{269}$$

where

$$K=\sin\phi''/F \tag{270}$$

$$\tan\phi=(\tan\lambda''\cos\lambda_t-\cos i\sin\lambda_t)/(1-e^2)\sin i \tag{271}$$

If $\cos\lambda''=0$, (271) is indeterminate, but

$$\sin\phi=\pm\{[-A(\sin^2 i-K^2)+2A'[A' \pm K\,(1-e^2)^{\frac{1}{2}}(1-e^2\cos^2 i-K^2)^{\frac{1}{2}}]]/(A^2+4A'^2)\}^{\frac{1}{2}} \tag{246d}$$

where the two $\pm$ signs take the sign of $\sin\lambda''$,

$$A=(1-e^2)^2\cos^2 i-\sin^2 i+K^2e^2 \tag{246a}$$

$$A'=(1-e^2)\sin i\cos i \tag{246c}$$

If $i=0$, (271) is indeterminate, but

$$\sin\phi=K/[(1-e^2)^2+e^2K^2]^{\frac{1}{2}} \tag{248}$$

See step 1 following equation (119) for quadrant adjustment.

3. For ϕ'' and λ'' in terms of ϕ' and λ', using equations (261) and (262),

$$\phi''=\phi'+\phi''_o \tag{261}$$

$$\lambda''=\lambda'+(\lambda''_o-\lambda') \tag{262}$$

with equations (258) through (260),

$$\tan\lambda''_o=\tan\lambda_{t_o}\cos i+(1-e^2)\tan\phi_o\sin i/\cos\lambda_{t_o} \tag{258}$$

$$\lambda_{t_o}=\lambda_o+(P_2/P_1)\,\lambda''_o \tag{259}$$

$$\sin\phi''_o=\frac{F}{(1-e^2\sin^2\phi_o)^{1/2}}\,[(1-e^2)\sin\phi_o\cos i-\cos\phi_o\sin i\sin\lambda_{t_o}] \tag{260}$$

and equations (257), (93), and (133),

$$\phi_o = \arcsin(\sin\lambda' \sin i) + \arcsin\left[\frac{ae^2 \sin 2\phi_o}{2R_o (1-e^2 \sin^2\phi_o)^{1/2}}\right] \tag{257}$$

$$\lambda_o = \lambda_{t_o} - (P_2/P_1)\lambda' \tag{93}$$

$$\tan\lambda_{t_o} = \tan\lambda' \cos i \tag{133}$$

Equations (259) and (257) require iteration, and the use of Fourier equivalents for the combination of these equations is especially recommended.

4. For ϕ' and λ' in terms of ϕ'' and λ''. While the inverse of the above equations could be listed, the coefficients of the Fourier series are so small that they can be used for forward or inverse merely by changing signs, without significant difference. (See below under the heading "Fourier series.")

5. For ϕ'' and λ'' in terms of ϕ and λ, from equations (235), (7), and (230),

$$\tan\lambda'' = \tan\lambda_t \cos i + (1-e^2)\tan\phi \sin i/\cos\lambda_t \tag{272}$$

$$\lambda_t = \lambda + (P_2/P_1)\lambda'' \tag{273}$$

following type 2 under the section "Iteration procedures."

$$\sin\phi'' = \frac{F}{(1-e^2\sin^2\phi)^{1/2}}[(1-e^2)\sin\phi\cos i - \cos\phi\sin i\sin\lambda_t] \tag{274}$$

6. For ϕ'' and λ'' in terms of x and y from equations (249) and (250),

$$\frac{x+(S/J)y}{a} = \int_0^{\lambda''} \frac{HJ-S^2}{(J^2+S^2)^{1/2}}\,d\lambda'' + \frac{S}{J}\int_0^{\lambda''} \frac{S(H+J)}{(J^2+S^2)^{1/2}}\,d\lambda'' \tag{275}$$

$$\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi''\right) = \frac{F(J^2+S^2)^{1/2}}{J}\left[\frac{y}{a} - \int_0^{\lambda''} \frac{S(H+J)}{(J^2+S^2)^{1/2}}\,d\lambda''\right] \tag{276}$$

As stated before, the solution of (275) for λ'' is possible but totally impractical unless a Fourier series is substituted. This series is equation (298) below.

7. For the satellite groundtrack,

a. For λ' in terms of ϕ_o, using (236) and (256),

$$\sin\lambda' = \sin\phi_g/\sin i \tag{236}$$

where

$$\phi_g = \phi_o - \arcsin\left[\frac{ae^2\sin 2\phi_o}{2R_o(1-e^2\sin^2\phi_o)^{1/2}}\right] \tag{256}$$

b. For λ' in terms of λ_o, using (135) and (136) with iteration,

$$\tan\lambda' = \tan\lambda_{to}/\cos i \tag{135}$$

$$\lambda_{t_o} = \lambda_o + (P_2/P_1)\lambda' \tag{136}$$

c. For ϕ_0 in terms of λ', using (257),

$$\phi_0=\arcsin(\sin\lambda'\sin i)+\arcsin\left[\frac{ae^2\sin 2\phi_0}{2R_0(1-e^2\sin^2\phi_0)^{1/2}}\right] \quad (257)$$

d. For λ_0 in terms of λ', using (93) and (133),

$$\lambda_0=\lambda_{t_0}-(P_2/P_1)\,\lambda' \quad (93)$$

$$\tan\lambda_{t_0}=\tan\lambda'\cos i \quad (133)$$

See step 1 following equation (119) for quadrant adjustments.

Symbols ϕ, λ, P_2, P_1, and i are explained after equation (7). In addition,

a = major semi-axis of ellipsoid (see fig. 4). This is 6378206.4 m for the Clarke 1866 ellipsoid.

e^2 = square of eccentricity of ellipsoid (see equation (160)). The square is 0.00676866 for the Clarke 1866 ellipsoid.

R_0 = radius of circular satellite orbit. This is 7,294.69 km for Landsat.

Other symbols such as ϕ', λ', ϕ'' and λ'' are intermediate quantities calculated from formulas above.

Iteration procedures given for the spherical SOM, after equation (137), apply identically to the corresponding formulas for the ellipsoidal SOM. The only additional ellipsoidal formula requiring iteration is (257), but the iteration procedure for it is the same as that following equation (86): trying any value of ϕ_0 in the right side, solving for ϕ_0 on the left side, and using that as the next trial ϕ_0 on the right side, until negligible change takes place.

Figure 11 shows an enlargement of the second quadrant of the projection after the ascending node. This graticule is 10°, calculated for the ellipsoid, but almost imperceptibly different from the sphere at this scale.

FOURIER SERIES

The principle of the Fourier series calculations is given in the section under the spherical SOM beginning with equation (53). For corresponding ellipsoidal SOM formulas, the Fourier equivalents are very similar, except that the constants are somewhat different. Furthermore, since ellipsoidal equations are more complicated than the corresponding spherical equations, there is a benefit in converting some of the ellipsoidal terms to series, whereas the corresponding spherical terms should be left in the closed form. In any case, the repeating cycle is 360° or 2π.

Beginning with the integral term of (263), for x in terms of λ'', the Fourier equivalent is like the first part of equation (73), coupled with equations (75) and (76):

$$x_a/a=B\lambda''+A_2\sin 2\lambda''+A_4\sin 4\lambda''+\ldots \quad (277)$$

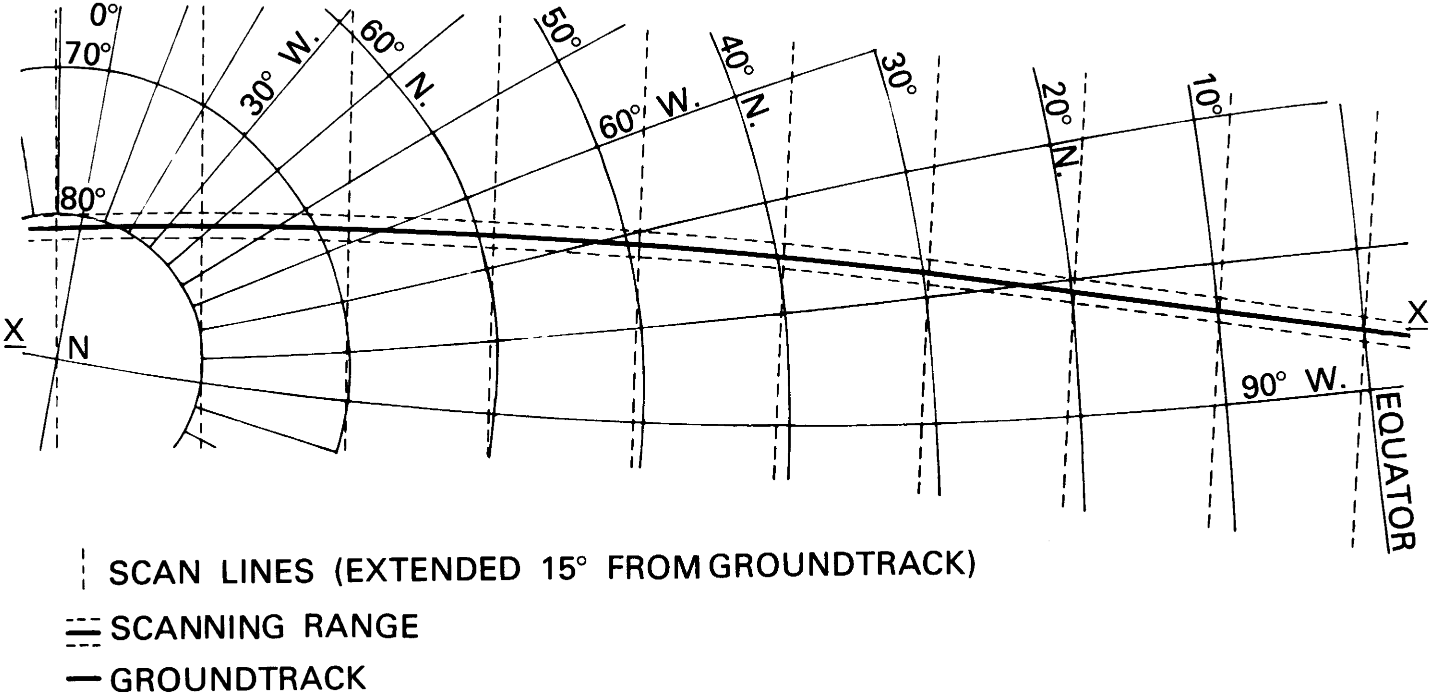

FIGURE 11.—Enlargement of second quadrant of Space Oblique Mercator projection applied to the ellipsoid. Heavy line indicates satellite groundtrack; dashed scan lines extended to ϕ' of 15°.

where

$$B=\frac{2}{\pi}\int_0^{\pi/2}\frac{HJ-S^2}{(J^2+S^2)^{1/2}}\,d\lambda'' \tag{278}$$

$$A_n=\frac{1}{\pi n}\int_0^{2\pi}\frac{HJ-S^2}{(J^2+S^2)^{1/2}}\cos n\lambda''\,d\lambda'' \tag{279}$$

As with the spherical cases, the B in the Snyder (1978b) form of equation (279) may be omitted.

Since the coefficient of the ln tan term in (263) involves calculating S and F from the lengthy equations (265) and (267) for each point, although numerical integration is not involved, a Fourier series is desirable here. Using the same principles,

$$\frac{S}{F(J^2+S^2)^{1/2}}=b_1\cos\lambda''+b_3\cos 3\lambda''+b_5\cos 5\lambda''+\ldots, \tag{280}$$

where

$$b_n=\frac{1}{\pi}\int_0^{2\pi}\frac{S}{F(J^2+S^2)^{1/2}}\cos n\lambda''\,d\lambda'' \tag{281}$$

Combining (277) and (280) with (263), the Fourier equivalent of the entire (263) is

$$\begin{aligned}x/a=&B\lambda''+A_2\sin 2\lambda''+A_4\sin 4\lambda''+\ldots\\&-(b_1\cos\lambda''+b_3\cos 3\lambda''\\&+b_5\cos 5\lambda''+\ldots)\ln\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi''),\end{aligned} \tag{282}$$

where B_n, A_n, and b_n are found from (278), (279), and (281), respectively.

Using the constants $P_2/P_1=18/251$, $i=99.092°$, and the Clarke 1866 ellipsoid,

$$\begin{aligned}B&=1.005798138 \text{ for } \lambda'' \text{ in radians}\\&=0.0175544891 \text{ for } \lambda'' \text{ in degrees}\\A_2&=-0.0010979201\\A_4&=-0.0000012928\\A_6&=-0.0000000021 \text{ (negligible)}\\b_1&=0.0721167880\\b_3&=-0.0000471816\\b_5&=-0.0000001313\\b_7&=-0.0000000003 \text{ (negligible).}\end{aligned}$$

Similarly, for equation (264),

$$\begin{aligned}y/a=&C_1\sin\lambda''+C_3\sin 3\lambda''+\ldots\\&+(\tfrac{1}{2}g_0+g_2\cos 2\lambda''+g_4\cos 4\lambda''+\ldots)\\&\ln\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi''),\end{aligned} \tag{283}$$

where

$$C_n=\frac{1}{\pi n}\int_0^{2\pi}\frac{S(H+J)}{(J^2+S^2)^{1/2}}\cos n\lambda''\,d\lambda'' \tag{284}$$

$$g_n=\frac{1}{\pi}\int_0^{2\pi}\frac{J}{F(J^2+S^2)^{1/2}}\cos n\lambda''\,d\lambda'' \tag{285}$$

With the above constants,

$C_1= 0.1434409899$
$C_3= 0.0000285091$
$C_5=-\ 0.0000000011$ (negligible)
$g_0= 2.000384416$
$g_2=-\ 0.0029599346$
$g_4=-\ 0.0000032363$
$g_6=-0.0000000052$ (negligible).

In equations (270), (260), and (274), there is some benefit in replacing F with the following:

$$F=\tfrac{1}{2}d_0+d_2\cos 2\lambda''+d_4\cos 4\lambda''+\ldots \tag{286}$$

where

$$d_n=\frac{1}{\pi}\int_0^{2\pi}F\cos n\lambda''\,d\lambda'' \tag{287}$$

Calculating,

$d_0=1.9970215476$
$d_2=0.0016545831$
$d_4=0.0000047966$
$d_6=0.0000000142$ (generally negligible).

For equation (261),

$$\phi''=\phi'+j_1\sin\lambda'+j_3\sin 3\lambda'+j_5\sin 5\lambda'+\ldots \tag{288}$$

where

$$j_n=\frac{1}{\pi}\int_0^{2\pi}\phi''_0\sin n\lambda'\,d\lambda' \tag{289}$$

$j_1=\ \ 0.0085556722$ for ϕ'' and ϕ' in degrees
$j_3=\ \ 0.0008178430$ "
$j_5=-\ 0.0000026304$ "
$j_7=-\ 0.0000000094$ " (negligible).

For equation (262),

$$\lambda''=\lambda'+m_2\sin 2\lambda'+m_4\sin 4\lambda'+m_6\sin 6\lambda'+\ldots \tag{290}$$

where

$$m_n = \frac{1}{\pi} \int_0^{2\pi} (\lambda''_0 - \lambda') \sin n\lambda' \, d\lambda' \tag{291}$$

$m_2 = -0.0238400531$ for λ'' and λ' in degrees
$m_4 = 0.0001060576$ ''
$m_6 = 0.0000001932$ ''
$m_8 = 0.0000000007$ '' (negligible).

For the inverses of (288) and (290), the signs are reversed:

$$\phi' = \phi'' - j_1 \sin \lambda'' - j_3 \sin 3\lambda'' - j_5 \sin 5\lambda'' - \ldots \tag{291a}$$

$$\lambda' = \lambda'' - m_2 \sin 2\lambda'' - m_4 \sin 4\lambda'' - m_6 \sin 6\lambda'' - \ldots \tag{291b}$$

While these are not true inverses, they are within 0.000003° and 0.000009°, respectively, of the true inverses of (288) and (290).

For equation (275), three Fourier series may be combined into two. The two integrals are equivalent to equations (277) and the first series of equation (283), while S/J (in two places) may be converted with the equations

$$S/J = D_1 \cos \lambda'' + D_3 \cos 3\lambda'' + D_5 \cos 5\lambda'' + \ldots \tag{292}$$

where

$$D_n = \frac{1}{J\pi} \int_0^{2\pi} S \cos n\lambda'' \, d\lambda'' \tag{293}$$

Substituting the three series into (275),

$$\begin{aligned} x/a + (y/a)\,(D_1 \cos \lambda'' + D_3 \cos 3\lambda'' + D_5 \cos 5\lambda'' + \ldots) = {} \\ B\lambda'' + A_2 \sin 2\lambda'' + A_4 \sin 4\lambda'' + A_6 \sin 6\lambda'' + \ldots \\ + (D_1 \cos \lambda'' + D_3 \cos 3\lambda'' + D_5 \cos 5\lambda'' + \ldots) \\ (C_1 \sin \lambda'' + C_3 \sin 3\lambda'' + C_5 \sin 5\lambda'' + \ldots) \end{aligned}$$

Multiplying the last two parenthetical expressions together and applying equation (84), a simplification similar to that of equation (85) may be made to give a right expression as follows:

$$\begin{aligned} x/a + (y/a)\,(D_1 \cos \lambda'' + D_3 \cos 3\lambda'' + D_5 \cos 5\lambda'' + \ldots) = {} \\ B\lambda'' + E_2 \sin 2\lambda'' + E_4 \sin 4\lambda'' + E_6 \sin 6\lambda'' + \ldots \end{aligned} \tag{294}$$

where

$$E_2 = A_2 + \tfrac{1}{2}(C_1 D_1 + C_3 D_1 - C_1 D_3 + C_3 D_3{}^* - C_3 D_5{}^*) \tag{295}$$

$$E_4 = A_4 + \tfrac{1}{2}(C_3 D_1 + C_1 D_3 + C_5 D_1 - C_1 D_5) \tag{296}$$

$$E_6 = A_6 + \tfrac{1}{2}(C_5 D_1 + C_1 \mathrm{D}_5 + C_3 D_3) \tag{297}$$

(Terms with asterisks are less than 10^{-10}.)

Transposing equation (294) gives the final equivalent to (275),

$$\begin{aligned} B\lambda'' = x/a + (y/a)\,(D_1 \cos \lambda'' + D_3 \cos 3\lambda'' + D_5 \cos 5\lambda'' + \ldots) \\ - E_2 \sin 2\lambda'' - E_4 \sin 4\lambda'' - E_6 \sin 6\lambda'' - \ldots \end{aligned} \tag{298}$$

Starting with a trial λ'' of (x/aB) on the right side, λ'' on the left is calculated and substituted for the previous trial value on the right until change in λ'' is minimal.

With the same basic constants, D_n and E_n are as follows:

$$D_1=0.0722098655$$
$$D_3=0.0000597921$$
$$D_5=0.0000000742$$
$$E_2=0.0040777482$$
$$E_4=0.0000040248$$
$$E_6=0.0000000040 \text{ (negligible).}$$

For equation (276), the first series of (283) can again replace the integral, but the expression ahead of the bracket can be replaced as follows:

$$\frac{F(J^2+S^2)^{\frac{1}{2}}}{J}=\tfrac{1}{2}G_o+G_2\cos 2\lambda''+G_4\cos 4\lambda''+G_6\cos 6\lambda''+\ldots \quad (299)$$

where

$$G_n=\frac{1}{\pi}\int_0^{2\pi}\frac{F(J^2+S^2)^{\frac{1}{2}}}{J}\cos n\lambda''\,d\lambda'' \quad (300)$$

Substituting into (276),

$$\ln\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'')=(\tfrac{1}{2}G_o+G_2\cos 2\lambda''+G_4\cos 4\lambda''+G_6\cos 6\lambda''+\ldots)$$
$$[(y/a)-(C_1\sin\lambda''+C_3\sin 3\lambda''+C_5\sin 5\lambda''+\ldots)]$$

Multiplying the parenthetical expressions with G's and C's and using equation (84) to combine the terms, the final Fourier equivalent of (276) may be written

$$\ln\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'')=(\tfrac{1}{2}G_o+G_2\cos 2\lambda''+G_4\cos 4\lambda''+G_6\cos 6\lambda''+\ldots)$$
$$(y/a)-L_1\sin\lambda''-L_3\sin 3\lambda''-L_5\sin 5\lambda''+\ldots \quad (301)$$

where G_n is as in (300) and

$$L_1=\tfrac{1}{2}(C_1G_o-C_1G_2+C_3G_2-C_3G_4+C_5G_4{}^*-C_5G_6{}^*)$$
$$L_3=\tfrac{1}{2}(C_3G_o+C_1G_2-C_1G_4-C_3G_6{}^*+C_5G_2{}^*)$$
$$L_5=\tfrac{1}{2}(C_5G_o+C_1G_4-C_1G_6+C_3G_2)$$
$$L_7=\tfrac{1}{2}(C_7G_o{}^*+C_1G_6+C_3G_4+C_5G_2{}^*)$$

(Terms with asterisks are less than 10^{-10}.)

For Landsat constants,

$$G_o=1.999624414$$
$$G_2=0.0029588261$$
$$G_4=0.0000076132$$
$$G_6=0.0000000212$$

$G_8 = 0.0000000001$ (negligible)
$L_1 = 0.1432018864$
$L_3 = 0.0002401662$
$L_5 = 0.0000005882$
$L_7 = 0.0000000016$ (negligible).

Although integrals are shown from 0 to 2π, the comments following equation (78) merit a reminder. For a circular orbit, all these Fourier coefficients which are not zero may be determined by integrating from 0 to $\pi/2$ and multiplying by 4, due to quadrant symmetry.

DISTORTION ANALYSIS

The ellipsoidal distortion analysis is also analogous to that for the spherical SOM, except that the formulas are, again, more complicated. Paralleling the discussion beginning with equation (94), but applying the principles to the ellipsoid, the distance on the map for a differential change of latitude is still

$$(\partial s/\partial\phi)_m = [(\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2]^{\frac{1}{2}} \tag{94}$$

and for a differential change of longitude is still

$$(\partial s/\partial\lambda)_m = [(\partial x/\partial\lambda)^2 + (\partial y/\partial\lambda)^2]^{\frac{1}{2}} \tag{95}$$

The corresponding distances on the ellipsoid, though, are ρ_m and $\rho_p \cos\phi$ as determined from equations (169) and (170):

$$(\partial s/\partial\phi)_e = a(1-e^2)/(1-e^2\sin^2\phi)^{\frac{3}{2}} \tag{302}$$

$$(\partial s/\partial\lambda)_e = a\cos\phi/(1-e^2\sin^2\phi)^{\frac{1}{2}} \tag{303}$$

Dividing (94) by (302) gives h, the scale factor along a meridian of the ellipsoid:

$$h = [(\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2]^{\frac{1}{2}}(1-e^2\sin^2\phi)^{\frac{3}{2}}/a(1-e^2) \tag{304}$$

Dividing (95) by (303) gives k, the scale factor along a parallel:

$$k = [(\partial x/\partial\lambda)^2 + (\partial y/\partial\lambda)^2]^{\frac{1}{2}}(1-e^2\sin^2\phi)^{\frac{1}{2}}/a\cos\phi. \tag{305}$$

Combining (304) and (305) for a general scale factor m, using finite increments,

$$m = \left[\frac{(\Delta x)^2 + (\Delta y)^2}{\frac{[a(1-e^2)\,\Delta\phi]^2}{(1-e^2\sin^2\phi)^3} + \frac{(a\cos\phi\,\Delta\lambda)^2}{1-e^2\sin^2\phi}}\right]^{\frac{1}{2}} \tag{306}$$

The discussion following equation (100) applies equally well here, through equation (104), with θ' defined the same way, but with h and k determined from (304) and (305). It is not useful to try to combine equations to obtain an equation for ω similar to (107) or (108); such a combination is about as complicated as calculating ω from (101) through (104) and (304) through (305), so the existing equations should be used instead.

To apply these ellipsoidal scale factors and maximum angular deformation to the SOM formulas, a sequence of differentiations is carried out. They are straightforward, subject to the inadvertent errors which can and did plague so many of the writer's initial derivations of foregoing equations. These have also been calculator tested and are believed to be free of errors.

In differentiating equation (265), the parenthetical expressions may be treated separately:

$$d(1+T\sin^2\lambda'')/d\lambda''=2T\sin\lambda''\cos\lambda''=T\sin 2\lambda''$$

For a calculating shorthand, let

$$(1+T\sin^2\lambda'')=T' \tag{307}$$

Then

$$dT'/d\lambda''=T\sin 2\lambda'' \tag{308}$$

Likewise, the shorthand may be used with expressions $(1+W\sin^2\lambda'')$ and $(1+Q\sin^2\lambda'')$. Then (265) may be rewritten

$$S=(P_2/P_1)\sin i\cos\lambda''\,(T'/W'Q')^{\frac{1}{2}} \tag{309}$$

$$\begin{aligned}dS/d\lambda''=&-(P_2/P_1)\sin i\sin\lambda''\,(T'/W'Q')^{\frac{1}{2}}+(P_2/P_1)\sin i\cos\lambda''\\&[T\sin 2\lambda''/2(T'W'Q')^{\frac{1}{2}}-\tfrac{1}{2}(T'/Q')^{\frac{1}{2}}\,W\sin 2\lambda''/(W')^{\frac{3}{2}}\\&-\tfrac{1}{2}(T'/W')^{\frac{1}{2}}\,Q\sin 2\lambda''/(Q')^{\frac{3}{2}}]\\=&\,S[\sin\lambda''\cos\lambda''\,(T/T'-W/W'-Q/Q')-\tan\lambda'']\end{aligned} \tag{310}$$

Rewriting (267), then differentiating,

$$F=(Q'/T')^{\frac{1}{2}}\,[1+U(Q')^2/W'T'] \tag{311}$$

$$\begin{aligned}dF/d\lambda''=&[1+U(Q')^2/W'T']\,[Q\sin 2\lambda''/2(T'Q')^{\frac{1}{2}}\\&-(Q')^{\frac{1}{2}}\,T\sin 2\lambda''/2(T')^{\frac{3}{2}}]\\&+(Q'/T')^{\frac{1}{2}}\,[2UQ'Q\sin 2\lambda''/W'T'\\&-U(Q')^2\,W\sin 2\lambda''/(W')^2\,T'\\&-U(Q')^2\,T\sin 2\lambda''/W'(T')^2]\end{aligned}$$

After some intermediate steps this simplifies to

$$\begin{aligned}dF/d\lambda''=\sin 2\lambda''\ \{&F\,(Q/Q'-T/T')/2\\&+[F-(Q'/T')^{\frac{1}{2}}]\,(2Q/Q'-W/W'-T/T')\}\end{aligned} \tag{312}$$

Differentiating (272) and (273),

$$\begin{aligned}\sec^2\lambda''d\lambda''=&\cos i\sec^2\lambda_t d\lambda_t+(1-e^2)\,(\sin i\sec^2\phi/\cos\lambda_t)\,d\phi\\&+(1-e^2)\sin i\tan\phi\sec\lambda_t\tan\lambda_t d\lambda_t\end{aligned} \tag{313}$$

$$d\lambda_t=d\lambda+(P_2/P_1)\,d\lambda'' \tag{314}$$

Combining (313) and (314), and separating differentials,

$$\sec^2\lambda''-(P_2/P_1)[\cos i/\cos^2\lambda_t$$
$$+(1-e^2)\sin i\tan\phi\sin\lambda_t/\cos^2\lambda_t]\ d\lambda''$$
$$=[\cos i/\cos^2\lambda_t+(1-e^2)\sin i\tan\phi\sin\lambda_t/\cos^2\lambda_t]\,d\lambda$$
$$+(1-e^2)(\sin i/\cos^2\phi\cos\lambda_t)\,d\phi \tag{315}$$

The partial derivatives are then

$$\partial\lambda''/\partial\phi=(1-e^2)\sin i/\cos^2\phi\,[\cos\lambda_t/\cos^2\lambda''-(P_2/P_1)[\cos i$$
$$+(1-e^2)\sin i\tan\phi\sin\lambda_t]/\cos\lambda_t] \tag{316}$$

$$\partial\lambda''/\partial\lambda=[\cos i+(1-e^2)\sin i\tan\phi\sin\lambda_t]/[\cos^2\lambda_t/\cos^2\lambda''$$
$$-(P_2/P_1)(\cos i+(1-e^2)\sin i\tan\phi\sin\lambda_t)] \tag{317}$$

Differentiating (274),

$$\cos\phi''d\phi''=[(1-e^2)\cos i\sin\phi-\sin i\cos\phi\sin\lambda_t]\,dF/(1-e^2\sin^2\phi)^{\frac{1}{2}}$$
$$+F\ \{(1-e^2\sin^2\phi)^{\frac{1}{2}}[(1-e^2)\cos i\cos\phi\,d\phi+\sin i\sin\phi$$
$$\sin\lambda_t\,d\phi-\sin i\cos\phi\cos\lambda_t\,d\,\lambda_t]-[(1-e^2)\cos i\sin\phi$$
$$-\sin i\cos\phi\sin\lambda_t]^{\frac{1}{2}}(1-e^2\sin^2\phi)^{-\frac{1}{2}}$$
$$(-2e^2\sin\phi\cos\phi\,d\phi)\}/(1-e^2\sin^2\phi)$$

Separating differentials, and then giving partial derivatives as before,

$$\frac{\partial\phi''}{\partial\phi}=B\quad A(\partial\lambda''/\partial\phi)+(1-e^2)\cos i+\sin i\tan\phi\sin\lambda_t$$
$$+e^2\sin\phi\,[(1-e^2)\cos i\sin\phi-\sin i\cos\phi\sin\lambda_t] \tag{318}$$

$$\frac{\partial\phi''}{\partial\lambda}=B\,[A(\partial\lambda''/\partial\lambda)-\sin i\cos\lambda_t] \tag{319}$$

where

$$A=(1/F)[(1-e^2)\cos i\tan\phi-\sin i\sin\lambda_t]\,dF/d\lambda''$$
$$-(P_2/P_1)\sin i\cos\lambda_t \tag{320}$$

$$B=F\cos\phi/\cos\phi''(1-e^2\sin^2\phi)^{\frac{1}{2}} \tag{321}$$

Differentiating (263),

$$\frac{dx}{a}=\frac{HJ-S^2}{(J^2+S^2)^{1/2}}\,d\lambda''-\frac{S}{F(J^2+S^2)^{1/2}}\sec\phi''d\phi''-\ln\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'')$$
$$\left\{\frac{1}{F}\left[\frac{(J^2+S^2)^{\frac{1}{2}}-\frac{1}{2}S(J^2+S^2)^{-\frac{1}{2}}2S}{J^2+S^2}\right]dS+\frac{S}{(J^2+S^2)^{1/2}}(-F^{-2})dF\right\}$$
$$=\frac{1}{(J^2+S^2)^{1/2}}\left[(HJ-S^2)\,d\lambda''-\frac{S}{F}\sec\phi''d\phi''-\left(\frac{J^2}{F(J^2+S^2)^{1/2}}\,dS\right.\right.$$
$$\left.\left.-\frac{S}{F^2}\,dF\right)\ln\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi'')\right] \tag{322}$$

Taking the partial derivative with respect to ϕ, using intermediate differentials derived earlier for S and F, then rearranging,

$$\frac{\partial x}{\partial \phi}=\frac{a}{F(J^2+S^2)^{1/2}}\Bigg\{\Bigg[F(HJ-S^2) - \left(\frac{J^2}{J^2+S^2}\frac{dS}{d\lambda''}-\frac{S\,dF}{F\,d\lambda''}\right)\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi''\right)\Bigg]\frac{\partial\lambda''}{\partial\phi} - \frac{S}{\cos\phi''}\frac{\partial\phi''}{\partial\phi}\Bigg\} \tag{323}$$

The formula for $(\partial x/\partial\lambda)$ is the same as (323), but with λ substituted for ϕ at all three places:

$$\frac{\partial x}{\partial \lambda}=\frac{a}{F(J^2+S^2)^{1/2}}\Bigg\{\Bigg[F(HJ-S^2) - \left(\frac{J^2}{J^2+S^2}\frac{dS}{d\lambda''}-\frac{S\,dF}{F\,d\lambda''}\right)\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi''\right)\Bigg]\frac{\partial\lambda''}{\partial\lambda} - \frac{S}{\cos\phi''}\frac{\partial\phi''}{\partial\lambda}\Bigg\} \tag{324}$$

Obtaining the same sort of relationships for y, equation (264) is differentiated:

$$\begin{aligned}\frac{dy}{a}&=\frac{S(H+J)}{(J^2+S^2)^{1/2}}\,d\lambda''+\frac{J}{F(J^2+S^2)^{1/2}}\sec\phi''\,d\phi'' \\ &\quad+\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi''\right)\cdot\left\{\frac{1}{F}\left[\frac{-S}{(J^2+S^2)^{3/2}}\right]dS + \frac{J}{(J^2+S^2)^{1/2}}(-F^{-2})\,dF\right\} \\ &=\frac{1}{(J^2+S^2)^{1/2}}\Bigg[S(H+J)\,d\lambda''+\frac{J}{F}\sec\phi''\,d\phi'' \\ &\quad-\left(\frac{S}{F(J^2+S^2)}\,dS+\frac{J}{F^2}\,dF\right)\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi''\right)\Bigg]\end{aligned} \tag{325}$$

The partial derivative with respect to ϕ:

$$\frac{\partial y}{\partial \phi}=\frac{a}{F(J^2+S^2)^{1/2}}\left\{\left[FS(H+J)-\left(\frac{S}{J^2+S^2}\frac{dS}{d\lambda''}+\frac{J}{F}\frac{dF}{d\lambda''}\right)\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi''\right)\right]\frac{\partial\lambda''}{\partial\phi}+\frac{J}{\cos\phi''}\frac{\partial\phi''}{\partial\phi}\right\} \tag{326}$$

As with x, the formula for $(\partial y/\partial\lambda)$ is the same as (326), but with λ substituted for ϕ at all three places.

$$\frac{\partial y}{\partial \lambda}=\frac{a}{F(J^2+S^2)^{1/2}}\left\{\left[FS(H+J)-\left(\frac{S}{J^2+S^2}\frac{dS}{d\lambda''}+\frac{J}{F}\frac{dF}{d\lambda''}\right)\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi''\right)\right]\frac{\partial\lambda''}{\partial\lambda}+\frac{J}{\cos\phi''}\frac{\partial\phi''}{\partial\lambda}\right\} \tag{327}$$

The equations for distortion analysis of the ellipsoidal SOM have now been completed. There appears to be no advantage in further combining the equations, but each portion may be calculated separately as part of a multipart program and then combined. Specifically, paralleling the analysis for the spherical SOM,

1. For a given ϕ'' and λ'', calculate ϕ, λ_t and λ from equations (268) through (271), placing λ_t in the proper quadrant by the procedure detailed in step 1 following (119).

2. Calculate J, S, H, and F from equations (265) through (267), (198) through (200), (207), and (219).

3. Calculate $(dS/d\lambda'')$ from (310) and $(dF/d\lambda'')$ from (312), using the notation of (307), etc.

4. Calculate $(\partial\lambda''/\partial\phi)$, $(\partial\lambda''/\partial\lambda)$, $(\partial\phi''/\partial\phi)$, and $(\partial\phi''/\partial\lambda)$ from (316) through (321).

5. Calculate $(\partial x/\partial\phi)$, $(\partial x/\partial\lambda)$, $(\partial y/\partial\phi)$, and $(\partial y/\partial\lambda)$ from (323), (324), (326), and (327).

6. Calculate h, k, and ω from equations (304), (305), and (101) through (104).

While these calculations are obviously lengthy, they are important to make, but need to be made only once for a given satellite orbit (like Fourier constants) to establish the degree of accuracy. They do not involve computer time during mapping operations.

Table 4 lists h, k, and ω for each 15° of a quadrant of the ellipsoidal SOM for Landsat. The values are first given within 1° of the groundtrack to show normal ranges of errors, and then given up to 15° from the groundtrack (with fewer decimals) to show ranges of errors when the SOM is used for other satellites. While other orbital characteristics would affect these values slightly, they have been found to be representative of errors to be expected. In the 1° range, $\sin \frac{1}{2}\omega$ is also given, since it is the variation which occurs from the normal conformal scale factor at that position, due to the lack of perfect conformality of the projection. For example, at $\lambda''=15°$ and $\phi''=1°$, the scale varies up to 0.000019 more or less from the normal scale factor of about 1.000152, because of the maximum angular deformation of 0.0022°.

The nonsymmetry of the errors with respect to both λ'' and ϕ'' is interesting, due largely to the manner in which the radius of curvature changes away from the groundtrack. The slight but negligible variation of the groundtrack from true conformality and scale is due to the approximation used for equation (225) and deviates from unity in the crosstrack direction only.

TABLE 4.—*Scale factors and angular deviation away from geocentric Landsat groundtrack for the ellipsoidal Space Oblique Mercator projection*

At 1° from groundtrack					
λ''	ϕ''	h	k	ω	sin ½ ω
0°	1°	1.000154	1.000151	0.0006°	0.000005
	0	1.000000	1.000000	.0000	.000000
	−1	1.000154	1.000151	.0006	.000005
15	1	1.000161	1.000151	.0022	.000019
	0	1.000000	1.000000	.0001	.000000
	−1	1.000147	1.000151	.0011	.000010
30	1	1.000167	1.000150	.0033	.000029
	0	1.000000	1.000000	.0001	.000001
	−1	1.000142	1.000150	.0025	.000021
45	1	1.000172	1.000150	.0036	.000031
	0	.999999	1.000000	.0001	.000001
	−1	1.000138	1.000150	.0031	.000027
60	1	1.000174	1.000150	.0031	.000027
	0	.999999	1.000000	.0002	.000001
	−1	1.000136	1.000150	.0028	.000025
75	1	1.000174	1.000152	.0019	.000016
	0	.999999	1.000000	.0001	.000000
	−1	1.000135	1.000150	.0019	.000016
90°	1	1.000170	1.000156	.0008	.000007
	0	.999999	1.000000	.0000	.000000
	−1	1.000133	1.000151	.0010	.000009

Beyond 1° from groundtrack				
λ'	ϕ''	h	k	ω
0°	15°	1.03565	1.03490	.136°
	10	1.01559	1.01526	.060
	5	1.00386	1.00378	.015
	−5	1.00386	1.00378	.015
	−10	1.01559	1.01526	.060
	−15	1.03565	1.03490	.136
15	15	1.03589	1.03472	.157
	10	1.01570	1.01521	.074
	5	1.00390	1.00377	.022
	−5	1.00383	1.00378	.006
	−10	1.01551	1.01530	.041
	−15	1.03549	1.03503	.107
30	15	1.03616	1.03453	.162
	10	1.01583	1.01514	.080
	5	1.00395	1.00376	.027
	−5	1.00381	1.00378	.003
	−10	1.01548	1.01530	.024
	−15	1.03543	1.03510	.076
45	15	1.03636	1.03440	.147
	10	1.01594	1.01509	.075
	5	1.00398	1.00375	.027
	−5	1.00380	1.00377	.007
	−10	1.01549	1.01529	.014

TABLE 4.—*Continued*

Beyond 1° from groundtrack				
λ'	ϕ''	h	k	ω
45°	−15°	1.03546	1.03511	.052°
60	15	1.03635	1.03446	.112
	10	1.01597	1.01509	.059
	5	1.00400	1.00374	.022
	−5	1.00381	1.00376	.009
	−10	1.01552	1.01526	.015
	−15	1.03553	1.03510	.041
75	15	1.03596	1.03489	.061
	10	1.01583	1.01526	.032
	5	1.00398	1.00378	.012
	−5	1.00381	1.00376	.008
	−10	1.01556	1.01525	.018
	−15	1.03554	1.03514	.040
90	15	1.03553	1.03533	.011
	10	1.01561	1.01549	.007
	5	1.00391	1.00385	.004
	−5	1.00373	1.00384	.006
	−10	1.01524	1.01557	.019
	−15	1.03449	1.03571	.040

Notes: λ''=angular position along geocentric groundtrack, from ascending node.
ϕ''=angular distance away from geocentric groundtrack, positive in direction away from north pole.
ω=maximum angular deformation
sin ½ ω=maximum variation of scale factors from true conformal values.
h=scale factor along meridian of longitude
k=scale factor along parallel of latitude

INCORPORATION OF LANDSAT ROW AND PATH NUMBERS

Rather than using the number of orbits N discussed under the second type of equation in the section "Iteration procedures," the user of Landsat information is more likely to refer to the row and path numbers of a given Landsat orbit. A given Landsat orbit has the same path number from one north polar approach to the next north polar approach. Therefore, there are 251 path numbers before the orbits exactly repeat. The paths are numbered not in the order of the occurrence of the orbit, but consecutively for each path encountered as one follows a given parallel of latitude westward around the Earth from a specified starting point. Thus path 2 is 1/251 of the circumference, or 360°/251=1.434° of longitude west of path 1. Chronologically, each successive orbital journey of a Landsat satellite has a path number 18 greater than the previous one, subtracting 251 if the path number would exceed this limit. (With each satellite revolution, the Earth turns 18/251 of a complete rotation, or 18 path numbers.)

Since the path number changes near the north polar approach, the path cycle approximately covers a change in λ' of 90° to 450° if λ' is related to the preceding ascending node. The row numbering is proportional to the time along the orbit and therefore, proportional to λ'. Since 248 rows are used, each corresponds to 360°/248 or 45°/31=1.452° of λ'. Numbering of rows does not begin until a λ' of 94.355°, so that a λ' of 180° (the descending node) corresponds to row 60. It should be noted that the ascending node on a given path occurs at row 184 (60+248/2), when $\lambda'=360°$. The ascending node at a λ' of 0° occurs at a fictitious row number 184-248 or −64.

Relating the above information to formulas for the satellite groundtrack, or to arithmetic considerations:
For λ' in terms of r, the number of the row,

$$\lambda' = 90° + (45°/31)(r+2) \tag{328}$$

and conversely,

$$r = (31/45°)\lambda' - 64. \tag{329}$$

For ϕ_0 or λ_0 along the groundtrack in terms of the row r, use equations (257), (93), and (133), adjusted for quadrant as described in step 1 following equation (119).

To convert from longitude λ_G east (+) or west (−) of Greenwich to the longitude λ relative to the ascending node, row "−64" of a given Landsat path p, use the following:

$$\lambda = \lambda_G - 128.87° + (360°/251)p. \tag{330}$$

Conversely,

$$\lambda_G = \lambda + 128.87° - (360°/251)p. \tag{331}$$

This is the value of λ which must be used in the various SOM equations, not a value based on the later ascending node.

SUMMARY OF SIMPLIFIED EQUATIONS FOR THE ELLIPSOIDAL SOM FOR LANDSAT

While some simplifications and empirical treatment were used in the development of the foregoing equations, several factors make the equations appear even more complicated than necessary when compared with equations for other projections, such as the ellipsoidal transverse and oblique Mercator projections. By deleting the following functions, the equations may be somewhat simplified to a set of equations for converting from latitude and longitude to rectangular coordinates and vice versa, especially for Landsat:

1. Remove all equations for finding positions along the groundtrack.

2. Remove all Fourier constants which are not to eliminate numerical integration. Show calculation only for integration of one quadrant.
3. Eliminate the factor F, since it almost completely cancels out in the region of Landsat sensing, and has a very slight effect in broader areas.

The formulas can then be summarized as follows:

Constants to be calculated for the given satellite orbit:

$$B=(2/\pi)\int_0^{\pi/2}[H\,J-S^2)/(J^2+S^2)^{\frac{1}{2}}]d\lambda'' \tag{278}$$

$$A_n=(4/\pi n)\int_0^{\pi/2}[(H\,J-S^2)/(J^2+S^2)^{\frac{1}{2}}]\cos n\lambda'' d\lambda'' \tag{279a}$$

where $n=2$ and 4.

$$C_n=(4/\pi n)\int_0^{\pi/2}[S(H\ +J)/(J^2+S^2)^{\frac{1}{2}}]\cos n\lambda'' d\lambda'' \tag{284a}$$

where $n=1$ and 3.

Simpson's rule (equation (78)) is recommended for integration, using 9° steps for each of the above. For S, H, J, Q, T, and W, equations (265), (266), (207), (199), (198), and (200) may be used unchanged. They are shown together following (265). For Landsat, $i=99.092°$ and $P_2/P_1=18/251$.

Forward equations (geodetic to rectangular coordinates):

Using equation (330) to change λ_G to λ,

$$\lambda_t=\lambda+(P_2/P_1)\lambda'' \tag{273}$$

$$\lambda''=\arctan[\cos i\tan\lambda_t+(1-e^2)\sin i\tan\phi/\cos\lambda_t] \tag{272a}$$

iterating together, using type 2 under "Iteration procedures."

$$\phi''=\arcsin\{[(1-e^2)\cos i\sin\phi-\sin i\cos\phi\sin\lambda_t]/(1-e^2\sin^2\phi)^{\frac{1}{2}}\} \tag{274a}$$

using the final λ_t above.

$$x/a=B\lambda''+A_2\sin 2\lambda''+A_4\sin 4\lambda''-[S/(J^2+S^2)^{\frac{1}{2}}]\ln\tan(\pi/4+\phi''/2) \tag{282a}$$

$$y/a=C_1\sin\lambda''+C_3\sin 3\lambda''+[J/(J^2+S^2)^{\frac{1}{2}}]\ln\tan(\pi/4+\phi''/2) \tag{283a}$$

where S is found, as in the integration, from equation (265).

Inverse equations (rectangular to geodetic coordinates):

$$\lambda''=[(x/a)+(S/J)(y/a)-A_2\sin 2\lambda''-A_4\sin 4\lambda''-(S/J)(C\sin\lambda''+C_3\sin 3\lambda'')]/B \tag{275a}$$

with S found from equation (265), and iteration as described following equation (298).

$$\phi''=2 \arctan \{\exp[(1+S^2/J^2)^{\frac{1}{2}}(y/a-C_1 \sin \lambda'' \\ -C_3 \sin 3\lambda'')]\} -\pi/2 \quad (276a)$$

$$\lambda_t=\arctan \{[(1-\sin^2\phi''/(1-e^2)) \tan \lambda'' \cos i-(\sin \phi'' \sin i/\cos\lambda'') \\ ((1+Q \sin^2 \lambda'') \cos^2 \phi''-U \sin^2\phi'')^{\frac{1}{2}}]/[1-\sin^2\phi''(1+U)]\} \quad (269a)$$

adjusting λ_t for quadrant as described in step 1 following equation (119).

$$U=e^2 \cos^2 i/(1-e^2) \quad (219)$$

$$\lambda=\lambda_t-(P_2/P_1)\lambda'' \quad (268)$$

Conversion of λ to λ_G is carried out according to equation (331).

$$\phi=\arctan [(\tan \lambda'' \cos \lambda_t-\cos i \sin \lambda_t)/(1-e^2) \sin i] \quad (271)$$

using the quadrant-adjusted value of λ_t obtained above.

MODIFICATION OF ELLIPSOIDAL SOM EQUATIONS FOR NONCIRCULAR ORBITS

For a noncircular orbit, λ' is not proportional to time, but if the orbit is elliptical, λ' may be calculated by Keplerian formulas from the mean longitude in orbit of the satellite. The factor $(P_2/P_1)\lambda'$ of equation (7) is changed, and this change necessitates changes in the derivations of the earlier formulas. The changes are actually not very difficult for obtaining a correct groundtrack, but the proper curve to give this groundtrack on the plotted map becomes much more complicated, except for small orbital eccentricities of 0.05 or less. For larger eccentricities, see the next section, although several formulas in this section apply to an orbit of any eccentricity and are referred to in the next section.

In terms of the discussion following equation (28) and leading to equation (7), the "satellite-apparent" longitude λ_t will be displaced from the actual longitude λ by an angle proportional to time, called the mean longitude of the satellite along its orbit. If we call the mean longitude L for the satellite with respect to the perigee, and if the perigee of the orbit is at a geocentric angle ω with respect to the ascending node of the orbit, the difference between λ and λ_t becomes $(P_2/P_1)(L+\omega)$, or

$$\lambda_t=\lambda+(P_2/P_1)(L+\omega) \quad (332)$$

To relate λ' to L, standard Keplerian equations are used (Moulton, 1914):

$$\tan \tfrac{1}{2}(\lambda'-\omega)=(\tan \tfrac{1}{2}E')\sqrt{(1+e')/(1-e')} \quad (333)$$

where

$$E-e' \sin E'=L, \quad (334)$$

e' is the eccentricity of the satellite orbit,

E' is an intermediate angle, or the "eccentric anomaly",

and ω is the longitude of the perigee relative to the ascending node. E' may be found from L by iteration, first rearranging (334) to

$$E' = L + e' \sin E' \tag{335}$$

assuming a value for E', inserting it in the right side, solving for the left side, and substituting the new value of E' into the right side, until there is sufficient convergence. Convergence is very rapid when e' is small. This is the same technique used in much of the iteration in preceding formulas.

The radius vector of the elliptical orbit, R_o, is found from the following equation:

$$R_o = a'(1 - e' \cos E') \tag{336}$$

where a' is the mean distance of the satellite from the center of the Earth, in the same units as a.

Equations for the satellite groundtrack as plotted on the globe thus vary from those given for the ellipsoid, as follows:

Equation (236)—no change.

(256)—no change, except that R_o is found from (336).

(135)—no change.

(136) becomes $\lambda_{t_o} = \lambda_o + (P_2/P_1)(L + \omega)$. (337)

(257)—no change, except that R_o is found from (336).

(93) becomes $\lambda_o = \lambda_{t_o} - (P_2/P_1)(L + \omega)$. (338)

(133)—no change.

In deriving the general equations for the ellipsoidal SOM, the relationship between λ', L, and λ_t for the ellipsoid first affects the definition of BD following equation (181). Now,

$$BD = (P_2/P_1)\, a \cos \phi_o\, dL/(1 - e^2 \sin^2 \phi_o)^{\frac{1}{2}},$$

so that equation (182) becomes

$$dy = (P_2/P_1)\, a \cos \phi_o \sin i'\, dL/(1 - e^2 \sin^2 \phi_o)^{\frac{1}{2}},$$

and (183) becomes

$$dx = ds - (P_2/P_1)\, a \cos \phi_o \cos i'\, dL/(1 - e^2 \sin^2 \phi_o)^{\frac{1}{2}}.$$

The next change is at equation (190), where λ' is merely replaced with L. Equation (197) becomes

$$dy/adL = (P_2/P_1) \sin i \cos \lambda' [(1 + T \sin^2 \lambda')/ (1 + W \sin^2 \lambda')(1 + Q \sin^2 \lambda')]^{\frac{1}{2}}.$$

It is more convenient to change this to a function of $d\lambda'$:

$$dy/ad\lambda' = (P_2/P_1) \sin i \cos \lambda' [(1+T \sin^2 \lambda')/(1+W \sin^2 \lambda')(1+Q \sin^2 \lambda')]^{\frac{1}{2}} dL/d\lambda' = S \tag{339}$$

Equation (203) becomes

$$\frac{dx}{adL} = \frac{ds}{dL} - a\frac{P_2}{P_1}\frac{\cos i\,(1-e^2\sin^2\phi_o)^{1/2}}{(1-e^4\sin^2\phi_o\cos^2 i)^{1/2}}$$

$$= \frac{ds}{d\lambda'}\frac{d\lambda'}{dL} - a\frac{P_2}{P_1}\frac{\cos i\,(1-e^2\sin^2\phi_o)^{1/2}}{(1-e^4\sin^2\phi_o\cos^2 i)^{1/2}}$$

The only change this introduces into (204) is the insertion of $d\lambda'/dL$:

$$\frac{dx}{adL} = \sqrt{\frac{1+Q\sin^2\lambda'}{1+W\sin^2\lambda'}}\left[\frac{1+W\sin^2\lambda'}{(1+Q\sin^2\lambda')^2}\frac{d\lambda'}{dL} - \frac{P_2}{P_1}\cos i\right]$$

As in the case of equation (339), it is preferable to multiply by $dL/d\lambda'$:

$$\frac{dx}{ad\lambda'} = \sqrt{\frac{1+Q\sin^2\lambda'}{1+W\sin^2\lambda'}}\left[\frac{1+W\sin^2\lambda'}{(1+Q\sin^2\lambda')^2} - \frac{P_2}{P_1}\frac{dL}{d\lambda'}\cos i\right] = H \tag{340}$$

The value of $dL/d\lambda'$ is derived from equations (333) and (334):

Differentiating (334),

$$dL = (1 - e' \cos E')\, dE' \tag{341}$$

Differentiating (333),

$$\tfrac{1}{2}\sec^2 \tfrac{1}{2}(\lambda' - \omega) d\lambda' = \tfrac{1}{2}\sec^2 \tfrac{1}{2}E' \sqrt{(1+e')/(1-e')}\, dE'$$

$$dE'/d\lambda' = \sec^2 \tfrac{1}{2}(\lambda' - \omega) \cos^2 \tfrac{1}{2}E' \sqrt{(1-e')/(1+e')} \tag{341a}$$

Substituting into (341),

$$dL/d\lambda' = (1 - e' \cos E') \sec^2 \tfrac{1}{2}(\lambda' - \omega) \cos^2 \tfrac{1}{2}E' \sqrt{(1-e')/(1+e')} \tag{342}$$

From (333), however,

$$\tan^2 \tfrac{1}{2}(\lambda' - \omega) = (\tan^2 \tfrac{1}{2}E') [(1+e')/(1-e')]$$

$$\sec^2 \tfrac{1}{2}(\lambda' - \omega) = 1 + (\tan^2 \tfrac{1}{2}E') [(1+e')/(1-e')]$$

Substituting into (342),

$$dL/d\lambda' = (1 - e' \cos E') \left[\cos^2 \tfrac{1}{2}E' \sqrt{(1-e')/(1+e')} + \sin^2 \tfrac{1}{2}E' \sqrt{(1+e')/(1-e')}\right]$$

$$= \tfrac{1}{2}(1 - e' \cos E') \left[\frac{(1+\cos E')(1-e') + (1-\cos E')(1+e')}{(1-e'^2)^{1/2}}\right]$$

$$= \tfrac{1}{2}(1 - e' \cos E') \left[\frac{2 - 2e' \cos E'}{(1-e'^2)^{1/2}}\right]$$

$$= (1 - e' \cos E')^2/(1-e'^2)^{\frac{1}{2}} \tag{343}$$

where E' is found from λ' in equation (333).

As with both spherical and ellipsoidal equation derivations, the values of x and y calculated from equations (340), (339), and (343) will give the groundtrack true to scale, but the groundtrack must be bent to a different shape to give sufficiently accurate mapping of the region near the groundtrack. For small orbital eccentricities the bending of the groundtrack can follow exactly the same approach as that used in ellipsoidal equations (206a) through (213). The eccentricity of the orbit does not affect this, except for its incorporation into the new formulas for H (340) and S (339); J does not change. Therefore, equations (211) and (213) remain the same:

$$x'=a\int_0^{\lambda'}[(HJ-S^2)/(J^2+S^2)^{\frac{1}{2}}]\,d\lambda' \tag{211}$$

$$y'=a\int_0^{\lambda'}[S(H+J)/(J^2+S^2)^{\frac{1}{2}}]\,d\lambda' \tag{213}$$

An additional complication is introduced by the elliptical orbit, however. With x' and y' starting at (0,0) when λ' is zero, y' will not be zero when λ' is 360°, since the quadrants are not symmetrical. For convenience, it is preferred that y return to zero at each 360° repetition. This is accomplished by rotating the X' and Y' axes slightly so that y becomes zero at each 360° interval. Calling the rotational angle θ_2,

$$x=x'\cos\theta_2+y'\sin\theta_2 \tag{344}$$

$$y=y'\cos\theta_2-x'\sin\theta_2 \tag{345}$$

using standard formulas. The tangent of θ_2 is the ratio of y' at 360° to x' at 360°, or

$$\tan\theta_2=B_2/B_1$$

where

$$B_1=\frac{1}{2\pi}\int_0^{2\pi}[(HJ-S^2)/(J^2+S^2)^{\frac{1}{2}}]\,d\lambda' \tag{346}$$

$$B_2=\frac{1}{2\pi}\int_0^{2\pi}[S(H+J)/(J^2+S^2)^{\frac{1}{2}}]\,d\lambda' \tag{347}$$

dividing each equation by 2π to provide convenient numbers usable later, since 2π cancels out in their use for θ_2.

Then

$$\begin{aligned}\sin\theta_2&=\tan\theta_2/(1+\tan^2\theta_2)^{\frac{1}{2}}\\&=B_2/(B_1^2+B_2^2)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\cos\theta_2&=1/(1+\tan^2\theta_2)^{\frac{1}{2}}\\&=B_1/(B_1^2+B_2^2)^{\frac{1}{2}}\end{aligned}$$

Assigning symbols H_1 to $\cos\theta_2$ and S_1 to $\sin\theta_2$, and deleting a from (211) and (213) for simplicity, equations (344) and (345) become

$$x/a = x'H_1 + y'S_1 \tag{348}$$

$$y/a = y'H_1 - x'S_1 \tag{349}$$

where

$$H_1 = B_1/(B_1^2 + B_2^2)^{1/2} \tag{350}$$

$$S_1 = B_2/(B_1^2 + B_2^2)^{1/2} \tag{351}$$

B_1 and B_2 are found from (346) and (347), and x' and y' are found from (211) and (213) omitting the a.

In developing positions of points along the scanlines, the equations for calculating F (214) through (220) are unchanged by orbital eccentricity. Rotation of scanlines in the x' and y' frame occurs as before, but the above rotation of X' and Y' axes to X and Y must include the scanlines as well. Therefore, equations (221) and (222) undergo only a slight change to the following in place of equations (211) and (213):

$$x' = \int_0^{\lambda'} \frac{HJ - S^2}{(J^2 + S^2)^{1/2}}\, d\lambda' - \frac{S}{F(J^2 + S^2)^{1/2}} \ln \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'\right) \tag{352}$$

$$y' = \int_0^{\lambda'} \frac{S(H + J)}{(J^2 + S^2)^{1/2}}\, d\lambda' + \frac{J}{F(J^2 + S^2)^{1/2}} \ln \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'\right) \tag{353}$$

Subsequent derivations of equations for transformation of geodetic latitude and longitude into ϕ' and λ' are unchanged from the ellipsoidal form except for the substitution of $(P_2/P_1)\,(L + \omega)$ in place of $(P_2/P_1)\lambda'$ wherever the latter appears. This occurs in equations (7) and (136), and wherever the like formula is referred to for calculation of λ_t, etc. The iteration required for solving equations (235) and (7), or (135) and (136), remains as before using successive substitution, except that additional steps are involved.

Procedure (3) under the section "Iteration procedures" is followed, but the first trial $\lambda'(\lambda'_p)$ is used in place of λ' in equation (333), which is solved for E', for insertion into (334), which is solved for L. This value of L is then used in (332), solving for λ_t, and λ_t is in turn used in (235) or (135), from which a new value of λ' is found for use as the trial λ' in equation (333). This is repeated until sufficient convergence occurs.

For inverse equations for the noncircular SOM, the foregoing discussion coupled with approaches taken for circular inverse equations, makes the modifications fairly obvious:

For ϕ and λ in terms of ϕ' and λ', only equation (93) is changed, to

$$\lambda = \lambda_t - (P_2/P_1)\,(L + \omega) \tag{354}$$

the inverse of (332).

For ϕ' and λ' in terms of x and y, the axis rotation is the principal change. The inverse of (344) and (345) involves standard formulas, so that

$$x' = x \cos \theta_2 - y \sin \theta_2$$
$$y' = y \cos \theta_2 + x \sin \theta_2$$

which are converted to our terminology as the inverse of (348) and (349):

$$x' = (x/a) H_1 - (y/a) S_1 \tag{355}$$

$$y' = (y/a) H_1 + (x/a) S_1 \tag{356}$$

Equations (249) and (250) remain the same, except that x' and y' are substituted for (x/a) and (y/a), respectively:

$$x' + (S/J)y' = \int_0^{\lambda'} \frac{HJ - S^2}{(J^2+S^2)^{1/2}} d\lambda' + \frac{S}{J} \int_0^{\lambda'} \frac{S(H+J)}{(J^2+S^2)^{1/2}} d\lambda' \tag{357}$$

$$\ln \tan \left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'\right) = \frac{F(J^2+S^2)^{1/2}}{J} \left[y' - \int_0^{\lambda'} \frac{S(H+J)}{(J^2+S^2)^{1/2}} d\lambda' \right] \tag{358}$$

Solution using Fourier series is almost mandatory, as it was for the circular orbits.

To correct for vertical rather than geocentric scanning, the same approach is taken as with circular orbits, but with the revised formulas above. As before, this changes single primes to double primes in most of the revised formulas, but a slight modification is shown below in the formula summary.

In summary, for equations giving a true-to-scale groundtrack for any Keplerian elliptical orbit, and sufficiently accurate plotting of points near the groundtrack (see later discussion) for an orbital eccentricity of about 0.05 or below, the following formulas may be used:

1. For x and y in terms of ϕ'' and λ'' use equations (348), (349), (350), (351), and the following modifications of (333), (339), (340), (343), (346), (347), (352), and (353):

$$B_1 = (1/2\pi) \int_0^{2\pi} [(HJ - S^2)/(J^2 + S^2)^{\frac{1}{2}}] d\lambda'' \tag{359}$$

$$B_2 = (1/2\pi) \int_0^{2\pi} [S(H+J)/(J^2+S^2)^{\frac{1}{2}}] d\lambda'' \tag{360}$$

$$S = (P_2/P_1) L' \sin i \cos \lambda'' [(1 + T \sin^2 \lambda'')/ (1 + W \sin^2 \lambda'')(1 + Q \sin^2 \lambda'')]^{\frac{1}{2}} \tag{361}$$

$$H = \sqrt{\frac{1 + Q \sin^2 \lambda''}{1 + W \sin^2 \lambda''}} \left[\frac{1 + W \sin^2 \lambda''}{(1 + Q \sin^2 \lambda'')^2} - (P_2/P_1) L' \cos i \right] \tag{362}$$

$$L' = (dL/d\lambda'') = (1 - e' \cos E')^2 / (1 - e'^2)^{\frac{1}{2}} \tag{363}$$

$$\tan \tfrac{1}{2}E' = \tan \tfrac{1}{2}(\lambda'' - \omega)\,[(1-e')/(1+e')]^{\frac{1}{2}} \tag{364}$$

Equations (352) and (353) are altered only by replacing λ' and ϕ' with λ'' and ϕ'', respectively.

Equations (267), (207), (199), (198), (200), and (219) shown in sequence in the summary of circular-orbit equations for F, J, W, Q, T, and U, remain unchanged.

2. For ϕ and λ in terms of ϕ'' and λ'', use equations (269), (270), (271), (248), and (354) without change, where L is found from equation (334) and E' from (364).

3. For ϕ'' and λ'' in terms of ϕ' and λ', use equations (261), (262), (258), (260), (257), and (133) in the sequence shown in the circular-orbit summary, but equation (259) is replaced by (337), where L is calculated from (334) and (364) from λ'', and equation (93) is replaced by

$$\lambda_o = \lambda_t - (P_2/P_1)(L+\omega)$$

where L is calculated from (334) and (333) from λ'. The radius vector R_o is obtained from equation (336). Fourier series are strongly recommended here, as with the circular-orbit formulas, except that equations (289) and (291) must be supplemented with integrations involving $\cos n\,\lambda'$, as in equations (53) through (56).

4. For ϕ' and λ' in terms of ϕ'' and λ'', the same Fourier series as that used for ϕ'' and λ'' in terms of ϕ' and λ' may be used with reversal of signs, since the correction is so small.

5. For ϕ'' and λ'' in terms of ϕ and λ, use equations (272), (332), and (274), finding L from equation (334) and E' from (364).

6. For ϕ'' and λ'' in terms of x and y, use equations (355), (356), (350), (351), and (359) through (364) as they stand. Equations (357) and (358) are altered only by replacing λ' and ϕ' with λ'' and ϕ'', respectively.

7. For the satellite groundtrack,
   a. For λ' in terms of ϕ_o, use (236) and (256), but determine R_o from (336), and E' from (364), using λ' in place of λ''. Iteration by successive substitution is required for the group of equations.
   b. For λ' in terms of λ_o, use (135), (337), (334), and (364) with iteration and quadrant adjustment essentially as followed for circular orbits.
   c. For ϕ_o in terms of λ', use (257), determining R_o from (336) and (364).
   d. For λ_o in terms of λ', use (338) and (133), finding L from (334) and (364). For (b), (c), and (d), λ' should replace λ'' in (364).

A check of scale factors for various values of λ'', taken at constant ϕ'' (parallel to the groundtrack) and constant λ'' (along a scanline), shows correct scale along the groundtrack and along the scanline at the groundtrack, regardless of orbital eccentricity. The scale along the scanlines is also the same as it is for a circular orbit with the same inclination, with a slight variation from sec ϕ'' due to ellipsoidal approximations. The scale parallel to the groundtrack, but 1° away from it, deviates from sec ϕ'' by a quantity increasing with orbital eccentricity, when using the formulas in this section. These formulas are sufficient only if the eccentricity is as low as 0.05. For greater eccentricities the next section should be consulted.

In any case, as stated before, the groundtrack is shown correctly with respect to geodetic latitude and longitude and to scale. The fact that it is not curved quite correctly on the map affects the scale away from it, but not along it.

The rapidly converging and minimal number of Fourier constants sufficient for circular orbits is not the case with noncircular orbits. In the case of a fictitious orbit with i and P_2/P_1 the same as those for Landsat, and an orbital eccentricity of 0.1 and ω of 20°, it was found that 13 coefficients were needed for the x' integral and 12 for the y' integral, instead of 3 and 2 respectively for the corresponding functions of the circular orbit, to provide 6- to 7-place accuracy in computing x and y. Assuming that similar numbers of coefficients are needed for the non-integral functions, it would be more efficient to calculate the latter directly without Fourier coefficients and to reserve the Fourier for the integrals only.

The approach in determining the Fourier coefficients is similar, of course, to preceding derivations, except that more terms are involved. Taking the term under the integral in equation (352), using λ'' in place of λ',

$$F(\lambda'')=\frac{HJ-S^2}{(J^2+S^2)^{1/2}}=\tfrac{1}{2}a_0+\sum_{n=1}^{n} a_n \cos n\,\lambda''+\sum_{n=1}^{n} b_n \sin n\,\lambda''$$

where

$$a_0=\frac{1}{\pi}\int_0^{2\pi} F(\lambda'')d\lambda''$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} F(\lambda'')\cos n\,\lambda'' d\lambda''$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} F(\lambda'')\sin n\,\lambda'' d\lambda''$$

Integrating,

$$\int F(\lambda'')d\lambda''=\tfrac{1}{2}a_0\lambda''+\sum_{n=1}^{n}(a_n/n)\sin n\lambda''-\sum_{n=1}^{n}(b_n/n)\cos n\lambda''+C$$

where C is the constant of integration, which will be determined to make the integral zero when λ'' is zero, by making it the sum of the coefficients of cos $n\lambda''$. Replacing $\frac{1}{2}a_0$ with B_1, (a_n/n) with A_n, and (b_n/n) with A'_n,

$$\int_0^{\lambda'} \frac{HJ-S^2}{(J^2+S^2)^{1/2}}\,d\lambda''=B_1\lambda''+\sum_{n=1}^{n} A_n \sin n\lambda'' - \sum_{n=1}^{n} A'_n \cos n\lambda''+\sum_{n=1}^{n} A'_n \qquad (365)$$

where

$$B_1=\frac{1}{2\pi}\int_0^{2\pi}\frac{HJ-S^2}{(J^2+S^2)^{1/2}}\,d\lambda'' \qquad (359)$$

$$A_n=\frac{1}{\pi n}\int_0^{2\pi}\frac{HJ-S^2}{(J^2+S^2)^{1/2}}\cos n\lambda''d\lambda'' \qquad (366)$$

$$A'_n=\frac{1}{\pi n}\int_0^{2\pi}\frac{HJ-S^2}{(J^2+S^2)^{1/2}}\sin n\lambda''d\lambda'' \qquad (367)$$

It will be noted that B_1 is the same term as that used in the rotation of axes earlier. An analogous approach leads to Fourier constants for the integral term for y' in equation (353):

$$\int_0^{\lambda'} \frac{S(H+J)}{(J^2+S^2)^{1/2}}\,d\lambda''=B_2\lambda''+\sum_{n=1}^{n} C_n \sin n\lambda'' - \sum_{n=1}^{n} C'_n \cos n\lambda''+\sum_{n=1}^{n} C'_n \qquad (368)$$

where

$$B_2=\frac{1}{2\pi}\int_0^{2\pi}\frac{S(H+J)}{(J^2+S^2)^{1/2}}\,d\lambda'' \qquad (360)$$

$$C_n=\frac{1}{\pi n}\int_0^{2\pi}\frac{S(H+J)}{(J^2+S^2)^{1/2}}\cos n\lambda''d\lambda'' \qquad (369)$$

$$C'_n=\frac{1}{\pi n}\int_0^{2\pi}\frac{S(H+J)}{(J^2+S^2)^{1/2}}\sin n\lambda''d\lambda'' \qquad (370)$$

B_2 is the same as the B_2 used in axis rotation.

SPACE OBLIQUE MERCATOR PROJECTION FOR THE GENERAL ELLIPTICAL ORBIT

As stated in the preceding section, the semi-empirical formulas used for the curvature of the groundtrack with a circular or near-circular orbit are not satisfactory if the eccentricity of the orbit exceeds about 0.05. While the formulas relating ϕ and λ to ϕ'' and λ'', as derived above for the elliptical orbit, may be used here without change, it is important to incorporate the constraint pointed out by Junkins, namely that the groundtrack on the map should have the same radius of curvature as the groundtrack on the surface of the ellipsoid as viewed vertically above the surface.

Since tests using subsequent formulas show that the use of this constraint does not appreciably improve the scale accuracy of the SOM for the circular orbit over the formulas derived already in this work, the formulas based on radius of curvature are proposed only for the general elliptical orbit, to convert ϕ'' and λ'' to x and y and the inverse. They may be fairly readily simplified for the circular orbit and sphere as desired.

For at least two reasons the radius of curvature restraint (combined with a groundtrack true to scale) does not alone produce a perfectly conformal projection: (1) The position ϕ'' along the scanline is calculated by somewhat empirical formulas for the ellipsoid independent of the groundtrack curvature, and (2) the scanlines are assumed to be straight, but this is not exactly correct, even for the sphere.

In figure 12, line ABCE is a satellite groundtrack as plotted on a flat map with origin at A, the Y axis arbitrarily tangent to the groundtrack at A, and the X axis perpendicular. Element BC, of length ds, has a radius of curvature BF=CF=r_c. Angle $d\theta$, CFB, is

$$d\theta = ds/r_c \tag{371}$$

Angle θ is the cumulative or integral of $d\theta$'s from A to B:

$$\theta = \int_0^\theta d\theta$$
$$= \int_0^s (ds/r_c)$$

To relate θ to λ'', the expression is multiplied by $d\lambda''/d\lambda''$:

$$\theta = \int_0^{\lambda''} (1/r_c)\,(ds/d\lambda'')d\lambda'' \tag{372}$$

Element BC has a slope such that angle BCD equals θ, and since BD and DC are elemental changes of x and y, respectively, in moving from B to C,

$$\begin{aligned} dx &= \text{BD} = \text{BC} \sin\theta \\ &= ds \sin\theta \\ &= (ds/d\lambda'') \sin\theta \, d\lambda'' \end{aligned} \tag{373}$$

and

$$\begin{aligned} dy &= \text{DC} = \text{BC} \cos\theta \\ &= ds \cos\theta \\ &= (ds/d\lambda'') \cos\theta \, d\lambda'' \end{aligned} \tag{374}$$

Since $ds = [(dx)^2 + (dy)^2]^{\frac{1}{2}}$, for which the X and Y axes may be in any perpendicular orientation, we may use equations (339) and (340) or earlier derivations to show that

$$(ds/d\lambda'') = a\,(H^2 + S^2)^{\frac{1}{2}} \tag{375}$$

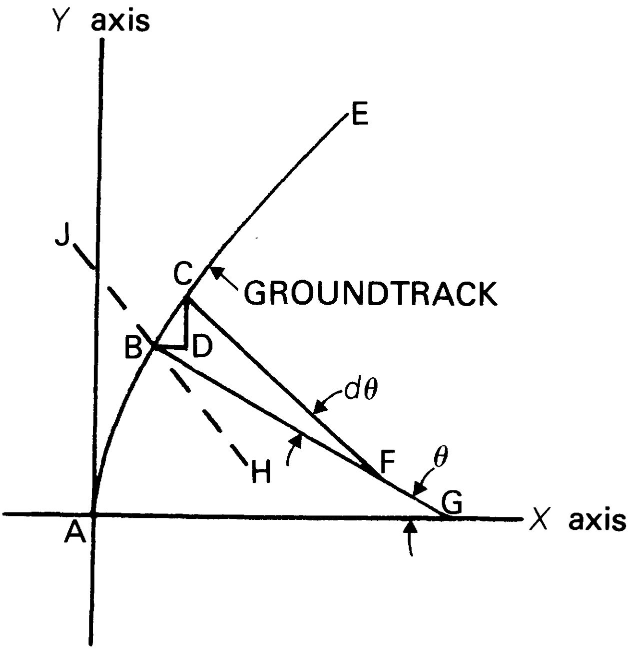

FIGURE 12.—Elements of groundtrack curvature.

where H and S are found for the non-circular orbit from (361) and (362) in terms of λ''.

Combining (372) through (375), and integrating, the basic equations for the groundtrack are found:

$$x/a = \int_0^{\lambda''} (H^2+S^2)^{\frac{1}{2}} \sin\theta \, d\lambda'' \tag{376}$$

$$y/a = \int_0^{\lambda''} (H^2+S^2)^{\frac{1}{2}} \cos\theta \, d\lambda'' \tag{377}$$

where

$$\theta = a \int_0^{\lambda''} [(H^2+S^2)^{\frac{1}{2}}/r_c] d\lambda'' \tag{378}$$

For reasons which will develop later, the X and Y axes will not be rotated from the orientation shown in figure 12, except for convenience in certain cases.

Formulas for H and S have been developed, but the radius of curvature r_c is yet to be derived for the general case. The practical evaluation of Fourier constants for the double integral also must be delineated. Before this is done, the adjustment for position along the scan lines will be derived.

The slope of the groundtrack in figure 12 at B is dy/dx, and from (373) and (374),

$$dy/dx = \cos\theta/\sin\theta = 1/\tan\theta$$

Thus

$$\angle CBD = \arctan(1/\tan\theta)$$

Angle JBC, between scanline JBH and the groundtrack, must be the same as that derived for the circular orbit, first for the sphere and then for the ellipsoid, except for using equations (361) and (362) for H and S. From equation (46) preceding either (47) or (206), since the slope of the groundtrack with vertical scanlines is (S/H),

$$\angle JBC = \arctan(H/S)$$

For the slope of the scanline with respect to the X axis of figure 12, we take the tangent of ∠JBD, which is the sum of ∠JBC and ∠CBD, or

$$\begin{aligned}(dy/dx)_{sl} &= \tan\angle JBD = \tan(\angle JBC + \angle CBD)\\ &= (H/S + 1/\tan\theta)/(1 - H/S\tan\theta)\\ &= (H\sin\theta + S\cos\theta)/(S\sin\theta - H\cos\theta) \qquad (379)\end{aligned}$$

This function is applied to equations (376) and (377) by adding cos ∠JBD times ln tan $(\pi/4+\phi''/2)$ to (376) and adding sin ∠JBD times ln tan $(\pi/4+\phi''/2)$ to (377). To obtain cos∠JBD,

$$\begin{aligned}\cos\angle JBD &= 1/(\tan^2\angle JBD + 1)^{\frac{1}{2}}\\ &= (S\sin\theta - H\cos\theta)/[(S\sin\theta - H\cos\theta)^2\\ &\quad + (H\sin\theta + S\cos\theta)^2]^{\frac{1}{2}}\\ &= (S\sin\theta - H\cos\theta)/(S^2\sin^2\theta - 2SH\sin\theta\cos\theta\\ &\quad + H^2\cos^2\theta + H^2\sin^2\theta + 2HS\sin\theta\cos\theta + S^2\cos^2\theta)^{\frac{1}{2}}\\ &= (S\sin\theta - H\cos\theta)/(H^2+S^2)^{\frac{1}{2}} \qquad (380)\end{aligned}$$

$$\begin{aligned}\sin\angle JBD &= \tan\angle JBD\cos\angle JBD\\ &= (H\sin\theta + S\cos\theta)(S\sin\theta - H\cos\theta)/\\ &\quad (S\sin\theta - H\cos\theta)(H^2+S^2)^{\frac{1}{2}}\\ &= (H\sin\theta + S\cos\theta)/(H^2+S^2)^{\frac{1}{2}}\end{aligned}$$

Equations (376) and (377) then become, for the complete projection,

$$\begin{aligned}x/a &= \int_0^{\lambda''} (H^2+S^2)^{\frac{1}{2}}\sin\theta\, d\lambda''\\ &\quad - [(H\cos\theta - S\sin\theta)/(H^2+S^2)^{\frac{1}{2}}]\ln\tan(\pi/4+\phi''/2) \qquad (381)\end{aligned}$$

$$y/a = \int_0^{\lambda''} (H^2+S^2)^{\frac{1}{2}} \cos\theta \, d\lambda''$$
$$+[(H\sin\theta + S\cos\theta)/(H^2+S^2)^{\frac{1}{2}}] \ln\tan(\pi/4+\phi''/2) \qquad (382)$$

These equations, looking simpler than they are in practical use, do not simplify when applied to the circular orbit for sphere or ellipsoid. Instead, the calculations of H, S, θ, ϕ'', and λ'' from ϕ and λ (as well as the Fourier equivalents) simplify for the latter cases.

Following is the derivation for r_c, the radius of curvature used to find θ in equation (378). This will follow the principles used to obtain equation (152) for the radius of curvature for the sphere and a circular orbit. First the equations for rectangular coordinates of the ellipsoidal orthographic projection will be obtained.

In figure 13, an octant of the ellipsoid is shown, with semi-major axis AD$=a$. The rectangular coordinates of random point K, with ellipsoidal coordinates ϕ and λ, and of the proposed center G of the tangent plane GL, at latitude ϕ_0 and longitude 0, are obtained relative to the X', Y', and Z' axes from earlier formulas, reversing the functions of λ since $\lambda=0$ on the X' axis rather than the Y' axis. For point K,

$$\mathrm{KJ} = x' = a\cos\phi\cos\lambda/(1-e^2\sin^2\phi)^{\frac{1}{2}} \qquad (176a)$$

$$\mathrm{SK} = y' = a\cos\phi\sin\lambda/(1-e^2\sin^2\phi)^{\frac{1}{2}} \qquad (223a)$$

$$\mathrm{JD} = z' = a\,(1-e^2)\sin\phi/(1-e^2\sin^2\phi)^{\frac{1}{2}} \qquad (177)$$

For point G, these formulas reduce to

$$x'_0 = a\cos\phi_0/(1-e^2\sin^2\phi_0)^{\frac{1}{2}} \qquad (383)$$

$$y'_0 = 0 \qquad (384)$$

$$z'_0 = a(1-e^2)\sin\phi_0/(1-e^2\sin^2\phi_0)^{\frac{1}{2}} \qquad (385)$$

On the tangent plane, with the Y axis pointing north and the X axis as shown in figure 13, the rectangular coordinates are as follows: the x coordinate is the same as y', or

$$x = a\cos\phi\sin\lambda/(1-e^2\sin^2\phi)^{\frac{1}{2}} \qquad (386)$$

while y equals GL in the equatorial view. To find this,

$$\tan\angle\mathrm{KGM} = \mathrm{KM}/\mathrm{GM}$$
$$= (z'-z'_0)/(x'_0-x')$$
$$\angle\mathrm{LGM} = 90° - \phi_0$$
$$\mathrm{GL} = \mathrm{GK}\cos\angle\mathrm{LGK}$$
$$= [(z'-z'_0)^2+(x'_0-x')^2]^{\frac{1}{2}}\cos(\angle\mathrm{LGM}-\angle\mathrm{KGM})$$
$$= [(z'-z'_0)^2+(x'_0-x')^2]^{\frac{1}{2}}/[1+\tan^2(\angle\mathrm{LGM}-\angle\mathrm{KGM})]^{\frac{1}{2}} \qquad (387)$$

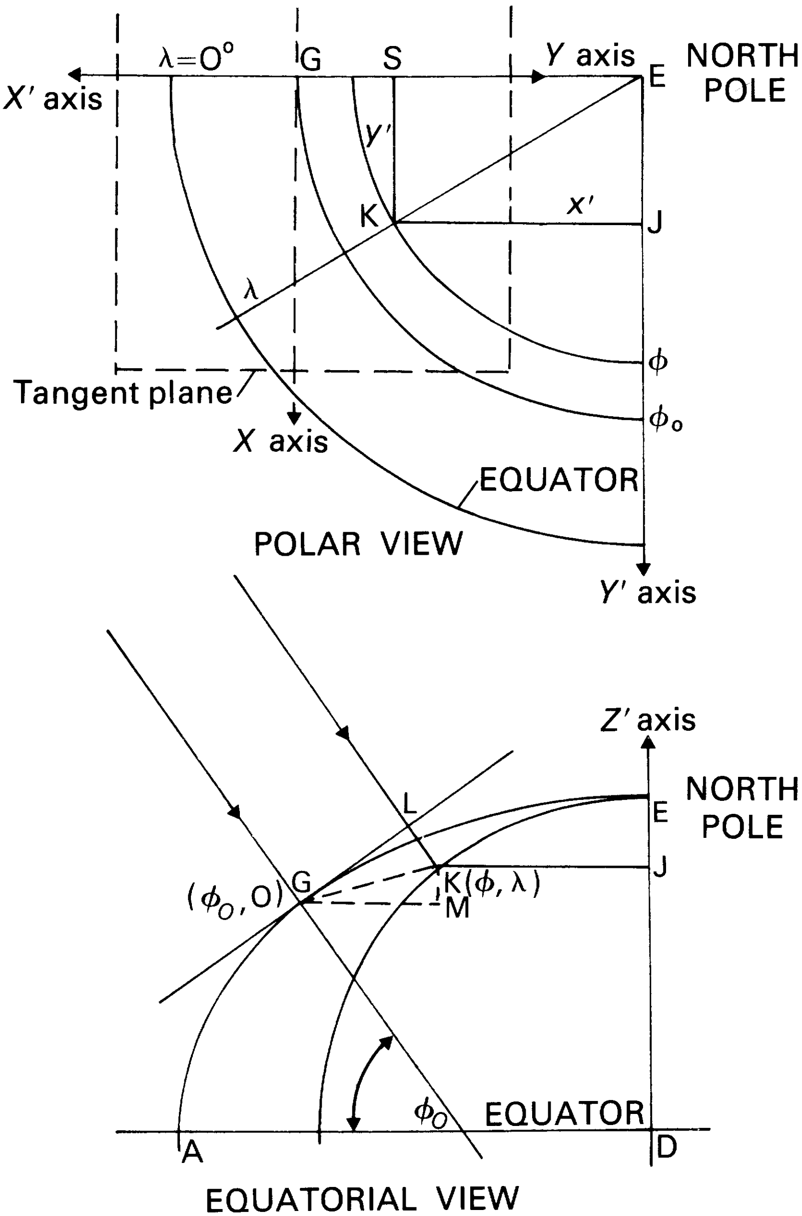

FIGURE 13.—Orthographic projection of the ellipsoid.

For the trigonometric term,

$$\begin{aligned}\tan(\angle LGM-\angle KGM) &= (\tan\angle LGM-\tan\angle KGM)/\\ &\quad (1+\tan\angle LGM\tan\angle KGM)\\ &=[1/\tan\phi_o-(z'-z'_o)/(x'_o-x')]/\\ &\quad [1+(z'-z'_o)/(x'_o-x')\tan\phi_o]\\ &=[(x'_o-x')-(z'-z'_o)\tan\phi_o]/\\ &\quad [(x'_o-x')\tan\phi_o+(z'-z'_o)] \qquad (388)\end{aligned}$$

Substituting into (387) and multiplying through by the denominator of (388),

$$\begin{aligned}GL=y=&[(z'-z'_o)^2+(x'_o-x')^2]^{\frac{1}{2}}[(x'_o-x')\tan\phi_o+(z'-z'_o)]/\\ &\{[(x'_o-x')\tan\phi_o+(z'-z'_o)]^2+[(x'_o-x')\\ &-(z'-z'_o)\tan\phi_o]^2\}^{\frac{1}{2}}\end{aligned}$$

Expanding the new denominator, canceling like terms and combining,

$$\begin{aligned}y=&[(z'-z'_o)^2+(x'_o-x')^2]^{\frac{1}{2}}[(x'_o-x')\tan\phi_o+(z'-z'_o)]/\\ &\sec\phi_o[(x'_o-x')^2+(z'-z'_o)^2]^{\frac{1}{2}}\\ =&(x'_o-x')\sin\phi_o+(z'-z'_o)\cos\phi_o\end{aligned}$$

Substituting from (176a), (177), (383), and (385), and canceling one pair of terms,

$$\begin{aligned}y=&[a/(1-e^2\sin^2\phi)^{\frac{1}{2}}][(1-e^2)\cos\phi_o\sin\phi-\sin\phi_o\cos\phi\cos\lambda]\\ &+ae^2\sin\phi_o\cos\phi_o/(1-e^2\sin^2\phi_o)^{\frac{1}{2}} \qquad (389)\end{aligned}$$

To satisfy the formula for radius of curvature,

$$r_c=[1+(dy/dx)^2]^{3/2}/(d^2y/dx^2) \qquad (138)$$

equations (386) and (389) are differentiated with respect to ϕ, and the latter derivative is then divided by the former for dy/dx (ϕ_o is a constant):

$$\begin{aligned}dx/d\phi=&a[-(1-e^2\sin^2\phi)^{\frac{1}{2}}\sin\phi\sin\lambda-(\cos\phi\sin\lambda)(\tfrac{1}{2})(1-e^2\sin^2\phi)^{-\frac{1}{2}}\\ &(-2e^2\sin\phi\cos\phi)]/(1-e^2\sin^2\phi)+a\cos\phi\cos\lambda(d\lambda/d\phi)/\\ &(1-e^2\sin^2\phi)^{\frac{1}{2}}\\ =&a[(1-e^2\sin^2\phi)\cos\phi\cos\lambda(d\lambda/d\phi)-(1-e^2)\sin\phi\sin\lambda]/\\ &(1-e^2\sin^2\phi)^{\frac{3}{2}} \qquad (390)\end{aligned}$$

$$\begin{aligned}dy/d\phi=&[a/(1-e^2\sin^2\phi)^{\frac{1}{2}}][(1-e^2)\cos\phi_o\cos\phi+\sin\phi_o\sin\phi\cos\lambda\\ &+\sin\phi_o\cos\phi\sin\lambda(d\lambda/d\phi)]+a[(1-e^2)\cos\phi_o\sin\phi\\ &-\sin\phi_o\cos\phi\cos\lambda](-\tfrac{1}{2})(1-e^2\sin^2\phi)^{-\frac{3}{2}}(-2e^2\sin\phi\cos\phi)\end{aligned}$$

$$\begin{aligned}dy/dx=&(dy/d\phi)/(dx/d\phi)\\ =&[(1-e^2)\cos\phi_o\cos\phi+\sin\phi_o\sin\phi\cos\lambda+\sin\phi_o\cos\phi\sin\lambda(d\lambda/d\phi)]\\ &(1-e^2\sin^2\phi)+[(1-e^2)\cos\phi_o\sin\phi-\sin\phi_o\cos\phi\cos\lambda]\\ &e^2\sin\phi\cos\phi\ /[(1-e^2\sin^2\phi)\cos\phi\cos\lambda(d\lambda/d\phi)\\ &-(1-e^2)\sin\phi\sin\lambda]\end{aligned}$$

$$=[(1-e^2)\cos\phi_o\cos\phi+\sin\phi_o\sin\phi\cos\lambda+\sin\phi_o\cos\phi\sin\lambda(d\lambda/d\phi)$$
$$-e^2(1-e^2)\cos\phi_o\cos\phi\sin^2\phi-e^2\sin\phi_o\sin^3\phi\cos\lambda$$
$$-e^2\sin\phi_o\cos\phi\sin^2\phi\sin\lambda(d\lambda/d\phi)+e^2(1-e^2)\sin^2\phi\cos\phi\cos\phi_o$$
$$-e^2\sin\phi_o\cos^2\phi\sin\phi\cos\lambda]/[(1-e^2\sin^2\phi)\cos\phi\cos\lambda(d\lambda/d\phi)$$
$$-(1-e^2)\sin\phi\sin\lambda]$$

$$=[(1-e^2)(\cos\phi_o\cos\phi+\sin\phi_o\sin\phi\cos\lambda)+(1-e^2\sin^2\phi)$$
$$\sin\phi_o\cos\phi\sin\lambda(d\lambda/d\phi)]/[(1-e^2\sin^2\phi)\cos\phi\cos\lambda(d\lambda/d\phi)$$
$$-(1-e^2)\sin\phi\sin\lambda] \qquad (391)$$

At the point of tangency, where the radius of curvature is being determined, $\phi_o=\phi$, and $\lambda=0$. Equation (391) simplifies to

$$(dy/dx)_o=(1-e^2)/(1-e^2\sin^2\phi)\cos\phi(d\lambda/d\phi) \qquad (392)$$

For (d^2y/dx^2), equation (391) rather than (392) must be differentiated with respect to ϕ and divided by $(dx/d\phi)$ from (390):

$$d^2y/dx^2=[d(dy/dx)/d\phi]/(dx/d\phi)$$
$$=\{[(1-e^2\sin^2\phi)\cos\phi\cos\lambda\,(d\lambda/d\phi)-(1-e^2)\sin\phi\sin\lambda]$$
$$\{(1-e^2)[-\cos\phi_o\sin\phi+\sin\phi_o\cos\phi\cos\lambda$$
$$-\sin\phi_o\sin\phi\sin\lambda\,(d\lambda/d\phi)]$$
$$-[\sin\phi_o\sin\phi\sin\lambda(d\lambda/d\phi)-\sin\phi_o\cos\phi\cos\lambda\,(d\lambda/d\phi)^2$$
$$-\sin\phi_o\cos\phi\sin\lambda\,(d^2\lambda/d\phi^2)](1-e^2\sin^2\phi)$$
$$+\sin\phi_o\cos\phi\sin\lambda\,(d\lambda/d\phi)(-2e^2\sin\phi\cos\phi)\}$$
$$-[(1-e^2)(\cos\phi_o\cos\phi+\sin\phi_o\sin\phi\cos\lambda)+(1-e^2\sin^2\phi)$$
$$\sin\phi_o\cos\phi\sin\lambda(d\lambda/d\phi)]\ (1-e^2\sin^2\phi)$$
$$[-\sin\phi\cos\lambda(d\lambda/d\phi)$$
$$-\cos\phi\sin\lambda\,(d\lambda/d\phi)^2+\cos\phi\cos\lambda\,(d^2\lambda/d\phi^2)]$$
$$+\cos\phi\cos\lambda\,(d\lambda/d\phi)(-2e^2\sin\phi\cos\phi)$$
$$-(1-e^2)[\cos\phi\sin\lambda+\sin\phi\cos\lambda\,(d\lambda/d\phi)]\}\}(1-e^2\sin^2\phi)^{\frac{3}{2}}/$$
$$a[(1-e^2\sin^2\phi)\cos\phi\cos\lambda(d\lambda/d\phi)-(1-e^2)\sin\phi\sin\lambda]^3$$

Since this equation does not need to be further differentiated, the value at tangent point $\phi=\phi_o$ and $\lambda=0$ may be determined now to reduce handling of the equation.

$$(d^2y/dx^2)_o=\{(1-e^2\sin^2\phi)\cos\phi\,(d\lambda/d\phi)[(1-e^2)(-\cos\phi\sin\phi$$
$$+\sin\phi\cos\phi)$$
$$+\sin\phi\cos\phi\,(d\lambda/d\phi)^2(1-e^2\sin^2\phi)]$$
$$-(1-e^2)(\cos^2\phi+\sin^2\phi)[(1-e^2\sin^2\phi)(-\sin\phi\,(d\lambda/d\phi)$$
$$+\cos\phi\,(d^2\lambda/d\phi^2))+\cos\phi\,(d\lambda/d\phi)(-2e^2\sin\phi\cos\phi)$$
$$-(1-e^2)\sin\phi\,(d\lambda/d\phi)]\}(1-e^2\sin^2\phi)^{\frac{3}{2}}/$$
$$a[1-e^2\sin^2\phi)\cos\phi\,(d\lambda/d\phi)]^3$$

After intermediate steps, this "simplifies" to

$$(d^2y/dx^2)_o=\{(1-e^2)[2\sin\phi\,(d\lambda/d\phi)-\cos\phi(d^2\lambda/d\phi^2)$$
$$-e^2\sin\phi(3\sin^2\phi(d\lambda/d\phi)-\sin\phi\cos\phi(d^2\lambda/d\phi^2)$$
$$-(d\lambda/d\phi))]+(d\lambda/d\phi)^3\,(1-e^2\sin^2\phi)^2\sin\phi\cos^2\phi\}/$$
$$a(1-e^2\sin^2\phi)^{\frac{3}{2}}\cos^3\phi(d\lambda/d\phi)^3 \qquad (393)$$

Before substituting (392) and (393) in (138), equation (138) for r_c should be inverted to give $1/r_c$, which shall be called $(-R_i)$. This results in finite values throughout, instead of infinity at the points of inflection of the groundtrack at the equatorial crossings or nodes. It is also advisable to multiply $(d\lambda/d\phi)$ and $(d^2\lambda/d\phi^2)$, derived subsequently, by $\cos \lambda''$ and $\cos^3 \lambda''$, respectively, and to multiply the numerator and denominator of the equation for R_i by $\cos^3 \lambda''$, thus avoiding indeterminate values or dividing by zero. The sign will also be reversed to give initial positive values of R_i consistent with figure 12. Subscript zero should be applied to ϕ and λ to denote groundtrack conditions. As a further simplification, let $a=1$.

Therefore, let

$$G = \cos \lambda''(d\lambda_0/d\phi_0) \tag{394}$$

$$M = \cos^3 \lambda''(d^2\lambda_0/d\phi_0{}^2) \tag{395}$$

$$C = 1-e^2 \sin^2 \phi_0 \tag{396}$$

$$K = 1-e^2 \tag{397}$$

Substituting,

$$\begin{aligned} R_i = -1/r_c = & -(d^2y/dx^2)_0/[1+(dy/dx)_0^2]^{\frac{3}{2}} \\ = & -\{K[2G \sin \phi_0 \cos^2 \lambda'' - M \cos \phi_0 - e^2 \sin \phi_0 \\ & (3G \sin^2 \phi_0 \cos^2 \lambda'' - M \sin \phi_0 \cos \phi_0 \\ & -G \cos^2 \lambda'')] + G^3C^2 \sin \phi_0 \cos^2 \phi_0\}/ \\ & G^3C^{\frac{3}{2}} \cos^3 \phi_0 \,(1+K^2 \cos^2 \lambda''/C^2G^2 \cos^2 \phi_0)^{\frac{3}{2}} \\ = & -\{K[G \sin \phi_0 \cos^2 \lambda''(2-3e^2 \sin^2 \phi_0 + e^2) \\ & -M \cos \phi_0 \,(1-e^2 \sin^2 \phi_0)] + G^3C^2 \sin \phi_0 \cos^2 \phi_0\}/ \\ & (G^2 \,C \cos^2 \phi_0 + K^2 \cos^2 \lambda''/C)^{\frac{3}{2}} \\ = & -\{K[G \sin \phi_0 \cos^2 \lambda''(3C-K) - MC \cos \phi_0] \\ & + G^3C^2 \sin \phi_0 \cos^2 \phi_0\}/(G^2C \cos^2 \phi_0 \\ & + K^2 \cos^2 \lambda''/C)^{\frac{3}{2}} \end{aligned} \tag{398}$$

This formula for the reciprocal of the radius of curvature applies to any path on the surface of the ellipsoid, if G and M can be determined for the specific path desired. It is now necessary to derive $(d\lambda_0/d\phi_0)$ and $(d^2\lambda_0/d\phi_0^2)$ for the elliptical satellite orbit. For simplicity, since the geocentric groundtrack rather than the vertical groundtrack has been used for determining ϕ'', λ'', H, and S, the same will be used here. The slight correction for the vertical groundtrack will be made as before, as described in the summary of equations for a slightly elliptical orbit in the preceding section.

For the geocentric groundtrack, equations for ϕ_0 in terms of λ'' are unchanged from (195) and (196), using λ'' in place of λ':

$$\tan \phi_0 = \tan \phi_g/(1-e^2) \tag{195}$$

$$\sin \phi_g = \sin i \sin \lambda'' \tag{196}$$

Equations for longitude for geocentric or vertical groundtrack were derived in the preceding section, again using λ'':

$$\lambda_o = \lambda_{t_o} - (P_2/P_1)(L+\omega) \tag{338}$$

$$\tan \lambda_{t_o} = \tan \lambda'' \cos i \tag{133}$$

$$L = E' - e' \sin E' \tag{334}$$

$$\tan \tfrac{1}{2} E' = \tan \tfrac{1}{2} (\lambda'' - \omega)[(1-e')/(1+e')]^{1/2} \tag{364}$$

$$L' = (dL/d\lambda'') = (1-e' \cos E')^2/(1-e'^2)^{1/2} \tag{363}$$

Combining (338) and (133) and differentiating,

$$\lambda'' = \arctan (\tan \lambda'' \cos i) - (P_2/P_1)(L+\omega)$$
$$(d\lambda_o/d\lambda'') = \cos i \sec^2 \lambda''/(1+\cos^2 i \tan^2 \lambda'') - (P_2/P_1)(dL/d\lambda'') \tag{399}$$

Rearranging (196) and differentiating,

$$\sin \lambda'' = \sin \phi_g/\sin i \tag{236}$$

$$\cos \lambda''(d\lambda''/d\phi_o) = (\cos \phi_g/\sin i)(d\phi_g/d\phi_o) \tag{400}$$

Differentiating (195),

$$\sec^2 \phi_o d\phi_o = [\sec^2 \phi_g/(1-e^2)]d\phi_g$$
$$(d\phi_g/d\phi_o) = (1-e^2)(\tan^2 \phi_o + 1)/(\tan^2 \phi_g + 1)$$
$$= (1-e^2)[\tan^2 \phi_g/(1-e^2)^2 + 1]/(\tan^2 \phi_g + 1)$$

Simplifying and assigning the symbol D,

$$D = (d\phi_g/d\phi_o) = [1-e^2(2-e^2) \cos^2 \phi_g]/(1-e^2) \tag{401}$$

Rearranging (400) and multiplying $(d\lambda_o/d\lambda'')$ from (399) by $(d\lambda''/d\phi_o)$ from (400),

$$(d\lambda_o/d\phi_o) = [\cos i/\cos^2 \lambda''(1+ \cos^2 i \tan^2 \lambda'') - (P_2/P_1)(dL/d\lambda'')]$$
$$(\cos \phi_g/\sin i \cos \lambda'')(d\phi_g/d\phi_o)$$

The first denominator may be changed to $(\cos^2 \lambda'' + \cos^2 i \sin^2 \lambda'')$, $(1-\sin^2 \lambda'' + \cos^2 i \sin^2 \lambda'')$, and $(1-\sin^2 i \sin^2 \lambda'')$. Substituting from (196),

$$(d\lambda_o/d\phi_o) = [\cos i/\cos^2 \phi_g - (P_2/P_1)(dL/d\lambda'')]\,(\cos\phi_g/\sin i \cos \lambda'')(d\phi_g/d\phi_o) \tag{402}$$

To find G of equation (394),

$$G = \cos \lambda''(d\lambda_o/d\phi_o)$$
$$= [\cos i/\cos^2 \phi_g - (P_2/P_1)(dL/d\lambda'')](\cos \phi_g/\sin i)(d\phi_g/d\phi_o) \tag{403}$$

where the differentials are found from (363) and (401) above.

For $(d^2\lambda_o/d\phi_o^2)$ equation (402) is differentiated:

$$\begin{aligned}(d^2\lambda_o/d\phi_o^2) = {} & [\cos i/\cos^2\phi_g-(P_2/P_1)(dL/d\lambda'')]\{\sin i\cos\lambda''\\ & [-\sin\phi_g(d\phi_g/d\phi_o)^2+\cos\phi_g(d^2\phi_g/d\phi_o^2)]\\ & +\cos\phi_g(d\phi_g/d\phi_o)\sin i\sin\lambda''(d\lambda''/d\phi_o)\}/\\ & \sin^2 i\cos^2\lambda''+[\cos\phi_g(d\phi_g/d\phi_o)/\sin i\cos\lambda'']\\ & [2\cos i\sin\phi_g(d\phi_g/d\phi_o)/\cos^3\phi_g\\ & -(P_2/P_1)(d^2L/d\lambda''^2)(d\lambda''/d\phi_o)]\end{aligned}$$

Substituting from (400) for the last differential, and combining some other terms,

$$\begin{aligned}(d^2\lambda_o/d\phi_o^2) = {} & [\cos i/\cos^2\phi_g-(P_2/P_1)(dL/d\lambda'')]\{-\sin\phi_g\\ & (d\phi_g/d\phi_o)^2+\cos\phi_g(d^2\phi_g/d\phi_o^2)\\ & +\sin\phi_g\cos^2\phi_g\,(d\phi_g/d\phi_o)^2/\sin^2 i\cos^2\lambda''\}/\\ & \sin i\cos\lambda''+2\sin\phi_g(d\phi_g/d\phi_o)^2/\tan i\cos\lambda''\cos^2\phi_g\\ & -(P_2/P_1)(d^2L/d\lambda''^2)\cos^2\phi_g(d\phi_g/d\phi_o)^2/\sin^2 i\cos^2\lambda''\end{aligned}$$

To obtain M in equation (395),

$$M=\cos^3\lambda''(d^2\lambda_o/d\phi_o^2)$$

$$\begin{aligned}M=\{ & [\cos i/\cos^2\phi_g-(P_2/P_1)(dL/d\lambda'')]\,[\cos\phi_g\cos^2\lambda''(d^2\phi_g/d\phi_o^2)\\ & +\sin\phi_g(d\phi_g/d\phi_o)^2(\cos^2\phi_g/\sin^2 i-\cos^2\lambda'')]\\ & +2\sin\phi_g\cos i\cos^2\lambda''(d\phi_g/d\phi_o)^2/\cos^2\phi_g\\ & -(P_2/P_1)(d^2L/d\lambda''^2)\cos^2\phi_g\cos\lambda''\,(d\phi_g/d\phi_o)^2/\sin i\}/\sin i\end{aligned}$$

The expression $(\cos^2\phi_g/\sin^2 i-\cos^2\lambda'')$ may be reduced by using (196):

$$\begin{aligned}\cos^2\phi_g/\sin^2 i-\cos^2\lambda'' & =(\cos^2\phi_g-\sin^2 i+\sin^2\lambda''\sin^2 i)/\sin^2 i\\ & =(\cos^2\phi_g-\sin^2 i+\sin^2\phi_g)/\sin^2 i\\ & =1/\tan^2 i\end{aligned}$$

Thus

$$\begin{aligned}M=\{ & [\cos i/\cos^2\phi_g-(P_2/P_1)(dL/d\lambda'')]\,[\cos\phi_g\cos^2\lambda''(d^2\phi_g/d\phi_o^2)\\ & +\sin\phi_g(d\phi_g/d\phi_o)^2/\tan^2 i]+2\sin\phi_g\cos i\cos^2\lambda''(d\phi_g/d\phi_o)^2/\\ & \cos^2\phi_g-(P_2/P_1)(d^2L/d\lambda''^2)\cos^2\phi_g\cos\lambda''(d\phi_g/d\phi_o)^2/\sin i\}/\sin i\end{aligned} \tag{404}$$

Solution of (404) requires two more second derivatives. First differentiating $(d\phi_g/d\phi_o)$ in equation (401) and assigning the symbol D', this is simply

$$D'=(d^2\phi_g/d\phi_o^2)=[2/(1-e^2)]e^2\,(2-e^2)\sin\phi_g\cos\phi_g(d\phi_g/d\phi_o) \tag{405}$$

Differentiating (363),

$$(d^2L/d\lambda''^2)=[2/(1-e'^2)^{\frac{3}{2}}](1-e'\cos E')\,e'\sin E'\,(dE'/d\lambda'') \tag{406}$$

Repeating equation (341a) in the previous section,

$$(dE'/d\lambda'')=\sec^2\tfrac{1}{2}\,(\lambda''-\omega)\cos^2\tfrac{1}{2}\,E'\,[(1-e')/(1+e')]^{\frac{1}{2}} \tag{341a}$$

From (333),

$$\tan^2\tfrac{1}{2}\,(\lambda''-\omega)=\tan^2\tfrac{1}{2}\,E'\,(1+e')/(1-e')$$

Thus $$\sec^2 \tfrac{1}{2}(\lambda''-\omega)=1+\tan^2 \tfrac{1}{2} E'(1+e')/(1-e'^2)$$

Substituting this in (341a) and the latter in turn in (406), and calling this L'',

$$\begin{aligned}L''=(d^2L/d\lambda''^2)&=[2/(1-e'^2)^{\frac{1}{2}}]\,(1-e'\cos E')\,e'\sin E'\cos^2 \tfrac{1}{2}E'\\ &\quad[(1-e')/(1+e')]^{\frac{1}{2}}\,[1+\tan^2 \tfrac{1}{2}E'(1+e')/(1-e')]\\ &=[2/(1-e'^2)^{\frac{1}{2}}]\,(1-e'\cos E')\,e'\sin E'\{\cos^2 \tfrac{1}{2}E'\,[(1-e')/\\ &\quad(1+e')]^{\frac{1}{2}}+\sin^2 \tfrac{1}{2}E'\,[(1+e')/(1-e')]^{\frac{1}{2}}\}\\ &=[1/(1-e'^2)^{\frac{1}{2}}]\,(1-e'\cos E')\,e'\sin E'\,[(1+\cos E')(1-e')\\ &\quad+(1-\cos E')(1+e')]/(1-e'^2)^{\frac{1}{2}}\\ &=2e'\sin E'\,(1-e'\cos E')^2/(1-e'^2) \qquad (407)\end{aligned}$$

Now the fundamental θ can be found with equation (378) by numerical integration: H and S are found in terms of λ'' from (361) and (362) and $(-1/r_c)$ or R_i from (398). To find (398) in terms of λ'', equations (396), (397), (196), (195), (401), (405), (364), (363), (407), (403), and (404) must be used, as well as (198), (199), and (200). These formulas will be assembled later, and they present no serious computing problem except that they are lengthy.

Fourier constants are again almost indispensable for practical computation. The usual procedure is described with equations (53) and those following. Simpson's rule is recommended for numerical integration. Instead of an integrating interval of 9°, as used for the circular orbit, an interval of 3° is recommended for the noncircular orbit, and integration must be carried out for the full 360° cycle due to lack of symmetry of the groundtrack for each quadrant.

Three types of constants must be determined, and none are zero for the general case, except that convergence leads ultimately to negligible coefficients:

$$\theta=g\lambda''+\Sigma\, d_n \sin n\lambda'' +\Sigma\, f_n(1-\cos n\lambda'') \qquad (408)$$

where
$$g=(1/2\pi)\int_0^{2\pi} R_i(H^2+S^2)^{\frac{1}{2}}d\lambda'' \qquad (409)$$

$$d_n=(1/\pi n)\int_0^{2\pi} R_i(H^2+S^2)^{\frac{1}{2}}\cos n\lambda'' d\lambda'' \qquad (410)$$

$$f_n=(1/\pi n)\int_0^{2\pi} R_i(H^2+S^2)^{\frac{1}{2}}\sin n\lambda'' d\lambda'' \qquad (411)$$

For circular orbits g and all d_n, coefficients are zero, as are coefficients f_n for even values of n. For elliptical orbits with perigee ω coinciding with either ascending or descending node, g is zero. The number of terms required for convergence increases with eccentricity: for a circular orbit f_1, f_3, and f_5 are sufficient for six-place accuracy; for a hypothetical eccentricity of 0.2, about 18 coefficients (g, d_n, and f_n) are required for the same accuracy, and for an eccentricity of 0.7, about 40.

Once the coefficients are determined, the second numerical integration may occur in order to evaluate the Fourier coefficients for the integrals of (376) and (377), or (381) and (382). If the coefficient g from (409) is zero, the coefficients are determined in the same way as the coefficients for θ or for circular orbits in earlier sections of this study. For the elliptical orbit, it is highly unlikely that the perigee will coincide with a node. Therefore, in general, the simple forms of (376) and (377) must be expanded as follows:

A requirement of Fourier coefficient determinations is the repeatability of the function with every cycle, as in equation (57). If g is not zero, θ does not return to its original value at each 360° cycle. It is necessary, then, to separate the repeatable portion of (408) from the nonrepeating portion. This may be done by letting

$$\theta=\gamma+g\lambda'' \tag{412}$$

where

$$\gamma=\Sigma\, d_n \sin n\lambda''+\Sigma\, f_n\,(1-\cos n\lambda'') \tag{413}$$

The integral for (x/a) from (376), for the groundtrack only, may thus be written

$$\begin{aligned}(x/a)_1&=\int_0^{\lambda''}(H+S^2)^{\frac{1}{2}}\sin(\gamma+g\lambda'')d\lambda''\\ &=\int_0^{\lambda''}(H^2+S^2)^{\frac{1}{2}}(\sin\gamma\cos g\lambda''+\cos\gamma\sin g\lambda'')d\lambda''\\ &=\int_0^{\lambda''}[(H^2+S^2)^{\frac{1}{2}}\sin\gamma]\cos g\lambda'' d\lambda''\\ &\quad+\int_0^{\lambda''}[(H^2+S^2)^{\frac{1}{2}}\cos\gamma\,]\sin g\lambda''\, d\lambda''\end{aligned} \tag{414}$$

The bracketed expressions repeat every 360°, and Fourier series may be determined for each. (A similar equation for y/a will follow later.) Let

$$(H^2+S^2)^{\frac{1}{2}}\sin\gamma\ =b+\Sigma\, h_n'\sin n\lambda''+\Sigma\, h_n\cos n\lambda'' \tag{415}$$

and

$$(H^2+S^2)^{\frac{1}{2}}\cos\gamma\ =b'+\Sigma\, c_n\sin n\lambda''+\Sigma\, c_n'\cos n\lambda'' \tag{416}$$

where

$$b=(1/2\pi)\int_0^{2\pi}(H^2+S^2)^{\frac{1}{2}}\sin\gamma\, d\lambda'' \tag{417}$$

$$b'=(1/2\pi)\int_0^{2\pi}(H^2+S^2)^{\frac{1}{2}}\cos\gamma\, d\lambda'' \tag{418}$$

$$h_n=(1/\pi)\int_0^{2\pi}(H^2+S^2)^{\frac{1}{2}}\sin\gamma\cos n\lambda'' d\lambda'' \tag{419}$$

$$h'_n = (1/\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}} \sin\gamma \sin n\lambda'' d\lambda'' \quad (420)$$

$$c'_n = (1/\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}} \cos\gamma \cos n\lambda'' d\lambda'' \quad (421)$$

$$c_n = (1/\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}} \cos\gamma \sin n\lambda'' d\lambda'' \quad (422)$$

Note that the left sides of (415) and (416) are not integrals, while θ of equations (378) and (408) involves an integral. Consequently the form of these coefficients is somewhat different from that of the constants in (409) through (411)., They may be determined, however, with the same application of Simpson's rule as that for all the other Fourier coefficients developed in this bulletin.

Now (415) and (416) may be substituted into (414):

$$\begin{aligned}(x/a)_1 = \int_0^{\lambda'} [&b \cos g\lambda'' + \Sigma\, h'_n \sin n\lambda'' \cos g\lambda'' \\ &+ \Sigma\, h_n \cos n\lambda'' \cos g\lambda'']d\lambda'' + \int_0^{\lambda'} [b' \sin g\lambda'' \\ &+ \Sigma\, c_n \sin n\lambda'' \sin g\lambda'' + \Sigma\, c'_n \cos n\lambda'' \sin g\lambda'']d\lambda''\end{aligned}$$

This equation may be integrated using standard trigonometric forms:

$$\begin{aligned}(x/a)_1 = \{&(b/g) \sin g\lambda'' + \Sigma\, h'_n [-\cos (n-g)\lambda''/2(n-g) \\ &-\cos (n+g)\lambda''/2(n+g)] + \Sigma\, h_n[\sin(n-g)\lambda''/2(n-g) \\ &+ \sin (n+g)\lambda''/2(n+g)] - (b'/g) \cos g\lambda'' \\ &+ \Sigma\, c_n[\sin (n-g)\lambda''/2(n-g) - \sin (n+g)\lambda''/2(n+g)] \\ &+ \Sigma\, c'_n[\cos (n-g)\lambda''/2(n-g) - \cos (n+g)\lambda''/2(n+g)]\}_0^{\lambda'} \\ = \{&(1/g)(b \sin g\lambda'' - b' \cos g\lambda'') \\ &- \Sigma\, [(-h_n+c_n)(\sin n\lambda'' \cos g\lambda'' + \cos n\lambda'' \sin g\lambda'') \\ &+ (h'_n+c'_n)(\cos n\lambda'' \cos g\lambda'' - \sin n\lambda'' \sin g\lambda'')]/2(n+g) \\ &- \Sigma\, [(-h_n-c_n)(\sin n\lambda'' \cos g\lambda'' - \cos n\lambda'' \sin g\lambda'') \\ &+ (h'_n-c'_n)(\cos n\lambda'' \cos g\lambda'' + \sin n\lambda'' \sin g\lambda'')]/2(n-g)\}_0^{\lambda'}\end{aligned}$$

$$\begin{aligned}= \{&(1/g)(b \sin g\lambda'' - b' \cos g\lambda'') - \Sigma\, [(-h_n+c_n) \cos g\lambda''/ \\ &2(n+g) - (h'_n+c'_n) \sin g\lambda''/2(n+g) \\ &+ (h_n+c_n) \cos g\lambda''/2(n-g) - (h'_n-c'_n) \sin g\lambda''/ \\ &2(n-g)] \sin n\lambda'' + \Sigma\, [(-h_n+c_n) \sin g\lambda''/2(n+g) \\ &-(h'_n+c'_n) \cos g\lambda''/2(n+g) + (h_n+c_n) \sin g\lambda''/ \\ &2(n-g) - (h'_n-c'_n) \cos g\lambda''/2(n-g)]\}_0^{\lambda'}\end{aligned}$$

$$=(1/g)(b \sin g\lambda''-b' \cos g\lambda'')+\Sigma\{[(nc_n'-gh_n')/(n^2-g^2)] \sin g\lambda''+[(nh_n+gc_n)/(n^2-g^2)] \cos g\lambda''\}\sin n\lambda''$$
$$-\Sigma\{[(nc_n+gh_n)/(n^2-g^2)] \sin g\lambda''+[(nh_n'-gc_n')/(n^2-g^2)] \cos g\lambda''\} \cos n\lambda''+b'/g$$
$$+\Sigma (nh_n'-gc_n')/(n^2-g^2) \qquad (423)$$

The last two terms are the negative value of the integral when $\lambda''=0$. It is convenient to determine new constants to replace the coefficients of (423):

$$B=b/g \qquad (424)$$

$$B'=b'/g \qquad (425)$$

$$A_n=(nh_n+gc_n)/(n^2-g^2) \qquad (426)$$

$$A_n'=(nc_n'-gh_n')/(n^2-g^2) \qquad (427)$$

$$C_n=(nc_n+gh_n)/(n^2-g^2) \qquad (428)$$

$$C_n'=(nh_n'-gc_n')/(n^2-g^2) \qquad (429)$$

$$C_x=B'+\Sigma C_n' \qquad (430)$$

These symbols may be substituted into (423), but since (423) is the Fourier equivalent of (376), the final term of (381) may also be attached for the complete equation for (x/a), the equivalent of (381):

$$x/a=B \sin g\lambda''-B' \cos g\lambda''+\Sigma (A_n' \sin g\lambda''+A_n \cos g\lambda'') \sin n\lambda''$$
$$-\Sigma (C_n \sin g\lambda''+C_n' \cos g\lambda'') \cos n\lambda'' + C_x$$
$$-[(H \cos \theta-S \sin \theta)/(H^2+S^2)^{\frac{1}{2}}] \ln \tan (\pi/4+\phi''/2) \qquad (431)$$

Using symbols which will also be found useful in later formulas, let

$$H'=H \cos \theta- S \sin \theta \qquad (432)$$

$$A''=\Sigma(A_n \sin n\lambda''-C_n' \cos n\lambda'') \qquad (433)$$

$$C''=\Sigma(A_n' \sin n\lambda''-C_n \cos n\lambda'') \qquad (434)$$

Then (431) may be rearranged and rewritten as follows:

$$x/a=(B+C'') \sin g\lambda''-(B'-A'') \cos g\lambda''+C_x$$
$$-[H'/(H^2+S^2)^{\frac{1}{2}}] \ln \tan (\pi/4+\phi''/2) \qquad (435)$$

In computer programing, care should be taken, of course, to minimize the number of calculations of repeated trigonometric terms.

The Fourier equivalent of equation (382) for (y/a) is derived in an analogous manner, so most of the steps are omitted here. From (377) and (412),

$$(y/a)_1=\int_0^{\lambda''}(H^2+S^2)^{\frac{1}{2}}\cos(\gamma+g\lambda'')d\lambda''$$
$$=\int_0^{\lambda''}[(H^2+S^2)^{\frac{1}{2}}\cos\gamma]\cos g\lambda''\,d\lambda''$$
$$-\int_0^{\lambda''}[(H^2+S^2)^{\frac{1}{2}}\sin\gamma]\sin g\lambda''\,d\lambda''$$

Substituting (415) and (416),

$$(y/a)_1=\int_0^{\lambda''}[b'\cos g\lambda''+\Sigma c_n\sin n\lambda''\cos g\lambda''+\Sigma c_n'\cos n\lambda''\cos g\lambda'']d\lambda''$$
$$-\int_0^{\lambda''}b\sin g\lambda''+\Sigma h_n'\sin n\lambda''\sin g\lambda''+\Sigma h_n\cos n\lambda''\sin g\lambda'']d\lambda''$$

With integration as accomplished for $(x/a)_1$, combining of terms, and final substitution into (382),

$$y/a=(B'-A'')\sin g\lambda''+(B+C'')\cos g\lambda''-C_y$$
$$+[S'/(H^2+S^2)^{\frac{1}{2}}]\ln\tan(\pi/4+\phi''/2) \qquad (436)$$

where coefficients have been previously defined, except for C_y and S':

$$C_y=B-\Sigma C_n \qquad (437)$$
$$S'=H\sin\theta+S\cos\theta \qquad (438)$$

Equations (435) and (436) have the interesting variation from equations for the circular orbit in that, aside from being much more complicated, the central line about which the sinusoidal-like groundtrack oscillates is no longer a straight line, but is now a circular arc. Its equation is found from (435) and (436) by letting $n\lambda''$ be any multiple of 360°, so that $\sin n\lambda''=0$ and $\cos n\lambda''=1$, while $\phi''=0$:

$$(x/a)_o=C_y\sin g\lambda''+C_x(1-\cos g\lambda'')$$
$$(y/a)_o=C_x\sin g\lambda''-C_y(1-\cos g\lambda'')$$

Transposing constants,

$$(x/a)_o-C_x=C_y\sin g\lambda''-C_x\cos g\lambda''$$
$$(y/a)_o+C_y=C_x\sin g\lambda''+C_y\cos g\lambda''$$

Multiplying the first by C_x, the second by C_y and subtracting, then multiplying the first by C_y, the second by C_x and adding, the two new equations may be squared and added to eliminate the angle. Simplifying the result,

$$[(x/a)_o-C_x]^2+[(y/a)_o+C_y]^2=C_x^{\,2}+C_y^{\,2} \qquad (439)$$

the equation of a circle with radius $(C_x^{\,2}+C_y^{\,2})^{\frac{1}{2}}$ and center at $(x/a=C_x)$, and $(y/a=-C_y)$. Generally, if Fourier constant g, equation (409), is positive, this "axis" curves to the right as the groundtrack proceeds upward (y increasing) from the origin, and left if g is negative. In examples checked, a positive g occurs if ω (longitude of perigee) is between 180° and 360°, and a negative g, if ω is in the first or second quadrants. Empirically, g is within 1 percent less than $[-(P_2/P_1)\,e' \sin \omega \sin i]$. The "axis" curves approximately 360° x g for each orbital revolution. An example of elements of the projection for the general elliptical orbit is shown in figure 14.

If g is zero, for the circular orbit or a perigee coinciding with a node (as stated earlier), equations (435) and (436) are indeterminate, since there is division by zero in (424) and (425). This affects the terms $B \sin g\lambda''$, $B' \cos g\lambda''$, and C_x of equation (435) and the corresponding terms of (436). To resolve this, (424) and (425) may be used to rewrite these terms as follows:

$$\begin{aligned} B \sin g\lambda'' &= b\,(\sin g\lambda'')/g \\ &= b\lambda'' \text{ as } g \text{ approaches zero.} \end{aligned}$$

$$\begin{aligned} -B' \cos g\lambda'' + C_x &= -(b'/g) \cos g\lambda'' + (b'/g) + \Sigma C_n' \\ &= b'(1-\cos g\lambda'')/g + \Sigma C_n' \end{aligned}$$

To evaluate $(1-\cos g\lambda'')/g$ as g approaches zero, the expression may be differentiated with respect to λ'':

$$d(1-\cos g\lambda'')/g d\lambda'' = g(\sin g\lambda'')/g = \sin g\lambda'' = 0, \text{ when } g \text{ is zero.}$$

Thus (435) is written as follows, if $g=0$.

$$x/a = b\lambda'' + \Sigma C_n' + A'' - [H'/(H^2+S^2)^{\frac{1}{2}}] \ln \tan (\pi/4 + \phi''/2) \qquad (440)$$

Similarly, (436) is written, for $g=0$,

$$y/a = b'\lambda'' + \Sigma C_n + C'' + [S'/(H^2+S^2)^{\frac{1}{2}}] \ln \tan (\pi/4 + \phi''/2) \qquad (441)$$

Since, when $g=0$, the groundtrack oscillates about a straight line, it is preferable to rotate the axes so that the central line is the X axis, or $y=0$ at each 360° of λ''. This may be done to obtain revised coordinates x' and y' as follows (derivation not shown, but conventional):

$$x' = (b'y + bx)/(b'^2+b^2)^{\frac{1}{2}} \qquad (442)$$

$$y' = (by - b'x)/(b'^2+b^2)^{\frac{1}{2}} \qquad (443)$$

For the circular orbit, h_n' and c_n are zero, and equations (442) and (443) may be combined with (440) and (441) to give the following equations. These correspond to (263), (264), (282a), and (283a) except that the latter are based on empirical curvature, while the following are based on calculated radius of curvature:

$$\begin{aligned} x'/a = {} & B''\lambda'' + \Sigma A_n'' \sin n\lambda'' \\ & + [(b'S' - bH')/B''(H^2+S^2)^{\frac{1}{2}}] \ln \tan (\pi/4 + \phi''/2) \end{aligned} \qquad (444)$$

$$\begin{aligned} y'/a = {} & \Sigma C_n'' \sin n\lambda'' \\ & + [(bS' + b'H')/B''(H^2+S^2)^{\frac{1}{2}}] \ln \tan (\pi/4 + \phi''/2) \end{aligned} \qquad (445)$$

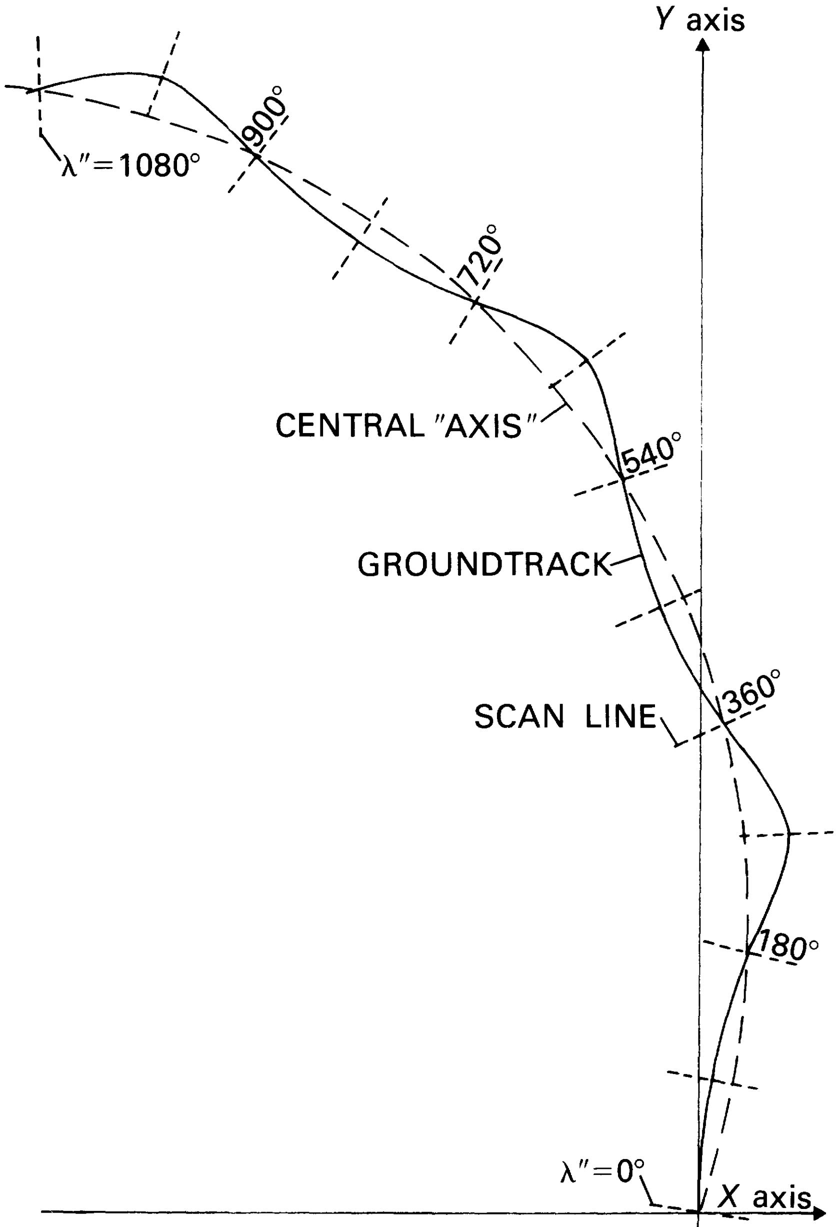

FIGURE 14.—Groundtrack and extended sample scanlines for Space Oblique Mercator projection, elliptical orbit. Eccentricity is 0.5, perigee at north polar approach, ($i=80°$, $P_2=275$ min., $P_1=1440$ min.). Three orbits are shown.

where $\quad B''=(b^2+b'^2)^{\frac{1}{2}}$ (446)

$A_n''=(bh_n+b'c_n')/Bn$ for even n only (447)

$C_n''=(bc_n'-b'h_n)/Bn$ for odd n only (448)

and other symbols are as calculated above. Note that $\gamma=\theta$ if $g=0$.

When Landsat constants are applied to these Fourier coefficient calculations ($P_2/P_1=18/251$; $i=99.092°$, Clarke 1866 ellipsoid), the following values are found:

$f_1=8.131909918°$
$f_3=0.0007134417°$
$f_5=0.0000101583°$
$f_7=0.0000000174°$
$B''=1.005806754$ for λ'' in radians
$A_2''=-0.0010940568$
$A_4''=-0.0000015160$
$A_6''=-0.0000000027$
$C_1''=0.1433202229$
$C_3''=0.0000327398$
$C_5''=0.0000000046$
$b=0.1422865425$
$b'=0.9956916017$

As check points in calculating R_i for the above, if $\lambda''=45°$, $R_i=0.0992990361$, and if $\lambda''=90°$, $R_i=0.1407675647$.

INVERSE EQUATIONS

Necessary forward equations have now been derived, including several in the preceding section. For inverse equations, the new equations necessary are those to determine ϕ'' and λ'' from x and y, or the inverses of (435) and (436), and the special inverses of (440), (441), (444), and (445).

For the former, following a procedure analogous to the earlier derivations for the circular orbit, ϕ'' in (435) and (436) may be eliminated by multiplying both sides of (435) by S' and both sides of (436) by H', and then adding the two equations. The terms containing ϕ'' cancel, and

$$S'x/a+H'y/a=(S'B+S'C''+H'B'-H'A'')\sin g\lambda''$$
$$-(S'B'-S'A''-H'B-H'C'')\cos g\lambda''+S'C_x-H'C_y$$

Since g is generally small, sin $g\lambda''$ will vary almost proportionately with λ'' for several orbits past the origin. It will not become insensitive to change of λ'' until $g\ \lambda''$ approaches 90°. Therefore, it is appropriate (and satisfactory from subsequent calculator tests) to separate out the terms containing sin $g\lambda''$ for a calculation of λ'' with few iterations. Conversely, cos $g\lambda''$ changes very little in the first few orbits and may be left where it is. Transposing and dividing by the coefficient of sin $g\lambda''$,

$$\sin g\lambda''=[(S'B'-H'B)\cos g\lambda''+S'(x/a-C_x-A''\cos g\lambda'') \\ +H'(y/a+C_y-C''\cos g\lambda'')]/[S'(B+C'')+H'(B'-A'')] \quad (449)$$

This iterates satisfactorily if (y/a) is chosen as the first trial λ'' in (433), (434), and the right side of (449), then sin $g\lambda''$ is found, from which a new λ'' may be determined for resubstitution, until the new λ'' varies from the preceding by less than a chosen convergence value. While the iterations are few, the calculation is lengthy because (433) and (434) have to be evaluated for each n at each new trial value of λ''.

After λ'' is determined, the final values of λ'', H, S, H', A'', and C'' may be substituted in a transposition of equation (435) to find ϕ'':

$$\ln \tan(\pi/4+\phi''/2)=[(H^2+S^2)^{\frac{1}{2}}/H'][C_x-x/a+(B+C'')\sin g\lambda'' \\ -(B'-A'')\cos g\,\lambda''] \quad (450)$$

Here the equation for x was transposed rather than the equation for y (used for the circular orbits) because the axes (figure 14) for the elliptical orbit are almost perpendicular for the first few orbits to those for the circular orbits, and x is therefore more sensitive to changes in ϕ'' than y is.

For inverse equations in the range $g\lambda''=90°$ and greater, equation (449) must be inverted to solve for cos $g\lambda''$, but correcting for quadrant of the circular central "axis." This has not been done in view of the unlikely need for the equations.

Inversion of equations (440) and (441) follows a similar pattern, more like the earlier inversions for the circular orbit. Again multiplying (440) by S', (441) by H', and adding, but now merely isolating the linear functions of λ'',

$$\lambda''=[S'(x/a-\Sigma C_n'-A'')+H'(y/a-\Sigma C_n-C'')]/(S'b+H'b') \quad (451)$$

in which λ'' is again found by successive substitution, beginning with a trial ($\lambda''=y/a$). Inverting (440) to find ϕ'' from the final λ'' and related parameters,

$$\ln \tan(\pi/4+\phi''/2)=[(H^2+S^2)^{\frac{1}{2}}/H'](b\lambda''+\Sigma C_n'+A''-x/a) \quad (452)$$

Similarly, equations (444) and (445), for a circular orbit, lead to the following, except that ϕ'' should be found from inversion of (445) because of rotation of the axes:

$$\lambda''=[(bS'+b'H')(x'/a-\Sigma A_n''\sin n\lambda'')+(b'S'-bH')(y'/a \\ -\Sigma C_n''\sin n\lambda'')]/B''(bS'+b'H') \quad (453)$$

solved by the same type of iteration, but starting with a trial ($\lambda''=x'/a$), and

$$\ln \tan (\pi/4+\phi''/2)=[B''(H^2+S^2)^{\frac{1}{2}}/(bS'+b'H')]\,(y'/a - \Sigma C_n'' \sin n\lambda'') \quad (454)$$

SUMMARY OF EQUATIONS

Since the formulas are scattered throughout the last few pages, they are assembled here in the order of calculation. First, for constants applying to the entire orbit, the following are calculated just once for all the integration:

$$Q=e^2 \sin^2 i/(1-e^2) \quad (199)$$

$$T=e^2 \sin^2 i\,(2-e^2)/(1-e^2)^2 \quad (198)$$

$$W=[(1-e^2 \cos^2 i)/(1-e^2)]^2-1 \quad (200)$$

$$K=1-e^2 \quad (397)$$

The following are calculated for each 3° of λ'', from 0° to 360°, to determine R_i, H, and S for numerical integration in subsequent formulas:

$$\phi_g=\arcsin (\sin i \sin \lambda'') \quad (196)$$

$$\phi_o=\arctan (\tan \phi_g/K) \quad (195)$$

$$C=1-e^2 \sin^2 \phi_o \quad (396)$$

$$D=[1-e^2 (2-e^2) \cos^2 \phi_g]/K \quad (401)$$

$$D'=2De^2 (2-e^2) \sin \phi_g \cos \phi_g/K \quad (405)$$

$$E'=2\arctan \{\tan \tfrac{1}{2}(\lambda''-\omega)\,[(1-e')/(1+e')]^{\frac{1}{2}}\} \quad (364)$$

$$L'=(1-e' \cos E')^2/(1-e'^2)^{\frac{3}{2}} \quad (363)$$

$$L''=2e' \sin E'\,(1-e' \cos E')^2/(1-e'^2) \quad (407)$$

$$G=D\,[\cos i/\cos^2 \phi_g-L'\,(P_2/P_1)] \cos \phi_g/\sin i \quad (403)$$

$$M=\{[\cos i/\cos^2 \phi_g-L'\,(P_2/P_1)]\,[D' \cos \phi_g \cos^2 \lambda'' +D^2 \sin \phi_g/\tan^2 i]+2D^2 \sin \phi_g \cos i \cos^2 \lambda''/\cos^2 \phi_g -L''D^2 \cos^2 \phi_g \cos \lambda''\,(P_2/P_1)/\sin i\}\,/\sin i \quad (404)$$

$$R_i=-\quad K[G \sin \phi_o \cos^2 \lambda''\,(3C-K)-MC \cos \phi_o] +G^3C^2 \sin \phi_o \cos^2 \phi_o\,/(G^2C \cos^2 \phi_o+K^2 \cos^2 \lambda''/C)^{\frac{3}{2}} \quad (398)$$

$$H=[(1+Q \sin^2 \lambda'')/(1+W \sin^2 \lambda'')]^{\frac{1}{2}}\,[(1+W \sin^2 \lambda'')/ (1+Q \sin^2 \lambda'')^2-(P_2/P_1)\,L' \cos i] \quad (362)$$

$$S=(P_2/P_1)L' \sin i \cos \lambda''\,[(1+T \sin^2 \lambda'')/ (1+W \sin^2 \lambda'')\,(1+Q \sin^2 \lambda'')]^{\frac{1}{2}} \quad (361)$$

Numerical integrations:

$$g=(1/2\pi)\int_0^{2\pi} R_i\,(H^2+S^2)^{\frac{1}{2}}\,d\lambda'' \tag{409}$$

$$d_n=(1/n\pi)\int_0^{2\pi} R_i\,(H^2+S^2)^{\frac{1}{2}}\cos n\,\lambda'' d\lambda'' \tag{410}$$

$$f_n=(1/n\pi)\int_0^{2\pi} R_i\,(H^2+S^2)^{\frac{1}{2}}\sin n\,\lambda'' d\lambda'' \tag{411}$$

$$\gamma=\Sigma d_n \sin n\,\lambda''+\Sigma f_n\,(1-\cos n\,\lambda'') \tag{413}$$

$$b=(1/2\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}}\sin\gamma\,d\lambda'' \tag{417}$$

$$b'=(1/2\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}}\cos\gamma\,d\lambda'' \tag{418}$$

$$h_n=(1/\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}}\sin\gamma\cos n\,\lambda'' d\lambda'' \tag{419}$$

$$h'_n=(1/\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}}\sin\gamma\sin n\,\lambda'' d\lambda'' \tag{420}$$

$$c'_n=(1/\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}}\cos\gamma\cos n\,\lambda'' d\lambda'' \tag{421}$$

$$c_n=(1/\pi)\int_0^{2\pi} (H^2+S^2)^{\frac{1}{2}}\cos\gamma\sin n\,\lambda'' d\lambda'' \tag{422}$$

Convenient terms:

$$A_n=(nh_n+gc_n)/(n^2-g^2) \tag{426}$$

$$A'_n=(nc'_n-gh'_n)/(n^2-g^2) \tag{427}$$

$$C_n=(nc_n+gh_n)/(n^2-g^2) \tag{428}$$

$$C'_n=(nh'_n-gc'_n)/(n^2-g^2) \tag{429}$$

If g is not zero (orbit elliptical and perigee not at a node):

$$B=b/g \tag{424}$$

$$B'=b'/g \tag{425}$$

$$C_x=B'+\Sigma C'_n \tag{430}$$

$$C_y=B-\Sigma C_n \tag{437}$$

If $g=0$ (perigee at a node or circular orbit), b, b', $\Sigma C'_n$, and ΣC_n are used separately as constants. See equations (444) - (448) for further reduction to circular orbits.

All the above produce constants for the projection. For an individual point, to find ϕ'' and λ'' in terms of ϕ and λ:

$$E' = 2 \arctan \tan \tfrac{1}{2}(\lambda'' - \omega)\, [(1-e')/(1+e')]^{\frac{1}{2}} \tag{364}$$

$$L = E' - e' \sin E' \tag{334}$$

$$\lambda_t = \lambda + (P_2/P_1)\,(L+\omega) \tag{332}$$

$$\lambda'' = \arctan\,[\tan \lambda_t \cos i + K \tan \phi \sin i / \cos \lambda_t] \tag{272}$$

The last four equations above are iterated together, following type (2) under "Iteration procedures."

$$\phi'' = \arcsin\{(K \sin \phi \cos i - \cos \phi \sin i \sin \lambda_t)/ (1-e^2 \sin^2 \phi)^{\frac{1}{2}}\} \tag{455}$$

This equation is like (274), but F is omitted as an unnecessary complication, not affecting accuracy significantly. To find x and y in terms of ϕ'' and λ'',

$$\theta = g\lambda'' + \Sigma d_n \sin n\lambda'' + \Sigma f_n\,(1-\cos n\lambda'') \tag{408}$$

$$A'' = \Sigma(A_n \sin n\lambda'' - C'_n \cos n\lambda'') \tag{433}$$

$$C'' = \Sigma(A'_n \sin n\lambda'' - C_n \cos n\lambda'') \tag{434}$$

$$L' = (1-e' \cos E')^2/(1-e'^2)^{\frac{1}{2}} \tag{363}$$

$$H = [(1+Q \sin^2 \lambda'')/(1+W \sin^2 \lambda'')]^{\frac{1}{2}}\,[(1+W \sin^2 \lambda'')/ (1+Q \sin^2 \lambda'')^2 - (P_2/P_1) L' \cos i] \tag{362}$$

$$S = (P_2/P_1) L' \sin i \cos \lambda''\,[(1+T \sin^2 \lambda'')/ (1+W \sin^2 \lambda'')\,(1+Q \sin^2 \lambda'')]^{\frac{1}{2}} \tag{361}$$

$$H' = H \cos \theta - S \sin \theta \tag{432}$$

$$S' = H \sin \theta + S \cos \theta \tag{438}$$

If g is not zero,

$$x/a = (B+C'') \sin g\lambda'' - (B'-A'') \cos g\lambda'' + C_x - [H'/(H^2+S^2)^{\frac{1}{2}}] \ln \tan\,(\pi/4 + \phi''/2) \tag{435}$$

$$y/a = (B'-A'') \sin g\lambda'' + (B+C'') \cos g\lambda'' - C_y + [S'/(H^2+S^2)^{\frac{1}{2}}] \ln \tan\,(\pi/4 + \phi''/2) \tag{436}$$

If $g=0$,

$$x/a = b\lambda'' + \Sigma C'_n + A'' - [H'/(H^2+S^2)^{\frac{1}{2}}] \ln \tan\,(\pi/4 + \phi''/2) \tag{440}$$

$$y/a = b'\lambda'' + \Sigma C_n + C'' + [S'/(H^2+S^2)^{\frac{1}{2}}] \ln \tan\,(\pi/4 + \phi''/2) \tag{441}$$

For the inverse equations, to find ϕ'' and λ'' in terms of x and y: If g is not zero, limited to several orbits from the origin,

$$\sin g\lambda'' = [(S'B' - H'B) \cos g\lambda'' + S'\,(x/a - C_x - A'' \cos g\lambda'') + H'\,(y/a + C_y - C'' \cos g\lambda'')]/[S'(B+C'') + H'\,(B'-A'')] \tag{449}$$

using iteration in which (y/a) is the first trial λ'', finding H', S', A'', and C'' for each trial λ'' from the formulas just above.

$$\ln \tan (\pi/4+\phi''/2)=[(H^2+S^2)^{\frac{1}{2}}/H'][C_x-x/a+(B+C'')\sin g\lambda'' -(B'-A'')\cos g\lambda''] \quad (450)$$

If $g=0$, without limit,

$$\lambda''=[S'(x/a-\Sigma C'_n-A'')+H'(y/a-\Sigma C_n-C'')]/(S'b+H'b') \quad (451)$$

iterated like (449) above.

$$\ln \tan (\pi/4+\phi''/2)=[(H^2+S^2)^{\frac{1}{2}}/H'](b\lambda''+\Sigma C'_n+A''-x/a) \quad (452)$$

For ϕ and λ in terms of ϕ'' and λ'' (ignoring F in (269)):

$$E'=2 \arctan\{\tan \tfrac{1}{2}(\lambda''-\omega)[(1-e')/(1+e')]^{\frac{1}{2}}\} \quad (364)$$

$$L=E'-e' \sin E' \quad (334)$$

$$U=e^2 \cos^2 i/K \quad (219)$$

$$\tan \lambda_t=\{[1-\sin^2 \phi''/(1-e^2)] \tan \lambda'' \cos i -(\sin \phi'' \sin i/\cos \lambda'')[(1+Q \sin^2 \lambda'')(1-\sin^2 \phi'') -U \sin^2 \phi'']^{\frac{1}{2}}\}/[1-\sin^2 \phi''(1+U)] \quad (269)$$

If $\cos \lambda''$ is not zero,

$$\tan \phi=(\tan \lambda'' \cos \lambda_t-\cos i \sin \lambda_t)/K \sin i \quad (271)$$

If $\cos \lambda''$ is zero,

$$A=K^2 \cos^2 i-\sin^2 i+e^2 \sin^2 \phi'' \quad (246a)$$

$$A'=K \sin i \cos i \quad (246c)$$

$$\sin \phi=\pm\{-A(\cos^2 \phi''-\cos^2 i)+2A'[A' \pm \sin \phi''[K(\cos^2 \phi''-e^2 \cos^2 i)]^{\frac{1}{2}}]\}^{\frac{1}{2}}/(A^2+4A'^2)^{\frac{1}{2}} \quad (246d)$$

taking the sign of $\sin \lambda''$ in both places.

The relationship between ϕ'', λ''. ϕ', and λ', and the formulas for ϕ_0 and λ_0 are identical to those described in the previous section for a slightly elliptical orbit.

To evaluate scale error, equations may be differentiated as they were for the ellipsoid (equations (302) through (327)), but it is simpler to evaluate scale factors as was done for the published paper (Snyder, 1978b). To do this, a quadrilateral 0.01° on a side is assumed: two sides 0.01° of ϕ'' and two sides 0.01° of λ'', located at any point for which the scale is to be determined. For example, the quadrilateral may extend from a λ'' of 45° to a λ'' of 45.01°, and from a ϕ'' of 1° to a ϕ'' of 1.01°. For the scale factor at constant λ'', values of x, y, ϕ and λ are calculated for (λ'', ϕ'')=(45°, 1°) and (45°, 1.01°). The numbers are inserted into equation (306), using the differences in place of the Δ's, and the arithmetical average of the two ϕ's for ϕ.

Likewise, for the scale factor at constant ϕ'', values of x, y, ϕ and λ are calculated for $(\lambda'', \phi'')=(45°, 1°)$ and $(45.01°, 1°)$, and inserted into (306). This does not provide ω, the maximum angular deformation, but the diagonal scale factors may also be evaluated for $(\lambda'', \phi'')=(45°, 1°)$ and $(45.01°, 1.01°)$, as well as for $(\lambda'', \phi'')=(45.01°, 1°)$ and $(45°, 1.01°)$, to give distortion information which the sides of the quadrilateral do not. This procedure may be followed for elliptical or circular orbits, and for the ellipsoid or sphere, and was used to verify the above formulas for the elliptical orbit.

REFERENCES CITED

Colvocoresses, A.P., 1974, Space Oblique Mercator: Photogrammetric Engineering, v. 40, no. 8, p. 921–926.

———1975, Evaluation of the cartographic application of ERTS-1 imagery: American Cartographer, v. 2, no. 1, p. 5–18.

Hotine, Martin, 1946, The orthomorphic projection of the spheroid: Empire Survey Review, v. VIII, no. 62, p. 300–311.

———1947, The orthomorphic projection of the spheroid: Empire Survey Review, v. IX, no. 63, p. 25–35; v. IX, no. 64, p. 52–70; v. IX, no. 65, 112–123; v. IX, no. 66, p. 157–166.

Junkins, J. L., and Turner, J. D., 1977, Formulation of a Space Oblique Mercator map projection: Final rept. UVA/525023/ESS77/105, Submitted to Computer Science Laboratory, U.S. Army Engineer Topographic Laboratories.

Maling, D. H., 1973, Coordinate systems and map projections: London, George Philip and Son, Ltd., p. 58–67. Specific formulas are (4.21), corrected for sign, (4.33), corrected for function, (4.51), and (4.59).

Moulton, F. R., 1914, An introduction to celestial mechanics, 2d ed.: New York, MacMillan, p. 159, 165.

Short, N. M., Lowman, P. D., Jr., Freden, S. C., and Finch, W. A., Jr., 1976, Mission to Earth: Landsat views the world: National Aeronautics and Space Administration, p. 437.

Snyder, J. P., 1978a, Equidistant conic map projections: Assoc. Am. Geog. Annals, v. 68, no. 3, p. 373–383.

——— 1978b, The Space Oblique Mercator projection: Photogrammetric Engineering and Remote Sensing, v. 44, no. 5, p. 585–596.

Thomas, P. D., 1952, Conformal projections in geodesy and cartography: U.S. Coast and Geodetic Surv. Spec. Pub. 251.

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