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# NATIONAL BUREAU OF STANDARDS REPORT 

A NEW METHOD OF ANALYZING EXTREME-VALUE DATA
by
Julius Lieblein

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# NATIONAL BUREAU OF STANDARDS REPORT NBS PROJECT 

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A NEW METHOD OF ANALYZING EXTREME-VALUE DATA
by
Julius Lieblein
Statistical Engineering Laboratory

Research supported by the National Advisory Committee for Aeronautics
NACA Project W-5367

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## FOREWORD

This report devolops various aspects of a new method of treating extreme $-v a l u e ~ d a t a . ~ T h i s ~ m e t h o d, ~$ based on order statistics, has a number of advantages over existing procedures, and will be useful in the efficient handling of small or large sets of extreme gust-load data and extreme data in many other fields as well.

The present report completes work under a research project aimed at the improved application of the theory of extreme values to the analysis of gust loads of airplanes. This project, supported by the National Advisory Committee for Aeronautics, was carried out by Julius Lieblein under the general supervision of Dr. Churchill Eisenhart, Chief of the Statistical Engineering Laboratory. The Statistical Engineering Laboratory is Section 11.3 of the National Applied Mathematics Laboratories (Division 1l, National Bureau of Standards), and is concerned with the development and application of modern statistical methods in the physical sciences and engineering.
J. H. Curtiss Chief, National Applied Mathematics Laboratories

## A. V. Astin

Director
National Bureau of Standards

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A NEW METHOD OF ANALYZING EXTREME -VALUE DATA by

Julius Lieblein

1. SUMMARY

A new method is presented and proposed for analyzing extreme-value data which may arise in a wide variety of applications.

Classical applications of statistical methods, which usually concern average values, are inadequate When the quantity of interest is the largest for smallest) in a set of magnitudes. This is the sitm uation in a number of fields, eogos gust loads of an airplane in flight, highest temperatures or lowo est pressures in meteorology, floods and droughts in hydrology, areaking strengths in materials testo ing, breakdown voltage of capacitors, and human Ife spans, in all of which applications of methods for dealing with extremes have already been made。

Discussion of the proposed method is preceded by the necessary statistical theory (Section 4) which also furnishes a basis for evaluating the new method In relation to existing ones. The techniques deso cribed provide a simple means for estimating the

* Preparation of this report was sponsored by the National Advisory Comittee for Aeronauticso

$$
n
$$

necessary parameters, making predictions from the fitted curve, estimating the reliability, and evalo uating the efficiency of the method in relation to cther methods. Moreover, these quantities are all produced by a single set of computations involving just two workshee'ts. This background material is not essential to an application of the method, and may be omitted if desired. The method itself is sumarized for practical convenience and illustrated step-by-step in Section 5, and compared with present procedures in Section 6. The latter section also discusses the advantages of the proposed method, chier among which are:*
(i) For the first time there is available an unbiased estimator of known efficiency:
(2) The proposed estimator appears to be more efficient than a simplified form of the Gumbel estimator in many practical cases, namely, for samples

[^0]of about 20 or more, and $P=.95$ or more. The improvement in efficiency increases with increasing $P$ or increasing sample size。 When compared with the original Gumbel estimator, the proposed one is up to twice as efficient.
(3) The confidence intervals have a more valid theoretical basis and are in many cases narrower than the ones in the Gumbel method。

Inciuded in the report are several appendices giving mathematical developments not given in the text.

## 2．IN TRODUC TION

The statistical theory of extreme values has been found to have wide applicability in many diverse fields，for example，meteprological extremes，floods， drought＇s，breaking strength of textiles and other types of materials，span of human life，gust loads experienced by an airplane in flight，breakdown voltage of capacitors．

The two existing methods of analyzing extreme－ value data have several limitations，discussed in the body of this report．One of these methods is known as the method of maximum likelihood and has been described by B．F。Kimball（references 9，l0）。 The other，the method of moments，has been developed by E。J。Gumbel（references 2，3，6），and its application to gustoload problems has been discussed．in detail in a previous NACA Technical Note by Harry Press （reference 18）。

The present report gives a new method for dealing with the problem of analyzing extreme measurements, treated in the above Technical Note, which has certain advantages over the existing methods. The method of application is presented in detail, together with the necessary worksheets and other data, and the new method is compared with the method of moments previously in use. For definiteness, the discussion is at time presented in terms of application to gust loads, but the methods are also applicable to other fields where extreme values occur.
-

Note: By "Samples" are meant independent random samples from the extremes value distribution

Equation no., ets.
(4.17): entering into weights Table I of orderostatistics estimator for sample of $n$

| $\operatorname{cov}(\overline{\mathrm{y}}, \mathrm{s})$ or | Covariance of mean |
| :--- | :--- |
| $\sigma(\overline{\mathrm{y}}, s)$ | and standard devis |
|  | ation in samples of |
|  | n from reduced distri- |
|  | bution |

E(...) $\quad \begin{array}{ll}\text { Mathematical expectas Various, } \\ & \text { tion (or mean value) }\end{array}$ of ( 0,0 )

Efficiency of ordero
$(4.24):(4.19) ;$ statistics estimator Table III. for subgroups of $m$ observations, or for samples of $n$
$E(s)$
Mean value of stando ard deviation in samples of $n$ from reduced distribution

Page 109
(Appendix E)
(Appendix E)

| $E_{m}{ }^{8} E_{n}$ | Efficiency of ordero statistics estimator for subgroups of $m$ observations, or for samples of $n$ | $\begin{aligned} & (4.24) ;(4.19) \text {; } \\ & \text { Table III: } \\ & \text { Part B } \end{aligned}$ |
| :---: | :---: | :---: |
| $E(s)$ | Mean value of stando ard deviation in samples of $n$ from reduced distribution | Page 109 <br> (Appendix E) |

$$
\begin{equation*}
F(x)=F(x ; u, \beta)=\text { Probability (chmu。 } \tag{4.1}
\end{equation*}
$$

$\exp \left[-e^{-}(x-u) / \beta\right]$ lative function (codofo) of extreme-value distribution with two parame eters

$$
f(x)=\frac{d F(x)}{d \bar{x}}
$$

Density (or frequency) Page 11 ; function of extremen. Figure 1 value distribution $F(x)$

| k | Number of equal sub． groups of size m contained in sample of $n$ ． | Pages 33， 36 |
| :---: | :---: | :---: |
| m | Size of one of the $k$ equal subgroups con－ tained in sample of $n$ | Pages 33， 36 |
| $\mathrm{m}^{\text {P }}$ | Size of remainder sube group in sample of $n$ that is left after k equal subgroups of $m$ are taken，i。e。： $\mathrm{n}=\mathrm{km}+\mathrm{m}^{0}$ | Page 36 |
| MSE（．．．） | Mean square error of （o．．）：equals variance plus square of bias | （4．13） |
| n | Sample size（ N de－ notes sample size in Gumbel method）． | Page 23 |
| P | Probability level associated with a predicted value | Page 13 |
| $Q_{0}$ | Numerator in Cramér． Rao lower bound； $Q_{L B}=Q_{0} / n$ | $\begin{aligned} & (4.24): \\ & \text { Table III, } \\ & \text { Part \& } \end{aligned}$ |
| $Q_{\text {LB }}$ | Cramér $\quad$ Rao lower bound to variance of unblased estimator of parameter $\xi_{P}$（see $Q_{0}$ ） | Page 26 |
| $Q_{m}, Q_{n}$ | Variance of order statistics＂sube estimator＂for sub－ groups of $m$ ，or of estimator for sample of $n$ 。 | $\begin{aligned} & (4.23)^{\circ} \\ & (4.18)^{\circ} \end{aligned}$ |

.

| r | Rank of roth obser vation (counted from smalles't) in samples of $n$ when arranged in ascending order from smallest to largest observation | Page 54 |
| :---: | :---: | :---: |
| $\begin{gathered} R\left(T_{1}, T_{2}\right)= \\ \operatorname{MSE}\left(T_{2}\right) \end{gathered}$ | Relative efficiency of estimators $\mathrm{T}_{1}$ to $\mathrm{T}_{2}$ (greater than unity | (E.7) |
| $\operatorname{MSE}\left(\mathrm{T}_{1}\right)$ | when $T_{1}$ is more efficient) |  |
| $s=\sqrt{\frac{1}{n} \sum_{i=1}^{n}\left(y_{i}-\bar{y}\right)^{2}}$ | Standard deviation (s.d.) of sample of n from reduced dis. tribution | $\begin{aligned} & \text { Page } 109 \\ & \text { (Appendix E) } \end{aligned}$ |
| $s_{x}=\sqrt{\frac{1}{n} \sum_{i=1}^{n}\left(x_{i}-\bar{x}\right)^{2}}$ | S.d. of sample of $n$ original distribution | (4.11) |
| $T_{1}, i=1,2, \ldots \ldots k$ | Order statistics "sub-estimator" for ioth of the $k$ equal. size subgroups in samples of $n$ | (4.21) |
| $\bar{T}=\frac{l}{K} \sum_{i=1}^{k} T_{i}$ | Average of the subo estimators for the $k$ equal-size subgroups | (4.22) |
| T ${ }^{8}$ | Sub-estimator for remainder subgroup (see mi) | (4.25) |
| $t, t^{8}$ | Weights for $\bar{T}$ and $T^{3}$ in grand estimator for sample: $\xi_{P}=t \bar{T}+t^{8} T^{8}$ | $\begin{aligned} & \binom{4.26)}{(4.27)} \end{aligned}$ |
| $\hat{u}=x-\frac{\bar{y}_{n}}{\sigma_{n}} s_{x}$ | original <br> Gumbel ${ }^{1} \mathrm{~s} /$ estimator of mode $u$ for sample of $n$ | (E.1) |

- 

| $\hat{u}^{\prime}=\bar{x}-\sqrt{\frac{6}{\pi}} \gamma s_{x}$ | Simplified expres'sion used to represent Gumbel's estimator of mode u | (E.2) |
| :---: | :---: | :---: |
| u | Mode or location parameter of extremevalue distribution | Page 12 |
| X | ```Random variable ("unreduced") having extreme-value distribution F(x)``` | $\begin{aligned} & (4.1), \\ & (4.5) \end{aligned}$ |
| $x_{1}, x_{2}, \ldots, x_{n}$ | The $n$ order $-s t a t i s t i c s$ in sample of $\mathrm{n}_{\text {, }}$ i.e., the observations ranked in ascending order | Page 23 |
| $\begin{gathered} x_{\lambda n}, x_{\mu n}{ }^{2} x_{\nu n}{ }^{9} 0 \lambda<\mu<\nu<1 \end{gathered}$ | Three selected order* statistics in Mosteller me thod for very large samples of $n$ | Page 86 <br> (Appendix C) |
| $\overline{\mathrm{x}}$ | Sample mean in sample from original ("uno reduced") distribution | Page 20 |
| \#\% $\mathrm{y}_{\mathrm{P}}$ | Reduced variate | Page 14 |
| $\beta$ | Scale parameter of extreme + value dis tribution $F(x)$ | Page 12 |
| $\hat{\beta}=\frac{s_{x}}{\sigma_{n}}$ | original <br> Gumbelis/estimator of $\beta$ for sample of $n$ | (E.1) |
| $\hat{\beta}^{\prime}=\sqrt{\frac{\sqrt{6}}{\pi}} s_{x}$ | Simplified expression used ta represent Gumbel's estimator of $\beta$ | (E.2) |
| $\begin{array}{r} \gamma=0.57721, \\ 56649 \end{array}$ | Euler ${ }^{\text {s }}$ constant | (4.8) |


| $\Delta_{x, n}=1.141 \beta$ | Halfowidth of $68 \%$ confidence interval in Gumbel method | ( D 。1) |
| :---: | :---: | :---: |
| $\Delta^{8}=1.141 B_{P} \beta$ | Halfowidth of $68 \%$ confidence interval when modified by' probability factor | (D.11) |
| $\Delta$ 。 | Half-width of $68 \%$ confidence interval in method of order statistics' | Table VIII |
| $\mu_{x}=E(x)$ | First moment or mathematical expectation of random variable $x$ | $(4.6)$ |
| $\mu_{2}=\sigma_{y}^{2}$ | Variance of reduced distribution | (4.9) |
| $\xi_{P}=u+\beta y_{P}$ | The 100Popercent point of the extreme value distribution $F(x)$ | Page 14 |
| $\hat{\xi}_{G}$ | Simplified expression used to represent Gumbel estimator of $\xi_{P}$ | (E.3) |
| $\sigma^{2}(s)$ | Variance of standard deviation in samples of in from reduced diso tribution | Table IX |
| $\sigma_{x}^{2}, \sigma_{y}^{2}$ | Population variance of x ! y | (4.7) |
| $\Phi(x)=\frac{r}{n+1}$ | Plotting position of roth observation ranked from smallest | Page 54 |
| $\Phi(y)=\exp \left(-e^{-y}\right)$ | Codof. of reduced extreme-value distribution | (D.4): <br> Page 112 <br> (Appendix |

## 4. STATISTICAL THEORY

### 4.1 Extreme-value distribution and meaning of parameters.

The method of analysis presented herein is based upon the assumption that the observed maxima to be analyzed are independent observations from a statistical distribution of the form

$$
\begin{equation*}
F(x)=F(x ; u, \beta)=\exp \left(-e^{-(x-u) / \beta}\right) \tag{4.1}
\end{equation*}
$$

This is the cumulative (or ogive) form of the diso tribution, which expresses the chance that an observed extreme value (gust load, for example) will not exceed $x$ in value. The more familiar concept of frequency or density function, $f(x)=F^{\prime}(x)$, for this distribution may be obtained by differentiation but is rather cumbersome (see Appendix A) and is not needed for present purposes. The general shape of the density function $f(x)$ is shown in Figure 1。 The meaning of the various quantities indicated is ex. plained below. A more detailed graph for the case where the parameters are $u=0, \beta=1$ (the "reduced") extreme-value distribution) is plotted in Figure 2. The distribution (4.1) has been studied exten sively by Gumbel (among others) (references $2,3,6$ ) and
is known as the asymptotic distribution of largest values. We shall refer to it briefly as the extremee Value distribution. The significance of the term "asymptotic" is as follows*。 If the underlying dis= tribution of all (not merely the largest) gust loads (eogo, effective gust velocity, normal acceleration) is considered, then the largest values in repeated large samples from this distribution have a distrio bution of their own which, as the sample size becomes larger and larger, approaches closer and closer (in a certain sense) to a limiting distribution. This limiting distribution is, according; to evidence presented in reference 18 , of the form ( 401 ), with $1 / \beta$ replacing the parameter $\alpha$ used in the reference. The parameters of the extreme $\infty$ value distribuo tion are depicted in Figure 1 . The quantity $u$ is the mode or highest point of the (frequency) diso tribution. The quantity $\beta$ is a scale parameter. analogous to the standard deviation $\sigma$ in the case of the normal distribution. In facts $\beta$ equals $\sqrt{6} / \pi$ (about 3/4) times the standard deviation of the extreme-value distribution.

Although the two parameters $u_{9} \beta$ completely specify the distribution, it is désirable
to introduce another quantity $=u+\beta y$ which is a linear combination of the parameters $u$ and $\beta$
（and therefore，since we shall assign known values to $y$ ，itself a parameter）${ }_{y}^{\text {F．}}$ and makes it possible to estimate $u$ and $\beta$ simultaneously，rather than in terms of two separate problems．Thus if we can estimate as $a+b y$ with $a, b$ known，then we can read off at once the values $u=a, \beta=b$ 。 The parameter $\xi$ has another highly important meaning。 In Figure 1 the area $P$ under the distribu－ tion to the left of the ordinate erected at $\xi$ reprea sents the probability that a value larger than $\xi$ will not occur．If $\mathfrak{G}$ is very large，then $P$ very nearly equals the whole area，unity，which means an obsere vation is almost certain not to exceed $\xi$ ；in other words，a larger value of $\mathcal{F}$ will occur only very rarely． Thus if $P=09$ ，then the corresponding value of has a chance of only ol of being exceeded．To denote this dependence of $\xi$ upon the probability

[^1]$P$ we use a subscript：$\xi_{P}^{\circ}$ This parameter is called
a percentage point or the l00Papercent point of the （extreme－value distribution）。 If we can estimate $\xi_{P}$ for different probability levels such $P=.90$ ． 95 。 －99，etco，then these values are precisely the preo dictions we desire for，say，gust－load accelerations that will be exceeded（on the average）only 10，5， I，etco，respectively，times in 100 。

The explicit relationship between $\} \quad p$ and $P$ can be determined by means of formula（4．1）for the extremeqvalue distribution．If $x$ is put equal to $\xi P^{\prime}$ then $P$ ，the probability of not exceeding this value，is simply $F\left(\xi_{p}\right)$ 。 Thus

$$
P=F\left(\xi_{P}\right)=\exp \left(-e^{-\left(\xi_{P}-u\right) / \beta}\right)=\exp \left(-e^{-y}\right),(4.2)
$$

since $\xi_{P}=u+\beta y$ ．Hence，for a given（usually large）probability $P_{s}$ the corresponding $\xi_{P}$ is obtained by finding y from the relation（4．2），and then writo ing

$$
\xi_{P}=u+\beta y_{P}
$$

where the subscript $P$ has been added to $y$ to denote dependence on.P. Comparison of the right members of (4.1) and (4.2) shows that the quantity $y$ bears the following simple relation to the corresponding variable $x$ in (4.1):

$$
\begin{equation*}
y=\frac{x-u}{\beta}, \tag{4.4}
\end{equation*}
$$

or

$$
\begin{equation*}
x=u+\beta y \quad . \tag{4.5}
\end{equation*}
$$

Also, if in (4.1) we set $u=0, \beta=1$, then $x$ has the same distribution as given by the righto hand side of (4.2). In other words, y as defined by (4.4) or (4.5) has an extreme-value distribution whose parameters have the extremely simple values $u=0, \beta=1$. Thus $y$ is called the reduced variate ${ }^{*}$ and is perfectiy analogous to the standardized variate $t=(x-\mu) / \sigma$ of normal distribution theory. The distribution of $y$ in (4.2), called the reduced distribution, has been tabulated in Table 2 of reference 15, which also contains a table of the inverse function as well as a number of other tables

[^2]related to application of extreme-value theory.
From the above discussion it is evident that the solutions of both the problems of estimation and prediction are embodied in the one quantity $\xi_{P}=u+\beta y_{P}$. Estimation of this quantity will be the main objective of the remainder of this report.

### 4.2 Determination of method of estimation

To avoid confusion, we distinguish between a function of sample variables $x_{1}, x_{2}, \ldots . . x_{n}$ such as the sample mean $g\left(x_{1}, x_{2} \not \ldots x_{n}\right)=\bar{x}$ $=\left(x_{1}+x_{2}+\ldots,+x_{n}\right) / n$, and the numerical values $g_{0}=g\left(x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}\right)$ assumed by the function when the actual values of the observations $x_{i}=x_{i}^{0}$ are substituted into the function. If the function is used'to estimate a parameter, we shall call it an estimator of the parameter; the particular numerical value assumed in a given case shall be called an estimate.

In searching for estimators the first step
is to seek what are known as sufficient statistics.
A definition of this concept may be found in any
advanced text on statistical theory，for example reference $8, V o l$ ．II，p．81；but the feature of importo ance here is that given a set of joint sufficient statistics，that is，certain functions of the sample observations，it is often possible to deduce from theim an estimator with certain desirable properties，pro＝ vided that the number of such functions does not depend upon sample size。 If it turns out that the only set of sufficient statistics is the trivial set consisting of the $n$ functions $t_{i}\left(x_{1}, \ldots, x_{n}\right)=x_{i}$ ， $i=1, \ldots, n, i o e$ ，the $n$ sample observations theme selves，then obviously this furnishes no guide whato ever for constructing functions of the $x^{\prime}$＇s which are optimum estimator＂s．

Investigation reveals that，unfortunately， finte sufficient statistics do not exist for the two parameters of the extremervalue distribution．A prodf of this fact（which was conjectured，by Bo F。 Kimball，reference 9，p．299）has been discovered by I。Richard Savage of the Statistical Engineering Laboratory of the National Bureau of Standards，and is presented in Appendix A。
-

It may be noted that Kimball (ibid.) has studied a broader concept called "set of statistical estimation functions" whereby the estimators of the parameters are given, not by explicit formulas involving only the sample values, but implicitly as the solutions of a set of simultaneous equations, for example, the classical maximum-likelihood equations. Unfor. tunately, such estimators do not seem to lend themselves to the procedure referred to above for con structing optimum estimators, and there seems to be no analytical means of accurately evaluating the important characteristics of bias and efficiency, defined below, for such estimators in the case of finite samples. (Al though these estimators may be as ymptotically optimum, i.e., for $^{\text {infinitely large }}$ samples, this need not be the case for samples of finite size.)

A second method of approach to the problem of estimation is the classical one known as the method of moments. In the case of the extremeo value population this method is as follows.

The first two moments of the extreme-value population (4.1) are

$$
\begin{align*}
& \mu_{x}=E(x)=u+\beta E(y)  \tag{4.6}\\
& \sigma_{x}^{2}=E[x=E(x)]^{2}=\beta^{2} E[y \propto E(y)]^{2}=\beta^{2} \sigma_{y}^{2} \tag{4.7}
\end{align*}
$$

where $y$ has the reduced extreme-value distribution (4.2), E denotes mathematical expectation, and $\sigma^{2}$ is the variance, the second moment about the mean. Using the moments of the reduced distribution (sees eogos reference 8, Vol. Is page 221)。

$$
\begin{align*}
& E(y)=\mu_{1}^{\prime}=\gamma=.577216 \text { (Euleris constant) }(4.8) \\
& \sigma_{y}^{2}=\mu_{2}=\frac{\pi^{2}}{6}=1.644934 \tag{4.9}
\end{align*}
$$

we obtain

$$
\begin{equation*}
\mu_{x}=u^{\prime}+\gamma_{\beta}, \sigma_{x}=\frac{\pi}{\sqrt{6}} \beta \tag{4.10}
\end{equation*}
$$

relations which express the population moments in terms of the population parameters. Therefore, if we had good estimators of the population moments, we could readily find the parameters. This fact constitutes the essence of the method of moments. It consists in treating the sample as an adequate representation of the population, replacing the population moments in the expressions which relate
them to the parameters by the corresponding sample moments, eogos $\mu_{x} b y$ the sample mean $\bar{x}_{\rho}$ and $\sigma_{x} b y$ the sample standard deviation

$$
\begin{equation*}
s_{x}=\sqrt{\frac{1}{n} \sum_{i=1}^{n}\left(x_{i}-\bar{x}\right)^{2}} \tag{4.11}
\end{equation*}
$$

This gives $\bar{x}=u+\gamma_{,} s_{x}=(\pi / \sqrt{6}) \beta_{\text {, }}$ which yield the moment estimators of the parameters:

$$
\begin{align*}
& \text { for } \beta, \hat{\beta}=(\sqrt{6} / \pi) s_{x} ; \\
& \text { for } u, \hat{u}=\bar{x}-\gamma(\sqrt{6} / \pi) s_{x} \tag{4.12}
\end{align*}
$$

These are essentially the estimators which form the basis of Gumbel's method (reference 6, Lecture 3. eq. (3.29), with $\left.u_{n}=\hat{u}, 1 / \alpha_{n}=\hat{\beta}\right) *$. This method is justifled by the fact that under general conditions the estimator functions $\hat{u}=\hat{u}\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, $\hat{\beta}=\hat{\beta}\left(x_{1}, x_{2} \ldots 0, x_{n}\right)$ in (4.12) approach(in a certain sense) the values of the corresponding parameters u, $\beta$ as the sample size becomes infinite。

* The actual estimators used in the Gumbel method are slightly more complicated (reference 6。 Lecture 3, eq. (3.39), but the difference is not important at this point. (See Appendix E).
- 

This method has apparently given satisfactory results in practice。 It is，however，subject to an important limitation．In studying estimators it is highly desirable to know something about their probo ability distributionsooif not the exact density funco tions，then at least their means and variances．The mean value（mathematical expectation）of an estimator indicates whether on the average the estimates given by it are too high or too low relative to the actual values of the parameter estimated win other words， whether there is any bias in using the estimator． Similarly，the variance indicates how much the estimates scatter among themselves and is the basis for constructing a measure of efficiency which makes it possible to compare the performances of different estimators．A more useful concept for some purposes than variance is mean square error（abbreviated MSE） which measures how far the estimates deviate，on the average，not from their own mean，but from the quantity－ the parameteroowhich they are supposed to measure． There is a simple relationship between variance and MSE。 namely。

$$
\text { Mean square error }=\text { variance }+(\text { bias })^{2} \text {. (4.13) }
$$

Thus, for unbiased estimators, variance and MSE are identical, and for brevity we shall use the term "variance* in such cases. But it should be remembered that the concept in view is actualiy the MSE. This becomes especially important later when biased estio mators arediscussed (Appendix E), and variance and MSE are no longer identical.

If we try to determine the mean (or expected)
values of the estimators, $u, \beta$, in (4.12) we find that statistically these functions are quite complicated, leading to very difficult multiple integrals which apparently can be evaluated accurately only by largescale numerical integration." Thịs difficulty evidentiy persists if we are interested in the parameter $\xi_{p}=u+\beta y_{p}$ instead of $u$ or $\beta$ separately.

> 4.3 Ordermstatistics approach for small samples

Apparently the anly method of estimation which avoids this difficulty is the method of order statistics.

* Shorter methods of limited accuracy are possible and have been used in this report for comparisonpurposes. (See Section 6.1 below and Appendix E.)

If the values in a sample of $n$ observations are arranged in, say, increasing order of size, and de。 noted by $x_{1}, x_{2}, \ldots, x_{n} x_{1} \leqslant x_{2} \leqslant \ldots,<x_{n}$ then these values $\mathrm{x}_{\mathrm{i}}$ are called order statistics. The smallest is called the first order statistic; the middle one (if $n$ is odd), the median; the one which is one four th the way up from the bottom, the first quartile, etc. (If there are several equal ones then suitable modifications are made in the definitions). There is an extensive Iiterature on this subject. chief among which is the comprehensive survey in reference 20.

Order statistics provide rapid and practical methods of analyzing data. The range, $x_{n}-x_{1}$, is a very common illustration from quality control. It is simply the difference of two order statistics, the largest and smallest, and its properties have been extensively studied for samples from the normal distribution. The range has been found to yield estimates of the standard deviation of the population that of ten compare very favorably with the theoretically best obtainable. More general linear functions, $C_{1} x_{1}+C_{2} x_{2}+\ldots+C_{n} x_{n}$, which give weight to every

.

sample value, have also been studied (reference 19), and values of the coefficients have been found which make it possible to estimate very simply and remarkw ably well certain quantities which previously were obtained only by more complicated calculations.

We shall carry over and extend this procedure to the case of samples from the extremewalue distribution (4.1). The method will in many respects follow the general approach used in reference 19 for several other distributions. The aim is to determine the weights $w_{i}, i=1, \ldots . n_{\text {, for }}$ all the $n$ order statistics in a sample of size $n$ so that the linear estimator

$$
\begin{equation*}
L=\sum_{i=1}^{n} W_{i} X_{i} \tag{4.14}
\end{equation*}
$$

has the properties we desire, namely:
(1) The mathematical expectation equals the parameter to be estimated, i.e. the estimao tor is unbiased:

$$
\begin{equation*}
E(L)=\xi_{P} \tag{4.15}
\end{equation*}
$$

(2) The MSE, which in this case is the same as the variance, is as small as possible, consistent with (I):

$$
\begin{align*}
\operatorname{MSE}(L) & =\sigma^{2}(L)=E[L-E(L)]^{2} \\
& =a \text { mininum. } \tag{4.16}
\end{align*}
$$

An estimator $L$ which satisfies these two conditions will be denoted by $\xi_{p^{s}}$ a notation suggested by condition (1)。

Condition (2) is equivalent to saying that the estimator $\xi_{p}$ is as efficient as possible under the given conditions. This concept will be discussed below.

The mathematical formulation of this minimumo variance problem is developed in Appendix $B_{8}$ and the solutions (the weights) are shown in Table I for $\mathrm{n}=2$ to $\mathrm{n}=6$ 。 The case n greater than 6 is diso cussed in the next Section. For each given value of $n_{,} n$ weights $w_{1}, W_{2}, \ldots . W_{n}$ are determined that depend on the quantity $y_{p}$ that occurs in the parameter $\xi_{P}=u+\beta y_{P}$ to be estimated. The wise ach of the following form:

$$
w_{i}=a_{i}+b_{1} y_{P}, \quad 1=1,2, \ldots, n \quad(4017)
$$

Substituting these weights for given $n$ into (4016)
actuaily gives the minimum value, $Q_{n}$, that the variance can attain under the above conditions, and this value depends upon $\bar{Y}_{P}$ quadratically:
.

$$
\begin{equation*}
V_{\min }=Q_{n}=\left(A_{n} X_{P}^{2}+B_{n} X_{P}+C_{n}\right) \beta^{2} \tag{4.18}
\end{equation*}
$$

Table I gives the values $a_{i}$, $b_{i}$, $A_{i}$, $B_{n}$ 。 $C_{n}$, which have been found by exact computation methods as indicated in Appendix B and Table II there described. The quantities $V_{\text {min }}=Q_{n}$ are shown in Table III, Part A.

As sample size increases, the estimation is expected to improve and the variance to diminish. In order to have a convenient standard of comparia son, in the case of unbiased estimators, * we scale all variances by dividing into a theoretically specified variance $\left(Q_{L B}\right)$, known as the "Cramér-Rao lower bound" (reference $1_{3}$ pp. 480 , equation ( $32.3 .3 a$ ), ${ }^{* \%}$, which

F For biased estimators, see Appendix E.
\% This Cramér-Rao bound is given for the case where the distribution has only one parameter to be estimated. For the extreme-value distribution with the two parameters ( $\xi, \beta$ ) we can regard $\beta$ as a "nuisance parameter" and thus obtain a "Cramer-Rao bound" for ( $\xi$, the expression for which will involve $\beta$ (see first footnote to Table III). This procedure is based on the "method of nuisance parameters" discussed in reference ll. (See also footnote to page 13.)
-
is less than or at most equal to the variance of any (unbiased) estimator of the parameter in ques $=$ tion* . The result is then an absolute number between 0 and 1 which, when expressed as a percento age, is callea the efficiency of the estimator for samples of $n$ :

$$
\text { Efficiency }(L)=E_{n}(L)=\frac{Q_{L B}}{Q_{n}} \quad \text { (4.19) }
$$

The quantities $E_{n}$ which evidentiy depend upon $\mathrm{J}_{\mathrm{P}}$, and therefore upon $P_{\text {s }}$ are given for $n=2$ to 6 for selected values of the probability $P$ in Table III。 Part B. Part A contains the numerical values of the variances $Q_{n}$ and the lower bound $Q_{Q B}$ in terms of the parameter $\beta^{2}$. The expression for ${ }^{Q}$ LB has been implicitly given in reference 10, p. 113, and is indicated in the first footnote to Table III, Part A of this report.

* There may or may not exist estimators whose variances reach the lower limit $Q_{L B}$. If (as may happen) there exists a $Q^{\circ}=Q_{L B B}^{L B}$ such that the variance of every estimator is ${ }^{2} \sum^{B}$ (and of course $\Rightarrow Q_{T B}$, then $Q^{8}$ may be substituted for $Q_{L B}$ in the HBunerator of the above expression for efficiency (4.19) without the fraction exceeding I。 The investigation of the existence of $Q{ }^{0}$ is too com plex a matter for purposes of this report. However, the only effect of using lower bound $Q_{T B}$ which is too low is to understate the efficiency, so that the results are on the safe, conservative side.

Two points should be noted about the cholce of probability levels shown in Table III。 The value $P=.36788=1 / \theta$, which corresponds to $\mathrm{y}_{\mathrm{P}}=0$, is important because it gives the mode, one of the desired parameters of the distribution. This is evident from the fact that the parameter we are estimating is $\xi_{P}=u+\beta y_{P}=u$, the mode, for $y_{P}=0$. Similarly, the limiting value $P=1$ corresponds to the scale parameter $\beta$. This may be seen as follows. If $P$ approaches $l_{\text {, }}$ the $\xi_{P}$ and $y_{P}$ both become indeo finitely large, but their ratio $\xi_{P}^{\prime}=\xi_{P} / y_{P}=\left(u / y_{P}\right)+\beta$ may be considered to be a new parameter which approaches $\beta$ s since the mode $u$ remains fixed and finite (as does also $\beta$ ). Hence we may estimate $\beta$ by first estimating $\xi_{P}^{0}$ for arbitrary $P$ and then letting $P$ approach 1 。 Now from (4.14) and (4.17), the linear estimator $L=\hat{\xi}_{\mathrm{P}}$ is of the form

$$
\begin{equation*}
\hat{\xi}_{p}=f_{1}+y_{P} f_{2} \tag{4.20}
\end{equation*}
$$

where $f_{1}$ and $f_{2}$ are functions of the sample values which do not involve $J_{P}$. By the preceding remark. we can then estimate the parameter $\beta$ by writing down
the corresponding estimator of $\xi_{p}$,

$$
\hat{\xi}_{p}=\frac{2}{\bar{y}_{p}} \hat{\xi}_{p}=\frac{f_{1}}{\bar{y}_{p}}+f_{2}
$$

and letting $P$ approach $I$, obtaining

$$
\hat{\xi}_{p}=\tilde{f}_{2}
$$

as the corresponding estimator of $\beta$ 。
In other words, an estimator $\hat{\xi}_{\beta}$ of $\beta$ may be obtained by simply taking the coefficient of $y_{P}$ in $\hat{\xi}_{\mathrm{p}}$ when written in the form (4020). Similarly the variance of $\hat{⿷}_{\beta}$ is the coefficient of $y_{P}^{2}$ in the variance of $\hat{\boldsymbol{\xi}}_{P^{\circ}}$. This may readily be seen as follows. From (4.20), we have

$$
\begin{aligned}
\sigma^{2}\left(\hat{\xi}_{p}\right) & =\sigma^{2}\left(f_{1}\right)+2 y_{P} \operatorname{cov}\left(f_{1}, f_{2}\right) \\
& +y_{P}^{2} \sigma^{2}\left(f_{2}\right)=A+B y_{P}+C y_{P}^{2}
\end{aligned}
$$

where $A_{9} B, C$ are quantities which do not involve $X_{P}$ (though they may involve $\beta$, in general): thus, as $P$ approaches $l$ and $y_{P}$ increases without limit,

$$
\sigma^{2}\left(\hat{\boldsymbol{\xi}}_{\mathrm{P}}\right)=\frac{1}{\mathrm{y}_{\mathrm{P}}^{2}} \sigma^{2}\left({\hat{\boldsymbol{\xi}_{\mathrm{P}}}}\right)=\frac{A}{\mathrm{y}_{\mathrm{P}}^{2}}+\frac{B}{\mathrm{y}_{\mathrm{P}}}+C \rightarrow \mathrm{C}_{8}
$$

the coefficient of $y_{p}^{2}$ in $\sigma^{2}\left(\hat{\xi}_{p}\right)$. From this it follows also that the efficiency of the estimator $\hat{\xi}_{\beta}$, being a ratio of variances, is simply the ratio of the coefficients of $y_{P}^{2}$, the other terms being disregarded. These facts applied to the estimator $\hat{\xi}_{P}$ make it possible to avoid a separate treatment for the two parameters $u$ and $\beta$. Their estimators are each represented by a single line in a table (such as Table III) showing values for various probability

## levels:

$$
P=.36788 \text { (or } y_{P}=0 \text { ) gives u: } P=I \text { (or } X_{P}=\infty \text { ) }
$$

gives $\beta$.
The concepts of variance and efficiency have also a more concrete, practical significance. The lower bound to the variance $Q_{\text {LB }}$ has the form $Q_{L B}=Q_{0} / n_{0}$ where $Q_{0}$ is a quadratic function of $y_{P}$, but is independent of sample size n . For two samples of sizes $n^{\prime}$ and $n^{\prime \prime}$, the variances $Q^{\prime} L B^{\circ} Q_{\text {LB }}$ are in the ratio

$$
\frac{Q_{L B}^{0}}{Q_{L B}^{n}}=\frac{n^{n}}{n^{p}}
$$

i。e.я inversely proportional to sample size. Similarly, if we had two estimators for the same sample size, we could form the ratio of their variances and think of it as representing an (inverse) ratio of (hypoo thetical) sample sizes. Thus, if for a sample of 20, the variance $Q^{0}$ of one estimator were oneohalf the variance $Q^{\prime \prime}$ of an alternative estimator, then the first estimator would require a sample of only 10 to give as much information as could be obtained with the second from a sample of 20. This saving of half the number of observations is expressed by saying that the first estimator is twice as efficient as the second. In general, a saving of the fraction p of the observations makes one estimator $1 /(1 \propto p)$ times as efficient as a second. The efficiencies of the estimators $\hat{\bar{\xi}} \mathrm{P}$ in Table III are more conveniently compared in graphical form, as in Figure 3. The heavy horizontal line at the top indicates perfect or 100 percent efficiency, and the rising curves as $n$ increases show how closely the estimator is approaching the standard of perfection. The most outstanding fact is that, in marked contrast to a theoretical. perfect estimator, the efficiency of
the actual estimator: $\hat{\xi}_{p}$ depends upon the probability $P$, being largest for the middle ranges 040 to .60 and dropping considerably at the ends near 0 and 1 . Since analysis of extreme (largest) data is concerned chiefly with the larger magnitudes associated with very small probabilities of occurring or being exceeded, we shall limit our interest to the range above $P=: 90$ 。 For $n=6$ the efficiency exceeds the 80 percent level for all values of $P$ in this range that are apt to occur in paractice (i.e., $P<.999$ ). In view of the satisfactory values of efficiency, further calculation for $n>6$ did not appear warranted at this time, particularly since it became apparent that the labor of computation would increase out of all proportion to the rapidly diminishing improvement in efficiency. Of course, most samples of observations are larger than the trivial size of 6 , and the question arises of how to handle the larger samples. This is treated in the next subosection.

## 404 Extension to larger samples

The key to handing samples with more than six observations is to treat them as sets or subgroups of samples of 6 (or if necessary, 5). If a sample size is not an exact multiple of 6 or of 5 , then the sample may be treated as consisting either of subgroups of 6 with an odd group remaining having less than 6 items, or of subgroups of 5 with a remaining group of 6 . We first deal with the simpler case where $n$ is an exact multiple of 5 or 6 .

Case I. Sample size an exact multiple of 5 or 6 . Suppose, in general, $n=k m$, where $m$ is the size of subgroup, which need not be 6 , and $k$ is the number of subgroups in the sample. If the sample is so divided into subgroups that the observations in one subgroup may be considered to be statistically independent of those in any other subgroup, then it is legitimate to treat the sample as consisting of $k$ independent subsamples, each of size mo

One way of obtaining independent groups is by use of random numbers. This, however, will lose valuable information embodied in the order in which the data were actually observed. If the data are
truly random, so that, for example, there are no seasonal effects, then this implies that subgroups formed in the order in which the data are observedthe first $m$ values observed put into the first group, the next $m$ into the second, etc. should be independent. This assumption, of course, underlies the entire method of estimation described in this report, and we shall adopt it in our procedures.

From each subgroup we form the "sub-estimator"

$$
\begin{equation*}
T_{i}=\sum_{j=1}^{m} w_{j} x_{j}, \quad i=1,2, \ldots \ldots k, \tag{4.21}
\end{equation*}
$$

where the weights $w_{1}, w_{2}, \ldots . w_{m}$ are those taken from Table I for sample size $m$ and are the same for each subgroup of $m$ values (but, of course, are different for different sizes $m$ ). These $k$ subeestimators $T_{i}$ are then combined by simple averaging to form the grand sample estimator:

$$
\begin{equation*}
\bar{T}=\frac{1}{k} \sum_{i=1}^{k} T_{i} \tag{4.22}
\end{equation*}
$$

The . variance of this estimator is simply

$$
\begin{equation*}
\operatorname{Var}(\bar{T})=\frac{1}{\mathrm{k}} Q_{\mathrm{m}} \tag{4.23}
\end{equation*}
$$

since we are taking the variance of a mean of $k$ independent quantities. $T_{i}$, each of which has the same variance $* \operatorname{Var}\left(T_{q}\right)=Q_{m}^{\prime} ; Q_{m}$ denotes the variance tabo ulated in Table III, Part A, for $m=2,3,4,5,6$. The efficiency of $\bar{T}$ is, since $n=k m$,

$$
E=\frac{Q_{L B}}{\operatorname{Var}(\bar{T})}=\frac{\frac{1}{k m} Q_{0}}{\frac{1}{k} Q_{m}}=\frac{\frac{l}{m} Q_{0}}{Q_{m}}=E_{m} \quad, \quad \text { (4.24) }
$$

Where $Q_{L B}=Q_{0} / n=Q_{0} / k m$, and $Q_{O}$ is independent of $n_{0}$ Thus we have the important fact that if a sample is broken into equal-size subgroups, the efficiency of the order-statistics estimator depends only upon the size ( $m$ ) of the subgroup (and, of course, P). Since, according to Table III, Part $B_{\text {, }}$ efficiency increases with sample (or subgroup) size, it follows that when there is a choice, a sample should be broken into subgroups as large as possible for best efficiency, i.e. into subgroups of 6. If this is not possible, but if the sample size n is an exact multiple of 5 . then subgroups of 5 may be used with not much loss in

[^3]efficiency. The last two columns in Table III, Part $B$ s show that the loss is 2.4 percent ( $=.8647$ - . 8404) at $P=.95$, and rises to a maximum of 3.8 percent for the limiting value $P=1$.

Case II. Sample size not an exact multiple of 5 or 6 . In most cases, of course, the sample size will have a remainder when divided by both 5 and 6. There are then a great variety of choices as to how to partition $n$ into subgroups of 6 and 5 and perhaps other sizes. Many of these possibilities have been examined, the aim being to establish as simple rules as possible without too great a loss in efficiency. Fortunately, most of the methods of partitioning a sample of given size n do not lead to greatly different efficiencies. Thus the following rules $c$ an be laid down for $n \geq$ ? ( $n \leq 6$ does not involve breaking into subgroups) based on writing $n$ in the form either $6 k+m^{8}$ or $5 p+6$, where $m^{8}<6$ :
(a) $\underline{n}=7$ up to large values.
(i) Use $n=6 k+m^{n}$ : split up into $k$ subgroups of 6 and a remainder suba group of $m^{8}<6$ items, unless $n=31,61$. etcog i.e.o. a multiple of 30 plus $l_{0}$
(ii) If $\mathrm{n}=30 \mathrm{k}+1$, write it as $\mathrm{n}=(30 \mathrm{k}-5)+6=(6 \mathrm{k}-1) \times 5$ +6 , i.e. split sample into ( $6 k-1$ ) subgroups of 5 and a "remainder" subgroup of 6 。
(b) n extremely large. If sample size is of the order of several hundred or more, so that the number of subgroups is of the order of 50 or 100 , then the amount of computation becomes increasingly labo orious. For such very large samples of extremes, which are rather rare, a shorto cut method is available which is explained in Appendix C. While its efficiency is substantially less then the longer me thod presented heres it is nevertheless of practical value inasmuch as the loss in efficiency, which in practical terms means an effective loss in number of observations, is not very important when a very extensive amount of data happens to be available。

The variance and efficiency of an estimator for most sample sizes. (Rule (a)) can be discussed readily in general terms. Assume that $n=k m+m^{\prime}$ represents the separation of the sample into two parts: one consisting of $k$ equal subgroups of size $m=5$ or 6 ; the other consisting of the remainder subgroup of size $m^{\prime}<m$ except for the exceptional case where $m=5, m^{\prime}=6$ (Case II, rule (a)(ii)). The average, $\bar{T}$, is formed from the first part as described under Case $I$. Then a sub-estimator $T^{\prime \prime}$ is formed from the remainder subgroup of $m$ " values using the weights $w_{\dot{I}}^{\prime}$ for samples of size $m^{\prime}$ :

$$
\begin{equation*}
T^{\prime}=\sum_{i=1}^{m^{\prime}} w_{i}^{!} x_{i}^{\prime} \tag{4.25}
\end{equation*}
$$

where $x_{1}^{\prime}, i=1,2, \ldots . m^{\prime}$, denotes the $m^{\prime}$ values in the subgroup. Finally a weighted average of $\bar{T}$ and $T{ }^{\prime}$ is formed, and this is the grand sample estimator $\widehat{§}_{P}$ :

$$
\hat{\xi}_{P}=t \bar{T}+\hbar^{\prime} T^{\prime}
$$

where the multipliers are *

$$
\begin{equation*}
t=\frac{k m}{n}, \quad t i=\frac{m^{8}}{n}=1-t \tag{4.27}
\end{equation*}
$$

Since all the subgroups are independent, so are $\bar{T}$ and $T$; whence

$$
\operatorname{Var}\left(\hat{\xi}_{P}\right)=\frac{t^{2}}{k} Q_{m}+t^{2} Q_{m}^{\prime}
$$

since the variance of the mean $T$ is $\frac{\pi}{K} \cdot Q_{m}$.
From the above it is evident that once the partitioning of sample size $n$ into $n=k m+m^{8}$ is determined, the $v a r i a n c e$ and efficiency may be obtained except (in the case of the variance) for a factor $\beta^{2}$ which must be estimated from the data. Table IV lists for convenience the efficiencies at two probability levels, $P=.99$ and the limiting value $P=1$, for most of the sample sizes that
\# Other multipliers are possibie. In particular, there is an optimum set of multipliers which produces an unbiased estimator, with slightly smaller variances and hence slightly greater efriciency. The optimum multipliers are, however, less simple than the prom portional onesof for example, they are not constants but depend on Pas and the gein in eificiency is not great. This was shown by a number of trials and by the fact that in any event, the efficiency cannot, exceed that for the larger subgroup size; $E_{m}$ (or $E_{m}$, if $\mathrm{m}^{8}>\mathrm{m}$ ), and does not differ much from it if the total sample size $n$ is at all sizable, say $>20$.
may occur in practice with gustoload data, provided the sample is split up according to the above rules. The levels $P=.99$ and $P=1$ furnish a convenient basis for comparing the efficiencies of two differ o ent partitions of the sample size. At this end of the probability scale the difference between the two efficiencies decreases monotonically as $P$ decreases. Thus, if the difference in efficiencies is 3 percent at $P=.99$ and 4 percent at $P=1$, then the difference is between 3 and 4 percent at $P=.995$. say, and at $P=.95$ and under is apt to be substantially below 3 percent, a difference negligible for practical purposes. The partitions shown in Table IV are those recommended by rule (a) above In certain cases, the efficiencies of alternative partitions are shown in the footnotes to Table IV for use in ease the extra few percentage points in efficiency are considered to be worth a little loss of simplicity in computation. There are some useful a prior guides for judging the efficiency in any given case even beyond the imit $n=40$ of Table IV 。 Thus, if $n=k m+m{ }^{\circ}$ 。 It is clear that the efficiency cannot exceed that for the subgroup sizes $m$ and $\mathrm{m}^{\mathrm{P}}$, but must lie e
somewhere between the efficiencies corresponding to these two sample sizes. If $m$ and $m^{\prime}$ are not far apart, then, regardless of the number of subgroups $k$, the efficiency is determined between narrow limits. Again; if $k$ is substantial, say near 10 or more, then the efficiency is practically that for the larger sample size m. Of course the maximum efficiency obtainable by the procedure outlined here is for Case I when the sample size is an exact multiple of 6. For $P=.99$ the efficiency in such case is 83.2 , and for $P=1$, it is 76.8. If any given partition results in efficiencies within, say, 2 or 3 percent of these values, then there is nothing significant to be gained by using any other partition, unless it is such as to simplify the computation.

## 5. SUMMARY OF PROCEDURES

The above method of analysis will now be summarized for ease of reference. The use of the method has been considerably simplified by the constriction of specially designed worksheets, represented by the pair of blank specimen forms following this page. A completely filled-out pair (Worksheets 1 and 2) will be found immediately preceding the tables at the end of this report. With the aid of such works sheets about two hours should be sufficient for all the calculations for a moderate-size sample, such as the sample of 23 observations analyzed below, and it has been found that this period is even sufficient to include the graphical analysis also presented. The materials needed for application of the method, besides Worksheets 1 and 2 and a sheet of extreme probability paper, are, in the order in which needed:
(i) Table IV, showing efficiencies for various methods of splitting sample into subgroups.
(ii) Table $I$, giving the weights $a_{i}, b_{i}$ 。
(iii) Table III, furnishing the quantities $Q_{0}, Q_{2}, Q_{3}, Q_{4}, Q_{5}, Q_{6}$.

## ESTIMATION OF EXTREMES

Worksheet 1 - Determination of Estimators
Source: Computer:
$\qquad$
Date: $\qquad$

| Record |  |
| :---: | :---: |
| No. | observed <br> extremes |

I. SUBGROUP SIZES AND PROPORTIONALITY FACTORS:

$$
\begin{aligned}
& \mathrm{n}=\ldots=\mathrm{km}+\mathrm{m}^{\prime} \quad \mathrm{t}=\mathrm{km} / \mathrm{n}=0 . \\
& t^{\prime}=m^{\prime} / n=0 . \\
& t^{2}=0 \text {. } \\
& =\ldots_{-}{ }^{+}+ \\
& \begin{aligned}
=k m / n & =0 \\
t^{2} / k & =0 .
\end{aligned} \\
& \mathrm{k}=\ldots \quad \mathrm{m}= \\
& m^{\prime}=
\end{aligned}
$$

ILA. MAIN SUBGROUPS:

|  | Weights $a_{i}$, | $b_{i}$ | (from Table I) |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $i$ | $=$ | 1 | 2 | 3 | 4 | 5 | 6 | Check sum

Observations $x_{i}$ in increasing order from $i=1$ to $i=m$


LIB. REMAINDER SUBGROUP:

$$
\begin{aligned}
& \text { Weights } a_{i}^{\prime}, b_{i}^{\prime} \text { (from Table I) } \\
& \begin{array}{llllllll}
i= & 1 & 2 & 3 & 4 & 5 & 6 & \text { Check sum }
\end{array} \\
& \begin{array}{l}
\mathrm{a}_{\mathrm{i}}= \\
\mathrm{b}_{\mathrm{i}}=
\end{array} \\
& \text { Observations } x_{i}^{1} \text { in increasing order from } i=1 \text { to } i=m^{\prime} \\
& x_{1}^{\prime} \quad x_{2}^{\prime} \quad x_{3}^{\prime} \quad x_{4}^{\prime} \quad x_{5}^{\prime} \quad x_{6}^{\prime} \quad \text { Check sum } \sum a_{i}^{\prime} x_{i}^{\prime} \quad \sum b_{i}^{\prime} x_{i}^{\prime} \\
& T^{\prime}=\Sigma a_{i}^{\prime} x_{i}^{\prime}+\left(\Sigma b_{i}^{\prime} x_{i}^{\prime}\right) y_{P}= \\
& +
\end{aligned}
$$

III. ESTIMATORS:

$$
\begin{gathered}
\hat{\xi}_{\mathrm{P}}=t \overline{\mathrm{~T}}+t^{\prime} \mathrm{T}^{\prime}= \\
u=\ldots, \quad, \quad{ }^{+} \quad{ }^{\mathrm{y}} \\
\end{gathered}
$$

## ESTTMATION OF EXTREMES

Worksheet 2 - Predicted values, confidence band, efficiency, plotting positions

| P $\quad \mathrm{y}_{\mathrm{P}}$ | PREDICTED VALUES $\hat{\xi}_{\mathrm{P}}=\ldots+\ldots y_{P}$ | $Q_{m}=Q^{2}$ <br> (from Tab | $Q_{m^{\prime}}=Q_{-}$ <br> le III) | $\begin{gathered} \quad \operatorname{Var}\left(\hat{\xi}_{\mathrm{P}}\right) \\ =\frac{t^{2}}{k} \Theta_{\mathrm{m}}+t^{\prime}{ }^{2} Q_{\mathrm{m}} \end{gathered}$ | 68\%-CONFDENGE band hale-width $\sigma\left(\hat{\xi}_{\mathrm{p}}\right)=\sqrt{\operatorname{var}\left(\hat{\xi}_{\mathrm{P}}\right)}$ | $\begin{aligned} & Q_{L B}=Q_{0} / n \\ & \left(Q_{0}\right. \text { from } \\ & \text { Table itI) } \end{aligned}$ | EFFICIENCY $E=\frac{Q_{L B}}{\operatorname{Var}\left(\hat{\xi}_{\mathrm{P}}\right)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (1) (2) | (3) | (4) $t^{2} / k=$ | (5) $t^{\prime}{ }^{2}=$ | (6) | $\begin{gathered} (7) \\ \beta= \end{gathered}$ | (8) | (9) |
| $\begin{aligned} & \text { Estinate of } \left.\begin{array}{l} \text { u: } \\ .36788 \\ 0 \end{array}\right) \end{aligned}$ |  | $\beta^{2}$ | $\beta^{2}$ | $\beta^{2}$ |  | $\beta^{2}$ |  |
| . $50 \quad 0.36651$ |  | $\beta^{2}$ |  | $\beta^{2}$ |  | $\beta^{2}$ |  |
| . $90 \quad 2.25037$ |  | $\beta^{2}$ | $\beta^{2}$ | $\beta^{2}$ |  | $\beta^{2}$ |  |
| . $95 \quad 2.97020$ |  | $\beta^{2}$ | $\beta^{2}$ | $\beta^{2}$ |  | $\beta^{2}$ |  |
| . 994.60016 |  | $\beta^{2}$ |  | $\beta^{2}$ |  | $\beta^{2}$ |  |
| .999 6.90726 |  | $\beta^{2}$ | $\beta^{2}$ | $\beta^{2}$ |  | $\beta^{2}$ |  |
| $\begin{aligned} & \text { Estimate of } \beta \text { : } \\ & 1 \end{aligned}$ | --- | $y_{p}^{2} \beta^{2}$ | $y_{p}^{2} \beta^{2}$ | $\mathrm{y}_{\mathrm{p}} \mathrm{p}^{2}$ | --- | $\mathrm{y}_{\mathrm{P}}^{2} \mathrm{\beta}^{2}$ |  |

PLOTTING POSITIONS
Observed extremes in increasing rank from 1 to $n=$

| Rank r | Observed <br> Extreme | Plotting Position $r /(n+1)$ | $\begin{gathered} \text { Rank } \\ \mathrm{r} \end{gathered}$ | Observed <br> Extreme | Plotting Position $r /(n+1)$ | Rank r | Observed <br> Futreme | Plotting Position $r /(n+1)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 |  |  | 11 |  |  | 21 |  |  |
| 2 |  |  | 12 |  |  | 22 |  |  |
| 3 |  |  | 13 |  |  | 23 |  |  |
| 4 |  |  | 14 |  |  | 24 |  |  |
| 5 |  |  | 15 |  |  | 25 |  |  |
| 6 |  |  | 16 |  |  | - |  |  |
| 7 |  |  | 17 |  |  | - |  |  |
| 8 |  |  | 18 |  |  | - |  |  |
| 9 |  |  | 19 |  |  | n |  |  |
| 10 |  |  | 20 |  |  | Sum |  |  |

The assumptions upon which the method is based are that the data in the given sample farranged in the order in which observed") may be treated as independent random observations all from the same population

$$
F(x)=\exp \left(-e^{-(x-u) / \beta}\right)
$$

(in cummulative form), with constant unknown parameters $u, \beta$ to be estimated.

For concreteness, the rules below refer to an actual example, worked out in Worksheets 1, 2, and Figure 4, consisting of the 23 maximum positive acceleration increments observed in 23 flights of an airplane and identified as "NACA-Langley-Sample III;" which are listed in the column headed "Observed extremes $(+\Delta n)^{\prime \prime}$ in Worksheet 1. These data are assumed to be given in the order of observation, so that under the above assumptions this arrangement may be considered to be a random one.

[^4]\[

$$
\begin{aligned}
& 3 \quad 1=
\end{aligned}
$$
\]

$$
\begin{aligned}
& 110 \quad 1
\end{aligned}
$$

$$
\begin{aligned}
& -\% \cdot 8.9 \\
& \text { (1) }
\end{aligned}
$$

$$
\begin{aligned}
& \text { (ty }
\end{aligned}
$$

Each rule (except Nos. 2 and 7, which are subdivided) consists of a single paragraph and this is followed by a detailed explanation of its use, inserted for conveniencelfor the user. This makes the list unavoidably lengthy, but the rules themselves are brief and simple to apply.

Before starting the calculations, it is desirable to plot the data on special probability paper according to the directions in rule 6(a). "Graphical analysis," below, in order to obtain a crude judgement of how well the data fit the assumed distribution. In rearranging the data in order of size, however, care should be taken not to lose the record of the original order in which the data were taken; because randomness will then have to be reintroduced.

Determination of estimators - Worksheet 1.

1. Enter the observations in the second colum in tho order in which given. The first column is for identification parposes.
2. Determine partition of sample size (if ? or more, but not extremely large) and split sample into subgroups according to the following rules (a), (b) or (c). If $n$ is extremely large, say several hundred or more, see Appendix C.
(a) If $n$ is an exact multiple of 5 or 6 ; write $n=k \cdot 5$ or $n=k \cdot 6$; if both, use $\mathrm{n}=\mathrm{k} \cdot 6$.
(b) If $n$ is not an exact multiple of 5 or 6 , write $n=k \cdot 6+m^{\prime}, m^{\prime}<6$, unless $\mathrm{n}=31,61$, etc., i.e., one plus a multiple of 30 .
(c) If n is of the form $30 \mathrm{k}+\mathrm{l}$, write it as $\mathrm{n}=(30 \mathrm{k}-5)+6=(6 \mathrm{k}-1) 5+6$, i.e. split $n$ up into ( $6 \mathrm{k}-1$ ) subgroups of 5 and a "remaindér" subgroup of 6 . Once $k, m, m$ are determined the blanks in Section I can be filled in. At the same time, in Worksheet 2, the numerical values of $m, m^{\prime}$ should be entered as subscripts in the headings "Q" and "Q " for columns 4 and 5 , respectively. In the worked example,
$n=23=3.6+5$ (rule (b)), so the data are split into 3 main subgroups of 6 and a remainder subgroup of 5:

Number of decimal places. - As a result of considerable experimentation it is recommended ${ }^{\text {b }}$ that all computations be carried to exactly the number of places shown for each item on the two worksheets.
3. Find estimators for the parameters $\xi_{p}$, $u$, by filling in the blanks and following the directions indicated in Worksheet I, Sections IIA, IIB, III.

In Section IIA, obtain the weights
$a_{i}$, $b_{i}$ from Table I for $n=m$, the size of the main subgroups. Mark off the subgroups by any convenient means, " arrange the observations in increasing order within each subgroup and enter horizontaly opposite

* It was found convenient here to determine the subgroup size m before entering the data in the extreme left columns, so that the subgroups could be plainly indicated by means of a space after every m-th observation.
the proper subgroup number in Section
IIA. Obtain the two product sums $\sum_{i=1}^{m} a_{1} x_{i}, \quad \sum_{i=1}^{m} b_{i} x_{i}$ as indicated in two right-hand columns and sum all columns as shown. The two productsums evaluated for the line labelled "Sum" will serve as a check. Form the average, $\overline{\mathrm{T}}$, by dividing by the number, $k$, of main subgroups. The work in Section IIB is analogous, except that the weights $a_{i}^{\prime}, b_{i}^{\prime}$ are the $a_{i}$ and $b_{i}$ shown in Table I for $n=m^{\prime}$, the size of the remainder subgroup; also, since there is only one subgroup, averaging is unnecessary.
Section ITT combines the (suk)estim mators, 4 and $\because 1$ coefficients $t, t^{\prime}$, determined in Section $I$, to produce the final overall sample estimator

$$
\begin{aligned}
& \hat{\xi}_{p}=t \bar{T}+t^{\prime} T^{\prime}=.92946+.16774 y_{P} \text {, } \\
& \text { on collecting the coefficients of } y_{P}
\end{aligned}
$$

and the constant terms. The estimates of the parameters $u, \beta$ are read off at once from the coefficients of $\hat{\xi}_{P}$ and entered. This constitutes the fitting of an extreme-value distribution to the given data.

Predicted values (etc.) -Worksheet 2.
4. Compute the values of $\hat{\xi}_{P}$ in column 3 for the values of $P$ and $\nabla_{P}$ shown in columns 1, 2. These values constitute the set of predictions for the respective probability levels.

Additional probability levels may be inserted between those shown, if desired. The value of $y_{p}=-\log _{e}\left(-\log _{e} p\right)$ is found most convenientiy from Table 2 of reference 15.
5. Confidence-band half-width (68-percent control curves) are computed from the standard deviations as indicated.

The numerical values of the variances $Q_{m}, Q_{m}$ in columns 4 and 5 are found under these same headings in Table III, Part A, and entered as shown. The values of $t^{2} / k, t^{2}$ are entered above these values, as indicated, in order to facilitate computation of the variances of the overall estimator,

$$
\operatorname{Var}\left(\hat{\xi}_{p}\right)=\frac{t^{2}}{k} Q_{m}+t^{2} Q_{m}^{\prime}
$$

in column 6. Column 7 gives the standard deviation of the estimator $\hat{\xi}_{\mathrm{P}}$. It is most easily computed by taking the square root of the coefficient of $\beta^{2}$ in column 6 and by
multiplying the value $\beta$ found in Section III of Worksheet 1 . Thus $\sigma\left(\hat{⿷}_{P}\right)$ for $P=.50$ is $\sqrt{.0605 I}$ times the value $\beta=.16774$ (written at the top of column 7 for convenience), giving the value . 0413 shown.

The standard deviation of the estimator measures the reliability, that is the extent to which repeated application of the procedure
to repeated samples taken under the same conditions would give values clustering more or less closely about the unknown parameter value For example, for a ixired probability $P$, in about 68 percent of the time (when the assumptions are satisfied) the comruted intorval $\xi_{p}$ plus or minus one stondurd deviation will contain the trie unknown parameter $\xi_{P}=u+\beta J_{P}$. For two standard deviations the percentage rises to 25*. Two curved lines, one joining the lefthand endpoints of these intervals and one joining the rightmhand endpoints, are called control curves (see rule 7 for graphical analysis, below) and these two curves define a confidence band consisting of the area betweenthan. The interval of values of the abscissa $x=\xi$ p included between the

* These percentages are only approximate since they assume $G$, to be normally distributed. As indicated in Appemdix D, this assumption is sufficiently correct for practical purposes for samples of the order of 100 or more. This may, of course, not be the case for much smaller samples. However, normality assumptions of this kind must often be made in practice in the absence of large-scale investigations to establish more precise distributions. Results obtained in this manner have often been found to be satisfactory.
control curves, when $P$ is given a specific value, is called a confidence interval. The standard deviation in Column 7 is thus the half-widith of a 68-percent conw fidence band (or interval). If levels of 95 percent, etc., are desired, the values $c$ an be readily obtained by adding another column consisting of twice (ete.) the entries in column 7.

6. Efficiency is computed as follows. The values of $\dot{Q}_{0}$ for the indicated values of $P$ are taken from the column headed $Q_{0}$ in Table III, Part A, divided by the given sample size $n$, and entered in the $Q_{L B}$ column, 8, of Worksheet 2. The efficiency is obtained by dividing this by the corresponding entry in column 7, canceling the $\beta^{2}$ (which was one reason for carrying it along separately), and finally entering the result in column 9.
7. Graphical analysis consists of plotting the data on suitably ruled paper, drawing the estirated straight line, drawing.
in the control curves, and sesing how well the data fall within them. The rethod is essentially due to Gumbel (cf....reference 5).
(a) In the section of Worksheet 2 called Plotting Positions, arrange all n obserm vations in the sample in a singlé ascending series from smallest to largest and enter them opposite the rank numbers $r=1$ to $n$. Compute and enter the plotting positions $\phi(x)=\frac{r}{n+I}$. Then, on a sheet of Extreme Probability Paper* such as used in Figures 4 and 5, plot the points ( $x_{r}, \frac{r}{n+1}$ ). The observation $X_{r}$ is plotted on the uniform scale along the horizontal axis; the fraction $\frac{r}{n+I}$ is plotted along the nonuniform vertical scale $\phi(x)$. These points are plotted as shown in Figure 4.

[^5](b) After the points are plotted the estimated line $x=u+\beta y, i . e .$, $x=.9295+.1677 y$ (see Rule 3, above), is drawn through them. This is easily done from columns 2 and 3 (Worksheet 2), since column 3 gives the predicted values of $x\left(=\hat{\xi}_{p}\right)$ corresponding to the values of $y\left(=y_{P}\right)$ in column 2. An even simpler method is to take two or three widely separated values P in colum $l$ together with the corresponding values $\hat{\xi}_{p}$, plot them on the $\phi(x)$ and $x$ scales, respectively, and draw the line through them.
(c) The 68-percent control curves are obtained by measuring off horizontally, at each value of $P$ in column $l$, the distance $\sigma\left(\hat{\xi}_{\mathrm{P}}\right)$, taken from column 7 , to the right and left of the fitted line, and then joining all the right and all the left endpoints of the intervals so formed, as in Figure 4. The area included between the two control curves
is the 68 -percent confidence band. If most.or all of the plotted points fall wi.thin the band, as in Figure 4 . then we conclude that the fit is satisfactory and furnishes no evidence that any of the basic assumptions are violated.
(d) The fitted straight line provides the predictions for any desired probability level P.* For example, the prediction for $P=.995$, which means a value of acceleration increment which has only one chance in 200 of being exceeded, is obtained (in Figure 4) by reading across to the solid (fitted) line at $P=.995$, and down to find the value $x=I^{\prime \prime} 82 \mathrm{~g}$. This is sufficiently close to the value 1.8176 obtained by calculation, using the value $y .995=5.29581$. The 68-percent curves give a confidence interval for this value of approximately 1.66 to 1.98. This means that there
is a probability of about two-thirds that such an interval includes the true predicted value 廈 that we are trying to estimate. The efficiency associated with this estimated is between 80.3 percent and 82.6 percent (column 9), surficiently narrow limits for practical purmses. If a more accurate value for the prediction or measure of efficiency is desired, it can be readily obtained by inserting a "P $=.995$ line" in the first table on Worksheet 2 and performing the computations indicated in colums 2 through 9.
6. COMPARISON WITH METHOD IN PRESENT USE

It is of interest to compare the proposed order-statistics method with the method of moments of E. J. Gumbel which has been used up to now in extreme gust-load computations (reference 18). The comparison is presented in two aspects - theoretical, involving an empiricai attempt to evaluate the bias and efficiency*of the Gumbel estimator; and practical, showing how the two methods worle out in an actual example.

### 6.1 Theoretical Comparison

Only the general results will be indicated here, the details being furnished in Appendix $E$. The comparison consists in writing down the Gumbel estimator, a function of the observations involving the sample means standard deviation, and the probability'factor yps and then obtaining the bias and the relabive efficiency of the proposed orderestatistics estimator to the Gumbel estimator.

* For theoretical comparison of confidence bands, see Appendix D.

Of the two characteristics bias and efficiencys the main interest at this point is in determining the efficiency of the proposed method, since that is the important feature whereby possibillties of cost savings, through taking fewer observations, can arise. Bias is less important for this purpose, and its consideration is therefore limited to the appendix (E).

As shown in Appendix E, (Section 1. ")) relative efficiency involves the first two moments of the sample mean and sample standard deviation, and the covariance of the mean and standard deviation. Of these, only the first two moments of the 3 ample mean can be obtained readily by standard procedures, while the remaining three quantities would require a prohibitive amount of numerical integration to evaluate accurately.

* The present discussion compares the order-statistics estimator with the Gumbel estimator $\hat{\xi}_{G}=x+$ $\frac{f_{6}}{\pi}\left(y_{P}-\gamma\right) s_{x}$. As explained in Appendix $E$, this estimator is a simplified form of Gumbel's original estimator, and is used when the sample of extremes is large. Appendix E also considers the original Gumbel estimator, which is a more complicated expression used for small samples, and shows that this estimator is both more biased and much less efficient than the simplified estimator.

Resort was therefore had to a method whereby the theoretical extremewalue distribution was reprea sented by a large set of suitably constructed random numbers. By means of these numbers a large number of actual random samples were drawn and the results tabulated. This was carried out mechanically with high-speed IBM equipment. By using 12,000 random numbers, 1200 random samples of 10 were drawn and a single average figure for relative efficiency was computed for each set of 100 samples. All these computations were made for the single probability level $P=.95$. Other values of $P$ are considered below.

The results are shown in Table $V$ and portrayed in Figure 6. For samples of 10, the efficiency was greater for the proposed order-statistics estimator in 5 cases out of 12 (relative efficiency $R$ (column 8) greater than 1) and greater for the present moment estimator in 7 cases out of 12. The average of all 12 relative efficiencies was very nearly unity. These results suggest that, for samples of 10 , the two methods are equally efficient.

The entire procedure was repeated for samples of 20 , obtaining 6 (instead of the previous 12) values for the 6 sets of 100 samples each. As rable $V$ (column 9) and Figure 6 show, the balance now was 5 to 1 in favor of the proposed method, with the average boing 1.11, representing 11 percent greater average efficiency for the proposed method.

For samples of 30 , there were 4 sets of 100 samples each, and the results (column 10) were 3 to $l$ in favor of the proposed method. The average rel ative efficiency was 1.13, representing a 13 percent gain in average efficiency.

To see the effect of different probability levels on these results, computations were undertaken for several values of P beyond .95. However, in order to avoid needless calculation, in view of the fact that only qualitative conclusions are warranted, the above procedure was modified as follows. The sets of 100 samples were combined for each sample size, and a single overall average for relative efficiency was obtained for the 1200 samples of 30 , the computations being carried out for the selected probabilitios $P=.95, .99, .999$, and the
limiting value, unity. The results are shown in Table VI. In addition, theoretical calculations* were made to obtain the asymptotic relative efficiencies as sample size increases without limit. These values will be found at the bottom of column 9 of Table VI. The above additional results indicate that increasing the probabillty $P$ tends to increase the efficiency of the proposed method relative to Gumbel's,

It should be pointed out that these values obtained from the empirical sampling method are indicative, rather than conclusive, on account of the random variation inherent in the method, as manifest in the wide fluctuation in efficiencies shown in Table $V$ for the individual sets of 100 samples. Wevertheless, the above results do give strong indication for the following statement:

For samples of 10 , the pronosed orderstatistics method is about as efficient as the me thod of Gumbel, while for samples of 20 or 30 or more, the proposed method is rore efficient.

[^6]```
For P = .95 0: greater, this noroase in
efficiency is about i2 to 15 perocnt
for samples of }20\mathrm{ to 30, and ulifmately
rises to 25 to 30 percent for inderinitely
large samples.
```

If, in the comparison presented above, the simplified Gumbel estimator is replaced by the original form of the estimator (Appendix E, Section 2), then the comparison becomes much more fayorable to the proposed order-statistics method and it can be stated that:

> For samples of 10,20, and 30 , and $p=.95$ or more, the ordermstatistics methodis up to twice as efficient as the Gumbel method using the original estimator. Moreover, this loo per cent difference in efficiency between the two methods is of suficient mage nitude not to be significantly affected by the sampling errors inherent in the method of evaluation.

> 6.2 Comparison based on a sample of actual observations

We shall use the sane data lipeady analyzed by the order-statistics method, consisting of the 23 maximum acceleration increments listed in Worksheet 1. For convenience we shall use a standard form of worksheet, employed by the Environmental

Protection Section of the Office of the Quartermaster Genral, Department of the Army, (reference 17) for applying the method of moments of $E$. J. Gumbel. To avoid confusion with the Worksheats 1 and 2 discussed previously, these new worksheets shall be referred to as Table VIImPart $\mathbb{A}$ and Part B. The items are filled in on both parts as directed, except that the factor $N /(N-I)$ is ignored in Sections I and IV, since subsequent theoretical investigation has shown its use to be incorrect; also, the values $x_{10}$ and $x_{100}$ in Section III, and the entire Section $V$ are not needed for our purposes. The values of $\sigma_{n}$ and $\bar{y}_{n}$ in Section II are taken from a table supplied with the worksheets but omitted here. Comparison is best shown graphically, as in Figure 5. It will be seen that in this particular case the fitted lines given by the two methods are not greatly different, the predicted values difering by amounts varying from .03 g at the $\mathrm{P}=.95$ level ( 1 chance in 20 of being exceeded) to nearly .10 g for $\mathrm{P}=.999$ (1 chance in 1,000 of being exceeded). The most striking and significant feature about the comparison in Figure 5 is the narrowness

of the confidence band for the order-statistics methof compared with that of the Gumbel mthod. This is attributable mainly to that fact that in the case of the order-statistics estimator the confidence. band width is based on the standard deviation of the estimstor, computed by the methods indicated in this report, whereas in the case of the moment (Gumbel) estimator, the standard deviation, whose value is not known, is replaced by a standard deviation that can be readily calculated, but which results in an unnecessarily wide confidence band (for details see Appendix D).

> 6.3 Adyantages and Limitations of Proposed Method

From the discussion given herein it appears that the proposed orese-statistics method offers the following advantages over the method of moments now in use:
a. The proposed method provides for the first time an estimator known to be unbiased, whose efficiency can be simply and accurately evaluated.

$$
\begin{aligned}
& \therefore ?
\end{aligned}
$$

$$
\begin{aligned}
& \therefore \ddot{\square}-\ddot{y}
\end{aligned}
$$

$16=$
b. The new estimator is more efficient than a simplified form of the Gumbel estimator, for samples of about 20 or more and $P=.95$ and more. Compared with the original form of the Gumbel estimator, the new estimator is up to twice as efficient for the same $r$ ange of values of $P$ and for samples of 10 or more.
c. The calculations necessary for the proposed method are simple and unified, giving simultaneously (i) estimates of both parameters, (ii) the predicted values corresponding to assigned probabilities and the reliability of these values, and (ili) estimates of the efficiency of the method.
d. The proposed methoduses a more valid procedure for obtaining the reliability of predicted values, and this procedure yields smaller confidence intervals in many cases. (See Appendix D.)

The following two limitations of the proposed
me thod should be kept in mind:
a. As is true of any other method of analyzing data, its use is appropriate only when the assumptions upon which it is based may be considered to be approximately satisfied, namely: all the observations constitute an independent random sample from the same population $F(x)=\exp \left(e^{-(x-u) / \beta)}\right.$ (in cumalative form).
b. The assumption that the data are'to be available in the order in which observed is of some importance. For if the data are first rearranged, grouped or processed in any manner.
$181=1.2$


- $B$,

> their randomness must be cone sidered lost, In order to use the proposed method it will then be necessary to restore randomness by use of \& table of random num bers to rearrange the data. This is less desirable and the original order should therefore be oreserved if possible.

This necessity of avoiding preliminary
processing imposes a disadvantage on the proposed method, as compared with the Gumbel method of moments, when tine sample is very large (several hundred or more, say). In the latter method the data may be groupto, simplifying the computations. The method of order statistics, on the other hand, is not applicable with grouped data - each observation must be treated on an individual basis - and hence is not suitable for occasional enormous samples, as is the Gumbel method. However, for such masses of data an even simpler method, described in Appendix C, is availablo.

## 7. CONGLUDING MEMARKS

This report has developed and illustrated a new method of analyzing extreme-value data based on order statistics that is simple to use and offers certain important advantages over the method of moments of Gumbel now in use, as well as being subject to certain limitations (see Section 6.3).

In view of these considerations, this new method is recomended for practical use in place of the present method of estimation.

In developing an estimator intended to be useful and efficient a number of subsidiary questions were encountered and treated. The most important of these were (i) obtaining minimum-variance unbiased linear functions of order statistics for small samples, and (ii) finding the most feasible way of breaking up a large sample into subgroups small enough to take advantage of the results in (i). In addition, considerable attention was given to a number of theoretical points of difference between the present and proposed methods.

Such theoretical study showed that one
feature of the present Gumbel method, namely determination of the confidence intervals or control curves
for large values of $P$, does not have a suitable theoretical basis, and that certain adjustments should be made in the formulas. These adjustments would have the effect of replacing the parallel control lines by diverging curves in the regions of high values of $P$, resulting in smaller confidence intervals for the more common values of $P$ and larger intervals for the higher values of $P$ that occur less often in practice.

The solutions to the above two main auxiliary problems have been incorporated into a simple set of tables and a pair of unified worksheets designed so that the computations show at a glance the essential quantities of interest - the actual predictions; their reliability, and the efficiency of the method. The method includes provision for showing these results graphically.

The present study has also devoted some attention to a method involving empirical random sompling and IBM tabulating equipment in cases where direct numerical evaluation is prohibitive. The use of 12,000 random numbers and from 400 to 1200 random samples was found insufficient to yield
$1 \times 1-2$
-
$-\quad 1$
$-3$
-
accurate quantitative results for one form of the Gumbel estimator (the siriplified form) on account of sampling variation. However, definite qualitative results in favor of the proposed method were indicated in the case of samples of 20 and 30 and theoretical calculation showed that this advantage was considerably greater for indefinitely large samples,

As a result of the experience gained in these studies it seems likely that for accurate results perhaps ten times the number of samples used (or more) should be taken and the computations performed through specialized procedures on high-speed electronic computing equipment.

Further calculation showed that in the case of the original form of the Gumbel estimator, much more definite statements were possible concerning efficiency. In this comprison the proposed estimator turned out to be up to twice as efficient as that of Gunbel, not only for the sample sizes 20 and 30 , but down to samples of 10 as well. For very large samples this advantage dropped somewhat, but the proposed estimator remained at least 20 to 30 percent more efficient.

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1. Cramérs Harald: Mathomatical Nethods of Stem tistics. Princeton Universjty Press, Princeton, N. J., 1946.
2. Gumbel, Emil J.: Les Valeurs Extrêmes des Distributions Statistiques. Annales de 1'Institut Menri Poincaré, vol. 4, pt. 2, 1935. pp. 115-158.
3. Gumbel, Emil J:: The Return Period of Flood Flows. Annals of Mathematical Statistics, vol. 12, no. 2, June 1941, pp. 163-190.
4. Gumbel, Emil J.: On Serial Numbers. hnnals of Mathematical statistics, vol. 14, no. 2, June 1943, pp. 163-178.
5. Gumbel, Emil J.: The Statistical Forecast of Floods. The Graduate Faculty of Political and Social Science, The New School for Social Research. New York, December 1948.
6. Gumbel, Emil J.: The Statistical Thecry of Extreme Values and Some Practical Applications. (In process of publication by the National Bureau of Standards).
7. Johnson, N. L.: Estimators of the Probability of the Zero Class in Poisson and Certain Related Populations. Annals of Mathematical Statistics, VoI. 22, no. 1. March $2951, \mathrm{pp}$. 94-101.
8. Kendall, M. G.: The Advanced Theory of Statistics.

Vol. I and II, Charles Griffin and Co., Ltd. London. 1948.
9. Kimball, Bradford F: Sufficient Statistical Estimation Functions for the Parameters of the Distribution of Maximum Values. Annals of Mathematical Statistics, vol. 17, No. 3, September 1946, pp. 299-309.
10. Kimball, Bradford $F_{8}$ : An Approximation to the Sampling Variance of an Estimated Maximum Value of a Given Frequency Based on Fit of Doubly Exponential Distribution of Maximum Values. Annals of Mathematical Statistics. vol. 20, no. I, Merch 1949, pp. 110-113.
11. Lehman, E. H.: Notes on the Theory of Estimation, Ch. II, pp. 15-19. (Unpublished mimeographed notes of a series of lectures given at the University of Calffornia in 1949-50).
12. Lieblein, Julius: On the Exact Evaluation of the Variances and Covariances of Order Statistics in Samples from the ExtremeValue Distribution. (Accented for publication in the Annals of hathometiocl Statistics).
13. Mosteller, $F$.: On Some Useful "Tnefficient" Statistics. Annals of Mathematical Statistics, vol. I7. no. 2, December 1946, pp. 377-408.
14. National Bureau of Standards: Table of the First Moment of Ranked Extremes, NBS Report 1167, September 20, 1951. A special report submitted to the National Advisory Committee for Aeronautics.
15. National Bureau of Standards: Probability Tables ior the Analysis of Extreme-Value Data, Applied Matheratics Series, No. 22. (In press).
16. Newman, F. W。: The Higher Trigonometry; Supperrationals of Second Order. Macmillan and Bowes, Cambridge, 1892, pp. 64-65.
17. Office of the Quartermaster General, Department of the Army: Evaluation of Climatic Extremes, Environmental Protection Section Report No. 175, Washington, D. C.. March 1951.
18. Press, Harry: The Application of the Statistical Theory of Extreme Values to Gust-Load Problems. NACA TN 1.926, Washington, D. C., 1949.
19. Smith, J, H. and Jones, H. L.: The Weighted Mean of Random Coservations Arranged in Order of Size. (Unpublished paper).
20. Wilks, S. S.: Order Statistics. Bulletin of the American Mathematical Society, vol. 54, January 1948, pp. 6-50.

## APPENDIX A

PROOF THAT SUFFICIENT STATISTICS DO NOT EXIST FOR THE PARAMETERS OF THE EX TREME-VALUE DISTRIBUTION*

$$
\text { (Relates to Section } 4 \cdot 2 \text { of text) }
$$

Problem: We have a sample of $n$ from the extreme-ralue population whose density function** is

$$
f(x)=a e^{-a(x-u)-e^{-a(x-u)}}
$$

$\beta=1 / \alpha>0$ and $u$ are unknown and we wish to find sufficient statistics for them.

## Theory: (1) If $t=\left(t_{1}, \ldots, t_{k}\right)$ is sufficient

(i.e., is a set of jointly sufficient statistics) for $\theta=\left(\theta_{1}, \ldots, \theta_{m}\right)$ then the density function of $x=\left(x_{1}, \ldots, x_{n}\right)$ may written in the form

$$
P(x, \theta)=f(t, \theta) g(x)
$$

(2) If $t(x)=t\left(x^{2}\right)$ for sample points $x$ and $x^{\prime}$, then
$A \equiv \frac{P(x, \theta)}{P\left(x^{g}, \theta\right)}=\frac{g(x)}{g\left(x^{0}\right)}=h\left(x, x^{1}\right) \quad$.

* This Appendix has been prepared by I. Richard Savage of the Statistical Engineering Laboratory, National Bureau of Standards.
$\%$ For convenience the symbol $a$ is used in place of the parameter $(1 / \beta)$ of the Eext.
(3) Hence for all those points where $t(x)$ has a constant value the ratio $A$ is free of $\theta$, and thus we can find sufficient statistics by seeing for which point sets $A$ is constant.
(4) Evidentiy, if $\theta=g\left(\theta^{\prime}\right)$ (i.e. . . $\theta_{i}$
$\left.=g_{i}\left(\theta_{1}^{\prime}, \ldots, \theta_{k}^{\prime}\right), i=1, \ldots, k\right)$ is a non-singular troass. formation of the parameters, then also

$$
\frac{P\left(x, \theta^{\prime}\right)}{P\left(x^{\prime}, \theta^{\prime}\right)}=\frac{f\left(t, \theta^{0}\right) g(x)}{f\left(t, \theta^{\prime}\right)} g(x)=h\left(x, x^{?}\right)
$$

using the same (set of estimators) t as for $\theta$. In other words, if a set of statistics $t$ is sufficient for a set of parameters $\theta$, the same set $t$ is sufficient for any other set $\theta^{2}$ obtainad from $\theta$ by a non-singular transformation.

Results: We will now apply the above theory to the problem at hand and show that the largest point set on which A is constant contains n! points, that is, it takes $n$ functions to describe $t$, so that the resulting sufficient statistic is the trivial set, $t=\left(x_{1}, \ldots, x_{n}\right)$ or $\left(\frac{\sum x}{n}, \frac{\sum_{x}}{n}, \ldots, \frac{\sum_{x}^{n}}{n}\right)$. In other words, the only sufficient statistics are the $n$ observations themselves, so that we do not have a basis upon which to construct optimum estimators.

Analysis: For the distribution $f(x)$ we have
$A=\exp \left\{a\left(\sum_{i=1}^{n} x_{i}^{\prime}-\sum_{i=1}^{n} x_{i}^{i}\right)-\sum_{i=1}^{n}\left[e^{-a\left(x_{i}-u\right)}-a\left(x_{i}^{\prime}-u\right)\right]\right\}$.
If $A$ is free of the parameters $\beta$ and $u$ then it is also free of $a=1 / \beta$ and $u$, and so are $\log A$ and $\frac{\delta^{k} \log A}{\delta a^{k}} \cdot$ Hence

$$
\log A=n a(\bar{x}-\bar{x} \cdot)-\sum_{i=1}^{n}\left[e^{-a\left(x_{1}-u\right)}-e^{-a\left(x_{i}-u\right)}\right] .
$$

Let $u$ approach $-\infty$. We first find that $\bar{x}=\bar{x}$ ' in order to have $\log \mathrm{A}$ free of $u$ and $a$. Next,

$$
\begin{aligned}
\frac{\delta^{k} \log A}{\delta a^{k}} & =(-1)^{k+1} \sum_{i=1}^{n}\left[\left(x_{i}-u\right)^{k} e^{-a\left(x_{i}-u\right)}-\left(x_{i}^{\prime}-u\right)^{k} e^{-a\left(x_{i}^{\prime}-u\right)}\right] \\
& =0, k=2,3, \ldots
\end{aligned}
$$

and this is true for $k=1$ as well, since $\bar{x}=\bar{x}^{\prime}$.
Since this is an identity in let us set $u=0$. We then get

$$
\sum_{i=1}^{n} x_{i} e^{-a x_{i}}=\sum_{i=1}^{n} x_{i}^{k} e^{-2 x_{i}^{\prime}} .
$$

These are finite sums; and therefore, since they are identitites in $\underline{a}$, it is clear, since a may converge to zero, that

$$
\frac{\Sigma x_{i}^{k}}{n}=\frac{\Sigma x_{i}^{y k}}{n}
$$

Thus the largest set of points of constancy of A are those points which give the same sample moments, and this fact implies the desired result.

Note: Statement (4) above inplies that the result also holds if the parameter $u$ is replaced by $\xi_{P}=u+\beta y_{P}=u+y_{P} / a_{0}$

Example: To show how this method works for a familiar problem consider a sample of $n$ from a normal distribution; here

$$
A=e^{-1 / 2 \sigma^{2}\left[\Sigma\left(x_{i}-\theta\right)^{2}-\Sigma\left(x_{i}^{\prime}-\theta\right)^{2}\right]}
$$

$-2 \log A=\left[\Sigma\left(x_{i}^{2}-x_{i}^{\prime}{ }^{2}\right)-2 \theta \Sigma\left(x_{i}-x_{i}^{\prime}\right)\right] / \sigma^{2}$
and clearly the necessary and sufficient condition for $A$ to be constant for all $\sigma^{2}$ and $\theta$ is that $\sum x_{i}^{2}=$ $\Sigma x_{i}^{\prime 2} / \Sigma x_{i}=\Sigma x_{i}^{\prime}$, which is the classicel result that the first two moments are sufficient statistics.

## APPENDIX B

MATHEMATICAL FORMUTATION AND SOLUTTON OF MIINMUM-VARIANCE PROBLEM

$$
\text { (Relates to Section } 4 \cdot 3 \text { of text) }
$$

$$
\text { We consider an estimator of } \hat{\xi}_{F}=u+\beta Y_{P} \text { of }
$$

the form

$$
\begin{equation*}
L=\sum_{i=1}^{n} w_{i} x_{j} \tag{B.1}
\end{equation*}
$$

where $x_{1} \leq x_{2} \leq \cdots \leq x_{n}$ are the $n$ order statistics of a sample of $n$ from the extreme-value distribution (4.1), and seek to find the $W_{i}$ which minimize $\operatorname{Var}(\mathbb{L})$ subject to

$$
\begin{equation*}
E(L)=\xi_{P} . \tag{B,2}
\end{equation*}
$$

The estimator I in (B.8) below with weights so
determined is called the minimum-varionce, unbiased, (Inear) order-statistics estimator for sample size n。

Wri ting

$$
x=u+\beta y
$$

where $y$ is the reduced variable corresponding to $x$, we also have

$$
\begin{equation*}
x_{i}=u+\beta y_{i} \tag{B.4}
\end{equation*}
$$

where $\mathrm{y}_{1} \leq \mathrm{y}_{2} \leq \ldots \leq \mathrm{y}_{\mathrm{n}}$ are the n order statistics of a sample of size $n$ from the reduced distribution
$\exp \left(-e^{-\mathrm{Y}}\right)$, free of parameters. From the above it follows that

$$
\begin{equation*}
E\left(x_{i}\right)=u+\beta E\left(y_{i}\right) \tag{B.5}
\end{equation*}
$$

since $u$ and $\beta$, though unknown, are constants not subject to sampling variation when the operation of expectation is performed. The values $E\left(y_{1}\right)$ have been tabulated in reference 14 for $i=n(1)$ rinin(1, $n-25)$. $n=1(1) 10(5) 60(10) 100^{*}$.

These results give readily

$$
E(L)=\Sigma w_{i}\left(u+\beta E y_{i}\right)=\xi_{P}=u+\beta y_{P} \quad . \quad(B .6)
$$

This is required to be an identity for all values of the parameters $u, \beta$. Equating their coefficients gives the two conditions on the weights $w_{i}$ :

$$
\begin{equation*}
\sum_{i=1}^{n} w_{i}=1 \tag{B,7}
\end{equation*}
$$

$$
\sum_{i=1}^{n}\left(E y_{i}\right) w_{i}=y_{p}
$$

where $E y_{i}$ are the numerical values tabulated in reference 14.

[^7]Turning to the variance, we have

$$
\operatorname{Var}(L)=\sum_{i=1}^{n} w_{i}^{2} \sigma_{x_{i}}^{2}+\sum_{j=1}^{n} \sum_{i=1}^{n} w_{i} w_{j} \sigma_{x_{i} x}{ }_{j}
$$

## $i \neq j$

From (B. 4 ) and the properties of the variances and covariances of Invar estimators we have $\sigma_{x_{i}}^{2}=\beta^{2} \sigma_{Y_{i}}^{2}=\beta^{2} \sigma_{i}^{2}, \quad, \sigma_{x_{i} X j}=\beta^{2} \sigma_{y_{i} y_{j}}=\beta^{2} \sigma_{i j}$.
making an obvious simplification in notation, whence

$$
\begin{align*}
V_{n}=\operatorname{Var}(L) & =\left(\Sigma \sigma_{i}^{2} W_{i}^{2}+\Sigma \Sigma \cdot \sigma_{i j} W_{i} W_{j}\right) \beta^{2} \\
& =\text { minimum subject to (B.7). } \tag{B.8}
\end{align*}
$$

This is a constrained minimum problem for variation in the unknown $w_{i}$, and is equivalent to finding the (unconstrained) minimum of ${ }^{*}$. $G_{I}=\left(\Sigma \sigma_{i}^{2} W_{i}^{2}+\Sigma \Sigma \Sigma_{i}^{i} \sigma_{i} W_{i} W_{j}\right) \beta^{2}+\lambda_{1}\left(\Sigma w_{i}-1\right)+\mu_{1}\left[\Sigma\left(E y_{i}\right) W_{i}-J_{p}\right]$, where $\lambda_{I}$, $\mu_{1}$ are the Lagrange multipliers. Since $\beta^{2}>0$ is constant, though unknown, this is the same as minimizing $G=\frac{G_{1}}{\beta^{2}}=\Sigma \sigma_{i}^{2} w_{i}^{2}+\Sigma \Sigma \Sigma_{i j} \sigma_{1} w_{j}+\lambda\left(\Sigma w_{i}-1\right)$ н. $\mu\left[\Sigma\left(E y_{i}\right) w_{i}-y p\right]$,

* The temporary notation $\mu, \rho_{1}$ should not be confused with the symbols for moments.

Where $\lambda=\lambda_{1} / \beta^{2}, \beta=\mu_{1} / \beta^{2}$. Setting the derivatives wit respect to $w, k=I, 2, \ldots, n$, equal to 0 and dividing by 2 , we have

$$
\sigma_{k}^{2} W_{k}+\sum_{\substack{i=1 \\(i \neq k)}}^{n} \sigma_{i k^{W}}+\lambda+\mu \sum_{k}=0, k=1,2, \ldots, y_{k}, \quad(B, 9)
$$

These latter are $n$ linear equations which, with the two in ( $B .7$ ), form a simultaneous system of ( $n+2$ ) equations in the $(n+2)$ unknowns $W_{1}, W_{2}, \ldots, W_{n}, \lambda$, $\mu$. The values of $\lambda$ and $\mu$ are useful as a check, since if (B.9) is multiplied by $W_{k}$ and summed, the result, in view of $(B, 7)$ which the $w_{i}$ 's satisfy and $(B, 8)$ is

$$
\mathrm{V}_{\mathrm{n}, \min }+\lambda+\mu \mathrm{y}_{\mathrm{P}}=0
$$

that is, we should have

$$
\mathrm{v}_{\mathrm{r}_{\mathrm{g}, \min }}=-\lambda-\mu \mathrm{y}_{\mathrm{p}}
$$

The minimum value $V_{n, m i n}$ will be denoted by $n_{n}$. Before solving the set (B.7), (B.9) it is necessary to determine the coerficients in these linear equations. The values of Ey $\begin{aligned} & \text { are tabulated, }\end{aligned}$ as already mentioned. The variances and covariances $\sigma_{k}^{2}, \sigma_{i k}$ involve complicated integrals. The author has been successful in expressing these integraly

In terms of simpler ones already tabulated (reference 12). The results are, for the variances,*

$$
\begin{aligned}
& \sigma_{i}^{2}=E\left(y_{i}^{2}\right)-\left(E y_{i}\right)^{2} \\
& E\left(y_{i}^{2}\right)=\frac{n!}{(i-1)!(n-i) \overline{!}} \sum_{r=0}^{n=1}(-1) c_{r}^{n-1} g_{2}(i+r), \\
& i=1,2, \ldots, n,
\end{aligned}
$$

where

$$
g_{2}(i+r)=\frac{1}{i+\Gamma}\left[\frac{\pi^{2}}{6}+(6+\log \overline{i+r})^{2}\right]
$$

ard $\gamma=$ Euleris constant $=.5772156649 \ldots$ and for the covariances,

$$
\begin{aligned}
& \sigma_{i j}=E\left(y_{i} y_{j}\right)-\left(E y_{i}\right)\left(E y_{j}\right) \\
& E\left(y_{1} y_{i}\right)=\frac{n b}{(1-1)!(j-1-1)!(n-j)!} \quad \sum_{r=0}^{j-i-1} \sum_{s=0}^{n-j}(-1)^{r+s} \\
& { }^{-} C_{r}^{j-i-1} C_{s} n-j \not D(i+r, j-i-r+s) \text {. } \\
& i<j ; i, j=1,2, \ldots, n,
\end{aligned}
$$

Where the function $\psi$ is defined by
$\tan \emptyset(t, u)=(u-t) g_{2}(t+u)+t^{2}\left[g_{1}(t)\right]^{2}-2 L\left(1+\frac{u}{t}\right)+\frac{\pi^{2}}{6}$,
in which $g_{2}$ is the same function as before, $g_{1}(t)=\frac{l}{t}(\gamma+\log t)$, and

* All logarithms are natural logarithms, to the base e.

$$
\begin{aligned}
L(1+x) & =\int_{1}^{1+x} \frac{\log w}{n-1} d n=\frac{(-1)^{n+1}}{n^{2}} x^{n} \\
& =\frac{1}{2}(\log x)^{2}+\frac{n^{2}}{6} \quad L\left(1+\frac{1}{x}\right)
\end{aligned}
$$

is Spence's integral, which has oot extensively tabulated (to 12 places) in reference 16 . The function $g_{1}$ also occurs in an expression for the means:

$$
E\left(y_{i}\right)=\frac{n!}{(i-1)!(n-i)!} \sum_{r=0}^{n-1}(-1)^{r}{ }_{r}^{n-1} E_{1}(i+r) .
$$

The above formulas have been evaluated as far as $n=6$ and the results are listed in mable II. The values in the table are believed to be accurate to the number of places shown. Those for the means agree (to within a unit in the $7^{\text {th }}$ place) to the 7 places to which the means have previously boen tabulated.

Table II thus provides the coefficients in the system of equations (B.7), (B.9) in the $W_{i}$ and $\lambda$, $\mu$. The righthand sides of these $(n+2)$ equations are $1, y_{p}, 0, \ldots, 0$ and the solutions $w_{i}$, $\lambda, \mu$ are linear combinations of these with numerical coefficients which involve only $\sigma_{i}^{2}, \sigma_{i j}$, and $E y_{i}$, but not $y_{p}$.
Hence the solutions are all of the form

$$
\begin{aligned}
w_{i} & =a_{i}+b_{i} y_{P}, i=1,2, \ldots, n \\
\lambda & =c_{1}+d_{1} y_{P} \\
\mu & =c_{2}+d_{2} y_{P}
\end{aligned}
$$

Substituting these $w_{i}$ in (B.8) we have an expression of the form

$$
Q_{n}=V_{n, \min }=\left(A_{n} y_{P}^{2}+B_{n} y_{P}+C_{n}\right) \beta^{2} \quad . \quad(B .10)
$$

The quantities $a_{i}$, $b_{i}$ for the weights $w_{i}$, and the coefficients $A_{n}, B_{n}, C_{n}$, of $Q_{n}$, are given in Table I for $n=2$ to 6 . The solution of the system of equations became increasingly lengthy for increasing values of $n$, with correspondingly diminishing accuracy, so that the computations were discontinued beyond $\mathrm{n}=6$. The procedures for handing samples larger than $n=6$ are explained in the toxt of this report.

## APPENDIX C

SHORT-CUT METHOD FOR VERY LARGE SAMPLES
(Relates to Section 4.4 of text)

If we have a sample of several hundred or more extreme observations, as may sometimes be the case (e.g. reference 18, where a sample of 485 extremes was analyzed) it is possible to select just three out of all the observations and from them obtain useful estimators.

This technique is based on a method used by F. Mosteller (reference 13) for samples from the normal distribution. If the $n$ sample values from a (continuous) population whose density is $f(x)$ when arranged in ascending order are denoted by the order statistics $x_{1}, x_{2}, \ldots, x_{n}$, and $n$ is very large, the application of Mosteller's method involves taking the observations whose ranks are $\lambda n, \mu n, \nu n$, where $0<\lambda<\mu<\nu<I$ wi.th $\lambda, \mu, \nu$ suitably determined, and choosing $a$ and $b$ so that ${ }^{*}$

[^8]\[

$$
\begin{equation*}
\hat{\xi}=a x_{\mu n}+b\left(x_{\nu n}-x_{\lambda n}\right) \tag{c.1}
\end{equation*}
$$

\]

is an (asymptotically) unbiased estimator of the parameter $\xi_{p}=u+\beta y_{p}$. (The reason for choosing this particular form is discussed below).

The mean and variance of the estimator $\hat{\xi}$ in (C.I) are computed from the corresponding moments of order statistics of the form $X_{\lambda_{n}}$, with $n$ very large and $\lambda$ a proper fraction not too near 0 or 1 . Under these circumstances the theorem used by Hosteler states that in the limit, as $n$ increases indefinitely,
(i) $x_{\lambda n}$ becomes normally distributed, with mean and variance

$$
\begin{align*}
& E\left(x_{\lambda n}\right)=t_{\lambda}  \tag{C.2}\\
& \sigma^{2}\left(x_{\lambda n}\right)=\frac{\lambda(1-\lambda)}{n\left[f\left(t_{\lambda}\right)\right]}, \tag{C.3}
\end{align*}
$$

Where $t_{\lambda}$ is ciefined by $\lambda=\int_{-\infty}^{t} f^{\prime}(x) d x$; and
(ii) the covariance of any two order statistics

$$
\begin{aligned}
& x_{\lambda n} \text { and } x_{\mu n}, \lambda<\mu, \text { is given by } \\
& \qquad \operatorname{cov}\left(x_{\lambda n}, x_{\mu n}\right)=\frac{\lambda(1-\mu)}{n f\left(t_{\lambda}\right) f\left(t_{\mu}\right)}, \quad \text { (c.4) }
\end{aligned}
$$

where $t_{\mu}$ is defined similarly to $t^{\prime}$.

For $\hat{\xi}$ in (C.I) to be unbiased in the case where $f(x)$ is the extreme value distribution,

$$
\begin{equation*}
E(\hat{\xi})=\xi_{p} \equiv u+\beta y_{p} \tag{c.5}
\end{equation*}
$$

must be an identity in $u, \beta$. We first note that from previous discussion in the text (see Section 4.1), the parameter $\xi_{P}$ is precisely the abscissa of the ordinate which cuts off the area $P$ to the left. Hence we have simply

$$
\begin{equation*}
t_{\lambda}=\xi_{\lambda}=u+\beta y_{\lambda} . \tag{c.6}
\end{equation*}
$$

Equations (C.1), (C.2), and (C.5) then give

$$
a t_{\mu}+b\left(t_{\nu}-t_{\lambda}\right)=u+\beta y_{p}
$$

or

$$
a\left(u+\beta y_{\mu}\right)+b\left(y_{\nu}-y_{\lambda}\right) \beta \equiv u+y_{p} \beta,
$$

from which, on equating coefficients of $u$ and $\beta$,

$$
\begin{equation*}
a=1, b=\frac{y_{P}-y_{\mu}}{y_{2}-y_{\lambda}} \tag{C,7}
\end{equation*}
$$

In principle, the fractions $\lambda, \mu, \nu$ might be determined so as to minimize the variance of $\hat{\xi}$ and thus make its efficiency a maximum, but this would require very extensive computation which would not be warranted on account of the limited importance of efficiency when the available sample is very large. (For example, a 50 percent efficient estimator with a sample of

$$
\begin{aligned}
& \therefore \dot{r}+5
\end{aligned}
$$

$$
\begin{aligned}
& W+4=a^{3} \cdots \sqrt{3}+3
\end{aligned}
$$

$$
\begin{aligned}
& \therefore \quad+\quad+\cdots \cdots=B
\end{aligned}
$$

l,000 gives results equivalent to using a sample of very 500 - still a sufficiently-large sample.*) Instead we consider estimators of $\xi_{p}$ of the form

$$
\begin{equation*}
\hat{\xi}=\hat{u}+y_{p} \hat{\beta} \tag{0,8}
\end{equation*}
$$

where $\hat{u}, \hat{\beta}$ are estimators of the two parameters $u$, $\beta$, that involve the fevest possible number of order statistics $x_{s n}$ wif thout undue sacrifico in efficiency as computed for idenfinitely large sompes. The aim is to find, with a minimum mount of computation, separate unbiased estimators $\hat{u}, \hat{\beta}$, of the parameters $u, \beta$, each of which has minimum variance or best efficiency in some sense, in the hope that the linear combination (C.8), which will also be unbiased, will turn out to have efficiency which is not unreasonably small. This is a heuristic method, since the fact that $\hat{u}$ and $\hat{B}$ are efficient does not imply that their

[^9]combination $\hat{u}+y_{p} \hat{\beta}$ is efficient. Much better estimators probably exist, but we are content to obtain just one of reasonable efficiency.

It turns out that the modal parameter $u$ can be estimated by a single order statistic. E. J. Gumbel has shown (reserence 4, equation (50)) that the value of $u$ for which $x_{\mu n}$ best (i.e., with least variance or most efficiency) estimates u is $\mu=.20319$. For simplicity, we therefore replace $u \ln (C, 8)$ by

$$
\begin{equation*}
\hat{u}=x \cdot 20 n \tag{0.9}
\end{equation*}
$$

The scale parameter $\beta$ requires at least two order statistics, or rather their difference $x_{v n}-x_{\lambda n}$, for estimation, multiplied by a suitable unbiasing factor which will become absorbed in the expression for $b$ in $(C .7)$. A considerable number of trials indicates that the pair of values $\lambda=.03, \nu=.85$ gives an estimate of $\beta$ with efficiency probably close to the maximum, if not actually maximum. since we are not seeking very precise results, this pair of values is adopted here. Thus (C.I), in view of (C.7), becomes

$$
\hat{\xi}=x_{.20 n}+0.3256\left(y_{p}+0.4759\right)\left(x .85 n^{-x} .03 n\right) \cdot(c .10)
$$

The variance of this estimator is obtained from the
rule

$$
\operatorname{Var}\left(\sum_{i=1}^{m} a_{\underline{1}} x_{i}\right)=\sum_{i=1}^{m} a_{i}^{2} \sigma_{x_{i}}^{2}+2 \sum_{\substack{i=1 \\ i<j}}^{m} a_{i} a_{j} \operatorname{cov}\left(x_{i} x_{j}\right)_{g}
$$

Which after simplification gives

$$
\sigma^{2}(\hat{\xi})=8.6916 d^{2}=0.0681 a+1.5442,(0.22)
$$

where

$$
d=0.3256 \mathrm{~J}_{\mathrm{P}}+0.1549 .
$$

Since $\hat{E}$ is unbiased, a measure of te efficiency may be obtained by dividing its variance into the CramérmRao Iowerinound, $Q_{\text {LB }}$ (see equation (4.19) and accompanying text; numerical values are given in the Qo column of Table III, Part A. The results are as follows, for several values of $P$ of interest:
$p$
.95
.99
.999
I (limiting
value)

Efficiency of $\hat{k}$ .6450
0.349
.652
.660

Thus, this largemsample method of estimation is slightly less than twoothirds efficient. However, as noted above, such apparently low efficiency need not be a serious matter in practice.

For convenience, a summary of the method described above is given here
-

1. Arrange 211 n observations (assumed to be independent and from the same extremeryalue diso tribution) in order of increasing size, and then rank them from 1 to $n$.
2. By hand or mechanical sorting, select the three observations $x_{r}$ whose ranks are the nearestin tegers to. $03 n, .20 n$, and $.85 n$. Denote these by ${ }^{x} .03 n^{*} .20 n^{x} .85 n^{\circ}$
3. Compute the predicted values $\mathcal{E}$, for various probability levels $P$, by Pormula (c.20).
4. For each $P$ compute the variance from formula (C.II)
5. Take the square root of the variance to obtain the standard deviation. This gives the halfowidth of the 68-percent confidence band, since for Iarge samples the distribution of $\hat{\xi}$ approaones normality. Sinilarly, twice the standard deviation determines the 95 percent confidence band, and. 2.58 standard deviations determines the 99 pero cent band.
6. Obtain the efficiencies by dividing the variance into the CraméraRao lower bound $Q_{0}$ in Table III。 Parot Al。


## APPENDIX D

ANALYSIS OF CONFIDENCE INTERVALS IN ORDER-STATI SICS METHOD AND METHOD OF MOMENTS OF GUMBEL
(Related to Section 5, Rule 5, and Section 6.2 of text)

> I. Confidence intervals in orders statistics me thod (based on normality as sumption).

In the text, Section 5s Rule 5s the confidence intervals given for various confidence levels in the proposed method are obtained by laying off a certain number of standard deviations, computed for the estimator $\hat{\xi}_{P^{2}}$ on either side of the estimated value given by the fitted line. If this is done for different values of $F$ and the ends joined, as in Figure to a confidence band is obtained. The number of standard deviations given in the method $\alpha$ one for a confidence level of 68 percent, two for a level of 95 percent is based on the assumption that the estimator o $\hat{E}_{p}$ is normally distributed. The purpose of this section is to investigate this assumption more closely. It will be recalled that the estimator $\hat{e}_{p}$ obtained by splitting the sample into a number of equal groups with perhaps a remainder of different size (see text in connection with equations (4.22), ( 4.25 ) and (4.26) . Then $\hat{g}_{p}$ can be written. Yon
(4.26)

$$
\hat{\xi}_{P}=t \bar{T}+t^{0} \cdot T^{0}
$$

where $\bar{T}$ is the average of a certain linear function of the sample variables (equation (4.22)) taken over the $k$ subgroups, $T^{\prime}$ is another linear function, and $t_{g} t^{\prime}$ are constants. Thus $\hat{\xi}_{P}$ is the sum of two parts:
(1) an average of $k$ independent random variables $\left(t T_{i}\right)$ * all with the same distribution, and (2) a single variable (t'T) with a somewhat different distribution. By the Central Limit Theorem in prob= ability (reference I, page 215), according to which the average of a number of random variables having the same distribution (with first two moments existing) is asymptoticaily normal as the number of variables increases indefinitely, the first part is approximately normal for large $k$. In fact, extensive experience has shown that a normal distribution is often a reo markably close approximation even if the numbor of variables k is under 10. Furthermore, the first two moments (actually all) of each variable $T_{i}$ cortainly exist - in fact the proposed method is based upon

* These variables are independent because the subgrouns were assumed to be formed independently。
their computed values. Hence, it is safe to say that for $k=10$ or more the first part is very closely normal. The second part ( $t^{8} T^{p}$ ) is a variable which has the same general character as $T_{i}$ (a weighted sum of order statistics [equation (4.25)]) and hence is believed not to impair significantly the approximate normality of $\hat{\xi}_{\mathrm{P}}$. Its influence is likely to be small, especially if the number of other variables, $\mathrm{K}_{\mathrm{s}}$ is large。

For samples as large as 100, $\mathrm{k}=16$ if broken into subgroups of 6 , or $k=20$ if broken into sube groups of 5. Since these values of $k$ are considerably larger than 10, the preceding discussion shows that it is quite safe to assume normality for $\hat{\xi}_{p}$ for samples of 100 or more, so that the corresponding multiples of the standard deviation given above are sufficiently accurate in such cases. In fact, it is likely that the normal approximation remains good for practical purposes down to samples of 50 or 60 。 becoming, of course, worse as sample size decreases still fuxther. However, in the absence of knowledge about the exact distribution of the orderstatistics estimator $\hat{\xi}_{p}$ for smaller samples, the normal approximati
apparently the only simple one available for determ ing confidence limits. It may be noted that approximate methods are also involved in determination of confidence limits in the Gumbel method. This point is further discussed in the following section.

A comparison of the confidence intervals (or bands) for predicted extremes in the orderostatistics method and in the Gumbel method is of interest and is presented in Table VIII, columns 2 and 4,5 and 7 s 8 and 10. It is to be noted that for prediction probability $P$ not beyond .99 the method of order-statistics gives an appreciably narrower band for sample sizes 20 and 30, and a significantly wider one for samples of 10. On the other hand, for values of P much beyond -99, the order statistics band widens very rapidiy, as compared to a constant width for the band in Gumbel's method. However, there appears to be some question as to the theoretical validity of the Gumbel confidence $=$ band width for large values of $P$. This point is also considered in the following section.

2. Confidence intervals for largest extremes in Gumbel method.

> 2.1 Gumbel's derivation of confidence intervals.

The purpose of this section is to inquire
into the theoretical validity of the confidence ino tervals (or confidence band) given for extreme predictions in Gumbel's method.

In the Gumbel method the 68-percent confidenceinterval halfowidth for the largest in a sample of $n$ extremes and for all larger predicted values\% is, in Gumbel's notation (Table VII, Part B, Section IV)

$$
\begin{equation*}
\Delta_{x_{0} n}=\frac{1.141}{a}=1.141 \beta ; \beta=1 / a, \tag{D.I}
\end{equation*}
$$

where $\beta$ is the scale parameter (or rather, an estimate of it) of the extreme-value distribution from which the observations are assumed to come. To obtain the confidence interval for a given prediction probo ability $P \geq n /(n+1)$, the value $\Delta_{x, n}$ is added to, and subtracted from, the estimate given by Gumbel, deo noted by him by $x$ (Table VII, Part $B_{\text {, Section III) }}$

FThat is, for all values of $P$ beyond $n /(n+1)$, which is the probability assigned to the largest fralue in the sample, $x_{n}$. For smaller $P_{9}$ the confidepee interval is given by a different method with which we shall not be concerned inasmuch as the promary interest is in large values of $P$ correspondirg to extreme predictions.
and in this report by $\hat{\xi}_{G}$ Gumbelis ( 68 percent) confidence interval for predictions beyond the largest observed extreme $x_{n}$ is thus given by

$$
\begin{equation*}
\xi_{G} \pm 1.141 \beta_{8} \tag{0.2}
\end{equation*}
$$

Where $\beta$ is the scale parameter (or an estimate thereof) of the extreme-value population from which the observed extremes $x$ have been assumed to come:

$$
F(x)=\Phi(y)=\exp \left(\infty e^{\infty y}\right), y=(x-u) / \beta \quad \quad(D, 3)
$$

The multipler 1.141 used for the 68 percent confidence band is obtained by setting $C=.68$ and solving for I the equation

$$
\begin{equation*}
\Phi(y)-\Phi(-y)=C \tag{D.4}
\end{equation*}
$$

Which is parameterofree, and gives $y(C)=y(.68)$ $=1.14073$ (reference 5, page 6)。 Thus

$$
\begin{equation*}
y=-1.14073 \text { to } y=1.14073 \tag{0.5}
\end{equation*}
$$

is the interval for the reduced variate that cuts off (or corresponds to) a. central area $0 \mathrm{I}^{\circ} .68$ under the extreme-value density curve shown in Figure $z_{0}$ The corresponding interval that cuts off the same

* From the theory of extreme values the distribution of the largest of the observed values, $X_{\text {p }}$ in a sample of $n$ extremes, is exactly an extreme-value distribution that has the same scale parameter $\beta$.

3
area under the original ("unreduced ${ }^{\text {P }}$ ) $x$-distribution thus has width given by the values (D.5) multiplied by the scale factor $\beta$, since

$$
x=u+y \beta \quad .
$$

The halfowidth is therefore $1.14073 \beta$, ioe.s (D.1).

The following discussion indicates that this method of obtaining confidence intervals is inaccurate in two respects: (1) the confidence interval is of constant instead of increasing width for large $P$; (2) the scale parameter used is inappropriate.

> 2.2 Constant width of confidence interval

The abovemethod of Gumbel of obtaining conm fidence intervals ( $D .2$ ) treats the estimator $\hat{C}_{G}$ as though it has an extremeovaIue distribution with the same seale parameter $\beta$ as in the population underlying the observed extremes $x_{m}$ (including the largest extreme $x_{n}$ )。 This assumption cannot be considered strictly valid since it implies that the confidence width remains constant for all large values of $\mathbb{P}_{s}$ as (D.I) does not invoive $P_{\text {. }}$ In other words, this asserts that from a sample of 20 observations ore en 10 , for example, we can make
statements about events that will occur with probo ability l in a million or billion yet have the same uncertainty of only $\Delta_{X, n}$ in our estimate for $x$ as for predictions about events with probability, say, I in 100. It does not seem reasonable that a limited sample can tell anything at all meaningful about such extremely rare events, let alone predict them with the same amount of uncertainty no matter what the probability of occurrence.

This lack of agreement with common sense indicates that the Gumbel estimator $\hat{\varepsilon}_{G}=\hat{u}+y_{P} \hat{\beta}$ carmot be treated, for all large values of $P_{s}$ as if it has an extremem value distribution with constant scale parameter.

Besides these common-sense considerations, there is another reason why the extrememvalue distribution is not appropriate for the Gumbel estimator, at least for large samples of data. The estimator is a sample characteristic of the form

$$
\begin{equation*}
\hat{\xi}_{G}=\bar{x}+k_{P} s_{X} \tag{D.6}
\end{equation*}
$$

where $k$ is a constant for given values of $P$ and $n$. The appropriate distribution of such an expression is given by a general limit theorem in probability (reference $I_{s}$ page 367) to the effect that under broad
conditions any sample characteristic based on moments (such as $\hat{\xi}_{G}$ ) is, for large values of $n$, approximately normally distributed. Thus for large values of $n$ the Gumbel estimator ( $D .6$ ) should be considered to be approximately normal, with variance given by an expression which increases as $k_{p^{\prime}}^{2}$ Moreover, this would yield a confidence band that diverges with increasing $P$, avoiding the difficulty of the parallel curves mentioned above.

### 2.3 Inappropriate scale parameter

Little is known about the exact distribution of the Gumbel estimator $\hat{\xi}_{G}$, particularly for small sample sizes. Yet even if it were an extremeavalue distribution (of the form (D.3)), it would seem that its scale parameter would not be $\beta$, but a certain multiple of it, $B_{p}$, found below. This multiple may be determined by considering the relation between the variance of the distribution (assumed extremevalue) of $\hat{\xi}_{G}$ and the scale parameter $\beta_{1}$ of this distribution:

$$
\begin{equation*}
\sigma^{2}\left(\hat{E}_{G}\right)=\frac{\pi^{2}}{6} \beta_{I}^{2} \tag{D.7}
\end{equation*}
$$

But we have an approximate expression for the left side, namely, ( $E .6$ ) in Appendix $E$ below. This is of the form

$$
\begin{equation*}
\sigma^{2}\left(\hat{\xi}_{G}\right)=q\left(y_{P}\right) \beta^{2} \tag{D.8}
\end{equation*}
$$

where $\beta$ is the scale parameter of the original (extreme-value) x-distribution, and $q\left(y_{p}\right)$ is a quadratic expression in the probability factor $y_{p}$ with coefficients involving the quantities $\sigma^{2}(s)$ and $\operatorname{cov}\left(\overline{y_{0}} s\right)$. whose computation by empirical sampling is indicated in Appendix $E ; q\left(y_{p}\right)$ may be regarded as a known value, $S_{P^{9}}$ depending on $P_{0}$ Fence

$$
\begin{equation*}
\sigma^{2}\left(\hat{\xi}_{G}^{r}\right)=s_{P}^{2} \beta^{2} \tag{D.9}
\end{equation*}
$$

Suostituting in (D.7) gives

$$
\begin{equation*}
\beta_{I}=\left(\frac{\sqrt{6}}{\pi} S_{P}\right) \beta=B_{P} \beta, \tag{D.10}
\end{equation*}
$$

which defines the multiple $B_{p}$. Thus the confidenceinterval halfowidth (D.I) must be replaced by

$$
\begin{equation*}
\Delta 0=1.141 \mathrm{~B}_{\mathrm{P}} \beta \tag{D.12}
\end{equation*}
$$

Where now $\Delta^{\prime}$ is no longer constant with $P$. but on account of $B_{p}$ actually increases veroy rapidiy for large values of $\mathrm{y}_{\mathrm{P}}$ corresponding to values of $P$ near 1 . Thus we obtain a modified confidence band whose divergence states that the amount of uno certainty increases without linit as wo autemp
estimate increasingly improbable events. This also avoids the conflict with common sense mentioned in Section 2.2.

The actual values of $B_{p}$ are of interest and are given in the following table for several important values of $P$ and for the three sample sizes for which they were computed in Appendix E.

| P | $B_{P}=\frac{\sqrt{6}}{\pi} S_{P}=\frac{\sqrt{6}}{\pi}\left[\sigma^{2}\left(\xi_{G}\right) / \beta^{2}\right]^{\frac{1}{2}}$ |  |  |
| :---: | :---: | :---: | :---: |
|  | $\underline{n}=10$ | $\underline{n}=20$ | $\underline{n}=30$ |
| $\begin{aligned} & .95 \\ & .99 \\ & .999 \end{aligned}$ | $\begin{array}{r} .749 \\ 1.093 \\ 1.593 \end{array}$ | $\begin{array}{r} .560 \\ .825 \\ 1.208 \end{array}$ | $\begin{aligned} & .458 \\ & .673 \\ & .986 \end{aligned}$ |

In this table the values of $B_{P}$ less than $I$ indicate that the modified confidence band (equation (D.1I)) is better (i.e. narrower) than the Gumbel confidence band and vice versa for the values of $B_{P}$ greater than 1. Thus, the modified $b$ and is indicated to be considerably better in the region $P=.95$ to .99 for samples of 20 and 30. For samples of 10, the advantage is less at $P=.95$ and becomes reversed in favor of the original Gumbel confidence band at $P=.99$.

The above comparison remains exactly the same for any other confidence level, it being merely necessary to replace 1.41 in equations (D.1) and (D.11) by the corresponding value $y(C)$ determined from (D.4). Thus, for the 95 percent level, $y(.95)=3.06685$ (reference 6, Lecture 3, Table 3.1). At each level the confidence intervals of the two methods are affected in the same ratio by such multipliers, that is, their ratio to each other remains $B_{p}$, regardless of confidence level C.
3. Comparison of confidence intervals in the Gumbel method and the me thod of order statistics

Table VIII shows the actual confidence intervals (in terms of the scale parameter $\beta$ ) for the two levels $C=.68$ and $C=.95$ for the Gumbel me thod and as modified by the factor $B_{P}$, and also compares these (where applicable) with the intervals given by the order-statistics method. Except for samples of 10, for which the Gumbel interval is apt to be narrower, the modification denoted by $B_{P}$, discussed in the previous section, reduces the intervalowidth for $P=.99$ (and less) by significant amounts - by 'about onesixth or more for samples of 20 (columns 5, 6) and by
about one-third or more for samples of 30 (columns 8, 9). These results are of course implied by the values of $B_{p}$, given in the preceding section. Also, as noted in Section 1 above, the ordermstatistics confidence interval is narrower than the (unmodified) Gumbel interval in many cases, for $P$ not beyond.99, and sample size: not below 20. Howeverg it increases beyond the constant Gumbel width for larger probabilities, in agreement with theoretical requirements. At $P=.99$ or less, there are two additional features to be noted. (I) With increasing confidence level, numerical the/factor in the Gumbel interval $\Delta_{X_{0}}$ increases faster in either the modified interval $\Delta^{8}$ or in the order-statistics interval (denoted by $\Delta_{0}$ in Table VIII), so that both the modified method and the orderstatistics method reduce the confidence interval of the Gumbel method by constantly increasing percentages as the confidence level increases. For example, for $P=.99$ and for samples of 20 the ordermstatistics interval is about 11 percent narrower than the Gumbel interval for a confidence level of 68 percent and about 30 percent narrower for a level of 95 percent (columns 5. 7). (2) Similarly, the percentage
-
reduction increases with sample size. Thus, for $P=.99$ and a confidence level of 68 percent, the reductions are 11 percent for samples of 20 , and 29
percent for samples of 30 (columns 8, 10).

## APPENDIX E

DETAILS OF THEORETICAL GOMPARISON BETWEEN ORDERSTATISTICS ESTIMATOR AND MOMENT ESTIMATOR OF GUMBEL (Related to Section 6.1 of text)

Since the order-statistics estimator has been fully discussed in the text, the remaining problem in making the above comparison is, essentially, to develop the characteristics of the Gumbel estimator.

The method of moments of Gumbel in present use provides the following estimators for the parameters $u_{8} \beta$ (reference 18, page 11, equations (26), (27); also reference 5, page 10, equation (29), but read $\left(\infty \bar{y}_{n} / a\right)$ for $\left.\left(+\bar{y}_{n} / a\right)\right)$ :

$$
\begin{equation*}
\hat{u}=x-\frac{\bar{y}_{n}}{\sigma_{n}} s_{x} \cdot \hat{\beta}=\frac{s_{x}}{\sigma_{n}} \tag{E.I}
\end{equation*}
$$

where $\bar{x}_{g} s_{X}$ are the mean and standard deviation of the given sample of size $n ; \bar{y}_{n}$ is a certain computed quantity, depending on the sample size $n$, which approaches Euler's constant $\gamma=.5772$... from below as $n$ becomes infinite; and $\sigma_{n}$ is another computed quantity, depending on $n$, which approaches $\pi / \sqrt{6}$ $=1.28255 \ldots$ from below as $n$ becomes infinite.

For sufficiently large samples the quantities $\bar{y}_{n}$ and $\sigma_{n}$ may be replaced by their limiting values*. This gives the somewhat simpler estimators, for computation purposes

$$
\begin{equation*}
\hat{u}^{v}=\bar{x}-\frac{\sqrt{6}}{\pi} \gamma s_{x}, \hat{\beta}^{8}=\frac{\sqrt{6}}{\pi} s_{x} . \tag{E.2}
\end{equation*}
$$

It is shown below that the net eifect of this simplification is to diminish the bias and to greatly understate the relative efficiency of the ordero statistics estimator to the Gumbel estimator. Since the asymptotic form (E.2) involves simpler notation and is occasionally used in practice, it has seemed desirable to present this case in detail below (Section 1) and also in the main text. The corresponding results for the original form (E.I) are indicated in Section 2 below and tabulated in Table VI.

1. Comparison with simplified Gumbel estimator

From the estimators (E.2) the following estimator of $\xi_{p}$ can be built up, which will be denoted by $\hat{\xi}_{G}$;

* This has been done, e.g.. in reference 3, pp. 181 ff. and 188, and in reference 18, po 10。


$$
\hat{\xi}_{G}=\hat{u}^{8}+\hat{\beta}^{\prime} y_{P}=\bar{x}+\left(y_{P}-\gamma\right) \cdot \frac{\sqrt{6}}{\pi} s_{x}
$$

This is a function of the $n$ sample values $x_{1}, x_{2}, \ldots$, $x_{n}$ and $i t$ is desired to find its mean and variance, and thence its bias and efficiency.

The mean is

$$
E\left(\hat{S}_{G}\right)=u+\gamma \beta+\left(y_{P}=\gamma\right) \frac{\sqrt{6}}{\pi} E(s) \beta \quad
$$

which can be rearranged to give

$$
\begin{equation*}
E\left(\hat{\xi}_{G}\right)=\xi_{P}+\left[\frac{\sqrt{6}}{\pi} E(s)-1\right]\left(y_{P}-\gamma\right) \beta \tag{E.4}
\end{equation*}
$$

where $\xi_{P}=u+y_{P} \beta, E(\bar{x})=u+\gamma \beta$, and $E(s)$ is the expected value of the sample standard deviation, $s$, when the sample is from the "reduced" extreme-value distribution $\exp \left(-e^{-\mathrm{y}}\right)$. Equation (E.4) shows that the Gumbel estimator is biased, (unless $E(s)=\pi / \sqrt{6}$ cor 211 sample sizes, which seems highly unlikely), with bias *An unbiased estimator analogous to $\hat{⿷}_{G}$ is

$$
\hat{E}_{0}=\bar{x}+\left(y_{P}-\gamma\right) s_{x} / E(s)_{,}
$$

for, as in equation ( $\mathrm{E} . \mathrm{H}_{4}$ )

$$
\begin{aligned}
\mathbb{E}\left(\hat{\xi}_{0}\right) & =u+\gamma^{\prime} \beta+\left(y_{P}-\gamma\right) E(s) \beta / E(s) \\
& =u+\beta y_{P}=\xi_{P}
\end{aligned}
$$

However, this estimator could not be used in an actual problem since $\mathrm{E}(\mathrm{s})$ is not known. Computation of this quantity was one of the aims of the IBM computing procedures discussed in the text.
$\mathrm{b}\left(\hat{\xi}_{G}\right)=\mathrm{E}\left(\hat{\xi}_{G}\right)-\xi_{P}=\left[\frac{\sqrt{6}}{\pi} \mathrm{E}(\mathrm{s})-1\right]\left(\mathrm{y}_{\mathrm{P}}-\partial\right)_{\beta}$.
The variance of the estimator $\hat{\xi}_{G}$ is

$$
\begin{align*}
\sigma^{2}\left(\hat{\xi}_{G}\right) & =\sigma_{\bar{x}}^{2}+\frac{6}{\pi^{2}}\left(y_{P}-\gamma\right)^{2} \sigma^{2}\left(s_{x}\right)+2\left(y_{P}-\gamma\right) \frac{\sqrt{6}}{\pi} \operatorname{cov}\left(\bar{x}_{,} s_{x}\right) \\
& =\left[\frac{\pi^{2}}{6 n}+\frac{6}{\pi^{2}}\left(y_{P}-\gamma\right)^{2} \sigma^{2}(s)+2\left(y_{P}-\gamma\right) \frac{6}{\pi} \operatorname{cov}\left(\bar{y}_{,} s\right)\right] \beta^{2}, \tag{E,6}
\end{align*}
$$

where $\sigma_{\bar{x}}^{2}=\frac{\pi^{2}}{6 n} ; \sigma^{2}(\mathrm{~s})$ is the variance of the sample standard deviation for samples from the reduced dis. tribution $\exp \left(-e^{-\mathrm{V}}\right)$; and $\operatorname{cov}(\overline{\mathrm{y}}, \mathrm{s})$ is the covariance of the mean and standard deviation in such samples. The efficiency of $\hat{\xi}_{G}$ could be evaluated by suitable generalization of equation (4019) to biased estimators. The variance $Q_{n}$ in the denominator would be replaced by the MSE (mean square error). * The numerator would have to be replaced by a complicated expression which, for unbiased estimators, would reduce to $Q_{L B}$. Instead of evaluating efficiency for the biased estimator $\hat{\xi}_{G}$, therefore, the discussion will be greatly simplified by limiting it to relative efficiency. The relative efficiency of one estimator

* For discussion of mean square error see equation (4.13) and accompanying text.
$\left(T_{1}\right)$ to another $\left(T_{2}\right)$ is defined as the ratio of mean square errors,

$$
\begin{equation*}
R\left(T_{1}, T_{2}\right)=\frac{\operatorname{MSE}\left(T_{2}\right)}{\operatorname{MSE}\left(T_{1}\right)} \tag{E.7}
\end{equation*}
$$

This ratio has been used as an index of comparison of two estimators (e.g. reference 7). Thus, the relative efficiency of the order-statistics estimator $\hat{\xi}_{P}$ to the Gumbel estimator $\hat{\xi}_{G}$ is, by (4.13) and the fact that the former estimator is unbiased.

$$
\begin{align*}
R\left(\hat{\xi}_{P}, \hat{\xi}_{G}\right) & =\frac{\operatorname{MSE}\left(\hat{\xi}_{G}\right)}{\operatorname{MSE}\left(\hat{\xi}_{P}\right)}=\frac{\sigma^{2}\left(\hat{\xi}_{G}\right)+\left(\text { bias }\left(\hat{\xi}_{G}\right)\right)^{2}}{\sigma^{2}\left(\hat{\xi}_{P}\right)} \\
& =\frac{(E .6)+(E .5)^{2}}{(1 / k) Q_{m}} \tag{E.8}
\end{align*}
$$

where $k$ is the number of subgroups of size $m$ into which the sample of n is partitioned ${ }^{* *}$ (equation (4.23), assuming there is no remainder subgroup), and the expressions needed for the numerator are given by the equation numbers indicated.

The key quantities needed in the calculation of relative efficiencies are, from equations (E.5) and ( $\mathrm{E}, 6$ ), $\mathrm{E}(\mathrm{s}), \sigma^{2}(\mathrm{~s})$, and $\operatorname{cov}(\overline{\mathrm{y}}, \mathrm{s})$. For general
F. Thus, $n=10=2 \times 5$ gives $k=2, m=5 ; n=20=$ $4 \times 5$ gives $k=4, m=5 ; n=30=5 \times 6$ gives $k=5$, $m=6$ 。
-
sample size $n$, their exact values are given by multiple integrals whose evaluation would apparently require a prohibitive amount of labor. Instead, the following method of empirical sampling was used with the aid of IBM calculating and tabulating equipment.

The universe of (reduced) extreme values $\Phi(y)$ $=\exp \left(-e^{-\mathrm{J}}\right)$ was approximated by constructing a population of 12,000 suitable random numbers and punching each number on an IBM punch card. These were then mechanically separated into 1200 random samples of size $n=10$ and for each sample the mean $\bar{y}$, standard deviations, and their product $\overline{\mathrm{y}}$ s were obtained. This was equivalent to having a "population" of 1200 means, one of 1200 standard deviations, and one of 1200 products of the mean and standard deviation. It was then assumed that the arithmetic mean of each of the three populations would be a close approximation to the mathematical expectations (averages) of the desired quantities, so that these approximations could be taken as estimates of the moments $\mathrm{E}(\mathrm{s})_{0}$ $E(\bar{y} s)$. From these values, and the relation

$$
E\left(s^{2}\right)=\frac{n-1}{n} \sigma_{y}^{2}=\frac{n-1}{n} \frac{\pi^{2}}{6}
$$

the variance

$$
\sigma^{2}(s)=E\left(s^{2}\right)-[E(s)]^{2}=\frac{n-1}{n} \frac{\pi^{2}}{6}-[E(s)]^{2}
$$

was computed, and also the covariance

$$
\operatorname{cov}(\bar{y}, s)=E(\bar{y} s)-E(\bar{y}) E(s)=E(\bar{y} s)-\gamma E(s)
$$

The five quantities $E(\bar{y}), \sigma^{2}(\bar{y}), E(s), \sigma^{2}(s)$, and $\operatorname{cov}\left(\vec{y}_{8} s\right)$ are shown in Table IX, together with the corresponding theoretical values that can be readily calculated.

In actual use this procedure was modified somewhat, since only one value of each of the desired quantities would be produced by the 12,000 cards and 1200 samples. This single value would be subject to the fluctuations of random sampling and would be difficult to rely on in making inferences. This difficulty was met by breaking the "population" of 1200 samples into 12 sets of 100 samples and obtaining 12 values of each of the desired moments instead of only one. These 12 values, although each was based on fewer samples, served to furnish an idea of how the single value based on 1200 samples was affected by sampling variation. Such analysis has provided a far firmer basis for judgement of relative efficiency.
-
-The above procedure resulted in moments cal. culated for samples of size $\mathrm{n}=10$. In like manner, 600 random samples of size $n=20$ were drawn, after starting afresh by putting all 12,000 cards together, but this time only 6 instead of 12 sets of 100 samples were available, resulting in 6 values of the desired quantities for comparisome Finally, the 12,000 cards were reprocessed to yíeld 400 samples of size $\mathrm{n}=30$, giving 4 values each based on a set of 100 samples.

The resulting sets of 12,6 , and 4 values each were substituted in the appropriate formulas ( E .5 ), (E.6), (E.8) in order to obtain the relative efficiency of the orderastatistics estimator to the (simplified) Gumbel estimator. These formulas, all of which depend upon $\mathrm{X}_{\mathrm{P}}$ were evaluated at the probability level $P=.95$. Wll these results are summarized in Table $V$ which shows the values of the bias, mean square errore and relative efficiency calculated for each set of 100 sampies of sizes 10,20 , and 30 , together with the corresponding average values obtained from all 1200 samples combined.
-

For ease of comparison, the relative efficiencies are also charted in Figure 6.

These results constitute the basis of the statement in the text that at the probability level $P=.95$, for samples of 20 and 30 , the proposed method has greater efficiency than the Gumbel method using ther simplified estimator, while for samples of 10 the efficiencies are about the same.

## 2. Comparison with original Gumbel estimator

The estimator corresponding to $\hat{\xi}_{G}$ in ( $E_{0}$ ) ) , built up from the estimators ( E .1 ) is

$$
\begin{equation*}
\hat{\boldsymbol{\xi}}_{G, n}=\hat{u}+\hat{\beta} \hat{y}_{P}=\bar{x}+k_{n} s_{x} \tag{E.9}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{n}=\frac{\bar{y}_{n}-y_{P}}{\sigma_{n}}=b_{n, P} \cdot d \tag{E,I0}
\end{equation*}
$$

Here
and

$$
\begin{equation*}
\mathrm{b}_{\mathrm{n}, \mathrm{P}}=\frac{\pi / \sqrt{6}}{\sigma_{n}} \cdot \frac{\mathrm{y}_{\mathrm{P}} \overline{\mathrm{y}}_{n}}{\mathrm{y}_{\mathrm{P}}{ }^{-j}} \tag{E.II}
\end{equation*}
$$

is the conversion factor for passing from the multiplier, $d_{\text {, of }} s_{x}$ in ( $\mathrm{E}_{\mathrm{o}}$ 3) to $\mathrm{k}_{\mathrm{n}}$ in ( E .9 ). It
is apparent from the discussion at the beginning of this Appendix that for infinitely large $n_{9} b_{n_{8}} P=1$, so that (E.9) includes the asymptotic case. For finite $n$, however, $\sigma_{n}<\pi / \sqrt{6}, \overline{\mathrm{y}}_{\mathrm{n}}<\gamma$. Hence, $\mathrm{b}_{\mathrm{n}_{8} \mathrm{P}}$, being a product of two factors each greater than 1 , may considerably exceed 1 , so that the multiplier $\mathrm{b}_{\mathrm{n}, \mathrm{P}}$ in (E.10) and (E.II) becomes appreciably larger than the multiplier in (E.3). Thus, for samples of 10, 20, 30, computation shows that, for $P=.95$.
.

$$
\begin{align*}
& k_{10}=1.397 \mathrm{~d} \\
& \mathrm{k}_{20}=1.234 \mathrm{~d} \\
& \mathrm{k}_{30}=1.173 \mathrm{~d} \tag{E.12}
\end{align*}
$$

Thè bias of $\hat{\xi}_{G, n}$ is, in manner similar to Section 1 , in view of (E.10).

$$
\begin{aligned}
b\left(\hat{\xi}_{G_{,},}\right) & =\left[-\left(y_{P} \sim \gamma\right)+k_{n} E(s)\right] \beta \\
& =\left(y_{P}-\phi\right)\left[\frac{\sqrt{6}}{\pi} E(s) b_{n_{\theta} P}-1\right] \beta \quad \text { (E.13) }
\end{aligned}
$$

Table VI (columns 2 and 3) indicates that the presence of the factor $b_{n, p}$ converts the small negative biases into larger positive ones.

For the variance we have from (E.9), analogously to Section 1,

$$
\sigma^{2}\left(\hat{\xi}_{G, n}\right)=\left[k_{n}^{2} \sigma^{2}(s)+2 k_{n} \operatorname{cov}(\bar{y}, s)+\frac{\pi^{2}}{6 n}\right] \beta^{2} \quad \text { (E.14) }
$$

The corresponding expression (E.6) may be written

$$
\sigma^{2}\left(\hat{\xi}_{G}\right)=\left[d^{2} \sigma^{2}(s)+2 d \operatorname{cov}(\bar{y}, s)+\frac{\pi^{2}}{6 \pi}\right] \beta^{2} \quad(E .15)
$$

Comparison of these two expressions shows, since $\operatorname{cov}(\bar{y}, s)$ was found to be positive, that replacement of $d$ by the larger value $k_{n}$ considerably increases the variance of the Gumbel estimator. Values of the variance for the original and simplified estimators are listed in columns 4 and 5 of Table Vi.

Comparison of these columns indicates that the variance of the original estimator can become more than half again as large as the variance of the simplified estimators, depending on sample size and probability P. The effect is most marked for the lower levels of $P$ and smaller sample sizes and disappears as shown When both these factors increase。

The result of the above increases in bias and variance is to greatly increase the mean square error (columns 6 and 7,Table VI), and thus to increase the relative efficiency of the order-statistics estimator (columns 8, 9)。 As a result the orderastatistics estimator is up to twice as efficient as the original Gumbel estimator even for samples as small as 10. This tremendous increase in efficiency falls off slowly, as shown, when sample size increases. For fixed sample size the efficiency increases for large values of $P$. These differences in efficiency are sufficiently large to completely outweigh any fluctuations of random sampling attributable to the empirical sampling method of evaluation used.

It must be concluded therefore, that the
original Gumbel estimator is both more biased and murk less efficient than its simplified formo A soutt,
comparison of the orderastatistics estimator with the original Gumbel estimator gives very conservative results and greatly understates the actual improve= ment in efficiency of the proposed method over the method in present use.

# ESTIMATION OF EXTREMES <br> (For instructions see Section 5) 

Worksheet 1 - Determination of Estimators
Source: NACA - Langley — Sample III $\quad$ Computer: $\frac{\mathrm{J}_{0} \text { L. }}{5 / 29 / 52}$

| Record <br> No. | + $\Delta n$ <br> Obeerved <br> extremee |
| :---: | :---: |
| 1 | .75 g |
| 2 | .90 |
| 3 | 1.08 |
| 4 | 1.20 |
| 5 | 1.38 |
| 6 | .81 |
| 7 | .80 |
| 8 | .75 |
| 9 | .90 |
| 10 | 1.20 |
| 11 | .88 |
| 12 | 1.08 |
| 13 | 1.15 |
| 14 | 1.00 |
| 15 | 1.31 |
| 16 | 1.43 |
| 17 | .98 |
| 18 | 1.02 |
| 19 | 1.01 |
| 20 | .93 |
| 21 | 1.15 |
| 22 | .75 |
| 23 | 1.16 |
| Sum | 23.62 |
|  |  |

I. SUBCROUP SIZES AND PROPORTIONALTTY FACTORS:

$$
\begin{array}{llrl}
n=23 & =k m+m^{\prime} & t=k m / n=0.78261 & t^{\prime}=m^{1 / n}=0.21739 \\
& =3 \times 6+\underline{5} & t^{2} / k=0.20416 & t^{\prime}=\underline{0.02726} \\
k=3 & m=6 & m^{\prime}=5
\end{array}
$$

IIA. MAIN SUBGROUPS:

| $i=$ | 1 | 2 | 3 | 4 | 5 | 6 | Creck sum |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{i}=$ | . 35545 | . 22549 | . 16562 | . 12105 | . 08352 | . 04887 | 1 |
| $\mathrm{b}_{1}=$ | -. 45928 | -. 03599 | . 07379 | . 12673 | .14953 | .14581 | -. .00002 |



IIB. REMAINDER SUBGROUP:
Weights $a_{1}^{\prime}, b_{1}^{\prime}$ (from Table I)

| 1 | $=$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{i}$ | $=$ | .47893 | .24628 | .16761 | .10882 | .05835 |
| $b_{i}$ | $=$ | -.50313 | .00654 | .13045 | .18166 | .18448 |
|  |  |  |  | Check sum |  |  |

Observations $x_{1}^{\prime}$ in increasing order from $1=1$ to $1=\mathrm{m}^{\prime}$

$$
\begin{aligned}
& \begin{array}{rrrrrccc}
x_{1}^{\prime} & x_{2}^{1} & x_{3}^{\prime} & x_{4}^{1} & x_{5}^{1} & x_{6}^{1} & \text { Check sum } & \sum a_{1}^{1} x_{1}^{1} \\
.75 & .93 & 1.01 & 1.15 & 1.16 & & 5.00 & .90535 \\
\hline 18 & .1840
\end{array} \\
& T^{\prime}=\Sigma a_{i}^{1} x_{1}^{\prime}+\left(Y 0_{1}^{\prime} x_{1}^{\prime}\right) Y_{P}=.90535+.18340 y_{P}
\end{aligned}
$$

III. ESTIMATORS:

$$
\begin{gathered}
\hat{\xi}_{P}=t T^{\prime}+t^{\prime} T^{\prime}=.92946+\frac{.16774}{} y_{P} \\
u=.92946, \beta=.16774
\end{gathered}
$$

.

ESTIMATION OF EXTREMES
Worksheet 2 - Predioted values, confidenoe band, efficiency, plotting positions


PLOTTING POSITIONS
Ooserved extremes in increasing rank from 1 to $n=23$

| $\begin{gathered} \text { Rank } \\ r \end{gathered}$ | Observed <br> Extreme | Plotting Position $r /(n+1)$ | $\begin{gathered} \text { Rank } \\ r \end{gathered}$ | Observed <br> Extreme | $\begin{aligned} & \text { Plotting Position } \\ & r /(n+1) \end{aligned}$ | $\begin{gathered} \text { Rank } \\ \mathbf{r} \end{gathered}$ | Observed Extrene | Plotting Position $r /(n+1)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | . 75 | . 0427 | 11 | 1.00 | . 4583 | 21 | 1.31 | . 8750 |
| 2 | . 75 | . 0833 | 12 | 1.01 | . 5000 | 22 | 1.38 | . 9167 |
| 3 | . 75 | . 1250 | 13 | 1.02 | . 5417 | 23 | 1.43 | . 9583 |
| 4 | . 80 | . 1667 | U | 1.08 | . 5833 | Sum | 23.62 |  |
| 5 | . 81 | . 2063 | 15 | 1.08 | . 6250 |  |  |  |
| 6 | . 88 | . 2500 | 16 | 1.15 | . 6667 |  |  |  |
| 7 | . 90 | . 2917 | 17 | 1.15 | . 7083 |  |  |  |
| 8 | . 90 | . 3333 | 18 | 1.16 | . 7500 |  |  |  |
| 9 | . 93 | . 3750 | 19 | 1.20 | . 7917 |  |  |  |
| 10 | . 98 | . 4167 | 20 | 1.20 | . 8333 |  |  |  |

Weights for minimu-varimze, unbiased, linear order-statistics estimator $I=\hat{\xi}_{P}=\sum_{i=1}^{n}\left(a_{i}+b_{i} y_{P}\right) x_{i, ~}$ of percentage points $\xi_{P}=u+\beta y_{P}$ and paxiance $\operatorname{Var}\left(\hat{\xi}_{P}\right)=Q_{n}=\left(A_{n} y_{p}^{2}+B_{n} \sum_{p}+G_{n}\right) \beta_{9}^{2}$ for sample size $n=2$ to $\sigma_{2} x_{2} \leqslant x_{2} \leqslant \ldots \leqslant x_{n}$ (Discussed in Section 4.3)

| $n$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  | $\mathrm{XI}_{2} \mathrm{~F}_{2} \mathrm{x}_{3}$ ( $\mathrm{x}_{4}$ | ${ }^{3} 5$ | $x_{6}$ |
| 2 | $\begin{aligned} & a_{0} \\ & b_{1} \\ & Q_{2} \end{aligned}$ | $\begin{aligned} & +0.42637 \quad+0.08363 \\ & -0.72135 \quad+0.721 .35 \\ & \left(0.71186 \bar{y}_{p}^{2}-0.13864 y_{p}+0.65955\right) \beta^{2} \end{aligned}$ |  |  |
| 3 | $\begin{aligned} & a_{i} \\ & b_{1} \\ & Q_{3} \end{aligned}$ | $\begin{aligned} & +0.65632+0.25571 \quad+0.08797 \\ & =0.63054+0.25582 \quad+0.37473 \\ & \left(0.34472 y_{\mathrm{P}}^{2}+0.04354 y_{\mathrm{P}}+0.40286\right) \beta^{2} \end{aligned}$ |  |  |
| 4 | $\begin{aligned} & a_{i} \\ & b_{i} \\ & Q_{4} \end{aligned}$ | $\begin{array}{lll} +0.58100+0.26394 & +0.15368 & +0.07138 \\ -0.55858+0.08590 & +0.22392 & +0.24880 \\ \left(0.22588 y_{p}^{2}+0.09938 \Psi_{p}+0.29346\right) \beta^{2} \end{array}$ |  |  |
| 5 | $\begin{aligned} & a_{1} \\ & b_{1} \\ & a_{5} \end{aligned}$ | $\begin{array}{lll} +0.41893 & +0.24686 & +0.16761 \end{array}+0.10882$ | $\begin{aligned} & +0,05835 \\ & +0,28448 \end{aligned}$ |  |
| 6 | $\begin{aligned} & a_{i} \\ & b_{i} \\ & Q_{6} \end{aligned}$ | $\begin{array}{lll} +0.35545 & +0.22549 & +0.16562 \end{array}+0.12105$ | $\begin{array}{r} +0.08358 \\ +0.14953 \end{array}$ | $\begin{aligned} & +0.04887 \\ & +0.14581 \end{aligned}$ |

TABLW II
Means, variances, and covariances of order statistics $y_{\text {g }}$ in samples of $n$ from the
reduced extremeavalue distribution $F(y)=\exp \left(-e^{-y}\right), n=2$ to $6, y_{I} \leqslant \Psi_{2} \leqslant \ldots \leqslant \mathcal{F}_{n}$

| n | 1. | MEANS Eyis | VARIANCES AND COVARIANCES, $\sigma_{i j}=\sigma_{j i}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $j=1$ | $j=2$ | $j=3$ | $j=4$ | $j=5$ | $j=6$ |
| 2 | 1 | $\begin{array}{r} 0.11593152 \\ +1.27036285 \\ \hline \end{array}$ | $0.68402804$ | $\begin{aligned} & 0.48045301 \\ & 1.64493407 \end{aligned}$ |  |  |  |  |
| 3 | $\begin{aligned} & 1 \\ & 2 \\ & 3 \\ & \hline \end{aligned}$ | $\begin{aligned} & 90.40361359 \\ & +0.45914263 \\ & +1.67582795 \end{aligned}$ | 0.44849796 | $\begin{array}{ll} 0.30137 & 114 \\ 0.65852 & 235 \end{array}$ | $\begin{aligned} & 0.24375810 \\ & 0.54629438 \\ & 1.64493407 \\ & \hline \end{aligned}$ |  |  |  |
| 4 | 1 2 3 4 | $\begin{aligned} & -0.57351263 \\ & +0.10608352 \\ & +0.81278175 \\ & +1.96351 \quad 003 \end{aligned}$ | 0.34402427 | $\begin{array}{ll} 0.22455 & 344 \\ 0.401553 & 113 \end{array}$ | $\begin{aligned} & 0.17903454 \\ & 0.33720966 \\ & 0.65180236 \end{aligned}$ | $\begin{aligned} & 0.15388 \\ & 0.29271 \\ & 0.57432 \\ & 1.64496 \\ & 407 \end{aligned}$ |  |  |
| 5 | $\begin{aligned} & 1 \\ & 2 \\ & 3 \\ & 4 \\ & 5 \\ & \hline \end{aligned}$ | $\begin{array}{rl} -0.69016 & 715 \\ -0.10689 & 454 \\ +0.42555 & 061 \\ +1.07093 & 582 \\ +2.18665 & 358 \\ \hline \end{array}$ | 0.28486447 | $\begin{aligned} & 0.18202536 \\ & 0.30849748 \end{aligned}$ | $\begin{aligned} & 0.14358737 \\ & 0.24676731 \\ & 0.40598 \\ & 292 \end{aligned}$ | $\begin{aligned} & 0.12257865 \\ & 0.21226644 \\ & 0.35267072 \\ & 0.64907319 \end{aligned}$ | 0.10901329 0.18967383 0.31716095 0.58991519 1.64493407 |  |
| 6 | 1 2 3 4 5 6 | $-0.77729368$ $-0.25453448$ $+0.18838534$ +0,66271588 <br> $+1.27504579$ <br> +2.36897513 | 0.2465820 | $\begin{aligned} & 0.1549674 \\ & 0.2485456 \end{aligned}$ | $\begin{aligned} & 0.1212161 \\ & 0.1 .967062 \\ & 0.2976159 \end{aligned}$ | $\begin{aligned} & 0.1029164 \\ & 0.1680628 \\ & 0.2561660 \\ & 0.40185 \end{aligned} 52$ | 0.0911619 0.1494532 0.2288790 0.3614555 0.6476996 | 0.08285 42 <br> 0.1361 .19 10 <br> 0.20925 46 <br> 0.33204 51 <br> 0.59985 67 <br> 1.664493 41 |

TABLE III
Variances and efficiencies of minimum-variance, unbiased, linear order-statistics estimator, $L=\xi$, for selected probability levels, $P_{9}$ for sample size $n=2$ to 6 (Discussed in Section 4.3 ; variances based on Table I)

| PART A: VARIANCES (Units of $\beta^{2}$ ) |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| P | $\mathrm{J}_{\mathrm{P}}$ | Q ${ }_{0}^{\text {² }}$ | Q 2 | Q ${ }_{3}$ | $Q_{4}$ | $Q_{5}$ | Q6 |
| . 36788 | 0 | 1.10866 | 0.65955 | 0.40286 | 0.29346 | 0.23140 | 0.19117 |
| . 40 | 0.08742 | 1. 15825 | 0.65374 | 0.40983 | 0.30125 | 0.23861 | 0.19766 |
| . 50 | 0.36651 | 1.37873 | 0.70802 | 0.46732 | 0.34915 | 0.27870 | 0.23189 |
| .60 | 0.67173 | 1.72827 | 0.89434 | 0.59168 | 0.44172 | 0.35225 | 0.29286 |
| . 70 | 1.03093 | 2.28472 | 1.28350 | 0.82030 | 0.60442 | 0.47859 | 0.39611 |
| . 80 | 1.49994 | 3.24743 | 2.06814 | 1.25271 | 0.90437 | 0.70829 | 0.58218 |
| .90 | 2.25037 | 5.34440 | 3.97502 | 2.26002 | 2.59046 | 1.22831 | 1.00065 |
| .95 | 2.97920 | 7.99866 | 6.55752 | 3.59108 | 2.48700 | 1.90349 | 1.54271 |
| . 975 | 3.67625 | 11.21444 | 9.80724 | 5.24369 | 3.59317 | 2.73352 | 2.20527 |
| -99 | 4.60016 | 16.33798 | 15.13171 | 7.92536 | 5.37994 | 4.07062 | 3.27230 |
| -999 | 6.90726 | 33.66365 | 33.73386 | 27.19133 | 11.52099 | 8.65174 | 6.92044 |
| $1 * *$ | $\infty$ | $0.60793 \mathrm{y}_{\mathrm{P}}^{2}$ | $0.71286 \mathrm{y}_{P}^{2}$ | $0.344717_{P}^{2}$ | $0.22528 \mathrm{~J}_{\mathrm{P}}^{2}$ | $0.16665 \mathrm{y}_{P}^{2}$ | $0.1 .3196 \mathrm{y}_{P}^{2}$ |

[^10]TABLE III，contimued
Variances and efficiencies of minimaspariance，unbiased，
（Discassed in Section 403）

| PART B：EFFICIENCIES |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| P | $\Psi_{P}$ | $\mathrm{E}_{2}=\frac{1}{2} \mathrm{Q}_{0} / \mathrm{Q}_{2}$ | $E_{3}=\frac{2}{3} Q_{0} / Q_{3}$ | $E_{4}=\frac{1}{4} Q_{0} / Q_{4}$ | $E_{5}=\frac{1}{5} Q_{0} / Q_{5}$ | $E_{6}=\frac{1}{6} Q_{0} / Q_{6}$ |
| $.36988 *$ | 0 | 0.8405 | 0.9173 | 0.9445 | 0.9582 | 0.9666 |
| ． 40 | 0.08742 | 0.8859 | 0.9421 | 0.9612 | 0.9708 | 0.9766 |
| ． 50 | 0.36651 | 0.9737 | 0.9834 | 0.9872 | 0.9894 | 0.9909 |
| ． 60 | $0: 67173$ | 0.9662 | 0.9737 | 0.9781 | 0.9813 | 0.9836 |
| .70 | 1.03093 | 0.8900 | 0.9284 | 0.9450 | 0.9548 | 0.9613 |
| －80 | 1.49994 | 0.7851 | 0.8642 | 0.8977 | 0.9170 | 0.9297 |
| ．90 | 2.25037 | 0.6722 | 0.7882 | 0.8400 | 0.8702 | 0.8901 |
| ． 95 | 2.97020 | 0.6099 | 0.7425 | 0.8040 | 0.8404 | 0.8647 |
| ． 975 | 3.67625 | 0.5717 | 0.7129 | 0.7803 | 0.8205 | 0.8475 |
| ．99 | 4.60016 | 0.5399 | 0.6872 | 0.7592 | 0.8027 | 0.8321 |
| －999 | 6.90726 | 0.4990 | 0.6527 | 0.7305 | 0.7782 | 0.8107 |
| I ${ }^{\text {\％}}$ | $\infty$ | 0.4270 | 0.5879 | 0.6746 | 0.7296 | 0.7678 |
| These lines give the variances and efficiencies for the ordermstatistics estimator of the parameter $u$ 。 |  |  |  |  |  |  |
| ＊Limiting efficiency as $P$ approaches 1 。 These are also the efficiencies for the estimator of the parameter $\beta$ 。 |  |  |  |  |  |  |

Efficiency of ordermstatistics estimatons fog vaxious sample sizes
$n=k m+m^{0}$ partitioned into subgroups as indicated for $P=099, P=1$
(Discussed in Section 4. 4 )

| n | $\mathrm{km}+\mathrm{mb}^{8}$ | EFFICIENCY |  | 88 | $\underline{x} m+m 8$ | EFFICIENCY |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $P=.9$ | $p=1$ |  |  | $P=.99$ | $\mathrm{P}=1$ |
|  |  |  |  |  |  |  | t) |
| 2 or $k \cdot 2$ |  | 54.0 | 42.7 | 29 | $3 x 6+3$ | 80.8 | 73.6 |
| 3 or ko3 |  | 68.7 | 58.8 | 22 | $3 \times 6+4$ | 81.8 | 74.9 |
| 4 or kol |  | 75.9 | 67.5 | 23 | $3 \times 6+5$ | 82.6 | 75.9 |
| 5 or k05 |  | 80.3 | 73.0 | 24 | $4 \times 6$ | 83.2 | 76.8 |
| 6 or kob |  | 83.2 | 76.8 | 25 | $5 \times 5$ | 80.3 | 73.0 |
| 73 | $1 \times 5+2$ | 70.5 | 60.7 | 26 | $4 \times 6+2$ | 79.9 | 72.3 |
| $8{ }^{\text {b }}$ | 1x6+2 | 73.3 | 63.8 | 27 | $486+3$ | 81.3 | 74.3 |
| 9 | 2x6+3 | 77.7 | 69.7 | 28 | $4 \times 6+4$ | 81.9 | 75.3 |
| 10 | $2 \times 5$ | 80.3 | 73.0 | $\begin{aligned} & 29 \\ & 30 \end{aligned}$ | $4 \times 6+5$ | $\begin{aligned} & 82.7 \\ & 83.2 \end{aligned}$ | 76.1 |
| 11 | $1 \times 6+5$ | 81.9 | 75.0 | 31 | $5 \times 5+6$ | 80.8 | 73.7 |
| 12 | $2 \times 6$ | 83.2 | 76.8 | 32 | $5 \times 6+2$ | 80.5 | 73.1 |
| 13 c | $2 \times 5+3$ | 77.3 | 69.1 | 33 | $586+3$ | 81.6 | 74.7 |
| 14 c | $288+2$ | 74.7 | 66.7 | 34 | $5 \times 6+4$ | 82.3 | 75.6 |
| 15 | $3 \times 5$ | 80.3 | 73.0 | 35 d | $7 \times 5$ | 80.3 | 73.0 |
| 16 | $2 \times 6+4$ | 81.3 | 74.2 | 36 | $6 \times 6$ | 83.2 | 76.8 |
| 17 | $2 \times 6+5$ | 82.3 | 75.6 | 37 | $7 \times 5+2$ | 78.2 | 70.3 |
| 18 | $3 \times 6$ | 83.2 | 76.8 | 38 | $6 \times 6+2$ | 80.9 | 73.7 |
| 19 | $3 \times 5+4$ | 79.3 | 71.7 | 39 | $6 \times 6+3$ | 81.9 | 75.0 |
| 20 | $4 \times 5$ | 80.3 | 73.0 | $40^{\circ}$ | $8 \times 5$ | 80.3 | 73.0 |
|  |  |  |  | 61 | $11 \times 5+6$ | 80.6 | 73.3 |

a If partition is $7=2 \pi 4$ - 3, then einiciencies ase:
$P=099$, effo $=72.79 \% \quad P=I_{\text {, }}$ effo $=63.4 \%$
b If partition is $8=2 \times 4$, then efficiencies are:

$$
P=099_{\text {, effo }}=75.9 \% \quad P=I_{g} \text { effo }=67.5 \%
$$

© If partition is $14=2 x 5+4$, then efficiencies are:

$$
\mathrm{P}=.99 \text { offo }=79.2 \% \mathrm{~F}=\mathrm{I}_{9} \text { effo }=71_{0} .3 \%
$$

If partition is $35=5 \times 6+5$, then efficiencies are:

$$
P=09, \text { effo }=82.8 \% \% \quad P=1, \text { effo }=76.2 \%
$$

$\ell$ If partition is $40=6 \times 6+4$, then efficiencies are:

$$
P=099_{\text {, effo }}=82.4 \% \% \quad P=I_{\text {, eff。 }}=75.7 \%
$$

TABLEE V
Biases, mean square errors, and relative efficiencies of proposed ordermstatistics $x^{\hat{\xi}} \mathrm{P}$ to Gumbel estinators $\mathrm{F}_{\mathrm{G}^{9}}$ based on empirical sampling results
ained from $N$ samples, for $P=095$ and sample size $n=10_{3} 20_{2} 30$
(Discussed in Appendix $E_{9}$ values of $R$ shown in Figure 6)

| ```Set No. (100 sample each)``` | (Averag set | BIAS value 200 samp | each Les) | $\begin{array}{r} \text { NEAN SQU } \\ \text { (ATerag } \\ \text { set } \\ \hline \end{array}$ | ARE ERROR <br> value <br> 100 <br> sam | (MSE) <br> each les) | RELAT | VE EFFICIEN $R=M S E / Q$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (1) | $n=10$ $(2)$ | $n=20$ $(3)$ | $n=30$ $(4)$ | $n=10$ (5) | $n=20$ $(6)$ | $n=30$ $(7)$ | $\begin{gathered} \mathrm{n}=10 \\ \mathrm{Q}=.95174 \beta^{2} \end{gathered}$ <br> (8) | $\begin{gathered} n=20 \\ Q=047587 \beta^{2} \\ 9) \end{gathered}$ | $\begin{aligned} & n=30 \\ & Q=.30834 \beta^{2} \\ & (10) \end{aligned}$ |
|  |  | nits of $\beta$ |  |  | nits of $\beta$ |  |  |  |  |
|  | $N=1200$ | $N=600$ | $N=400$ | $N=1200$ | $\mathrm{N}=600$ | $N=400$ | $N=1200$ | $N=600$ | $N=400$ |
| 1 | . -2.25968 | . .08075 | -.03772 | . 93875 | . 56707 | - 32762 | . 986 | 1.192 | 1.062 |
| 2 | -. 25375 | -. 05465 | -.06069 | 1.13608 | - 39516 | . 25458 | 2.194 | -830 | . 826 |
| 3 | 0.16007 | -. 17221 | . . 08280 | . 85744 | . 52061 | - 39850 | . 901 | 1.094 | 1.292 |
| 4 | -. 34090 | . .05595 | ¢.09250 | . 80648 | . 59028 | . 41688 | . 847 | 1.240 | 1.352 |
| 5 | -. 16138 | -. 20845 |  | . 81749 | . 58173 |  | . 859 | 1.222 |  |
| 6 | 0.25951 | -. 211583 |  | 1.13818 | . 52853 |  | 1.196 | 1.117 |  |
| ? | -.10497 |  |  | . 88266 |  |  | . 927 |  |  |
| 8 | -. 15613 |  |  | . 92989 |  |  | . 977 |  |  |
| 9 | -. 21475 |  |  | . 87389 |  |  | . 918 |  |  |
| 20 | -. 18457 |  |  | 1. 22356 |  |  | 1.286 |  |  |
| 11 | -. 29129 |  |  | 1.00460 |  |  | 1.056 |  |  |
| 12 | -.32256 |  |  | 1,02068 |  |  | 1.072 |  |  |
| Average | -. 21827 | -. 11464 | -. 06842 | . 96914 | . 53056 | . 34940 | 1.018 | 1.115 | 1.1 .33 |
| Proportion | of sets Iavorable to the prop |  |  | ed estin | ror ( $R$ ) | I): | out of 12 | out of 6 | out of 4 |

[^11]$\square$
.
TABIE VI

(asymptotic) form $\hat{\xi}_{G}$ for sample size $n=10,20,30$, and $n$ infinite, for $P=095$, 99 , I (Discussed in Appendix E)

For values of $Q_{9}$ see Table $V_{9}$ headings for cols. 8, 9, 10.
** For $p=1$, all quantities except selative efficiency are infinite, for finite sample size.
 values of $P_{0}$ $*$
$*$
,


TABLE VII - PART B PROBABILITIES OF EXTREMES
(Discussed in Section 6.2)
I. Mean and Standard Deviation:
$N=23 \quad \Sigma(x p)=23.62^{\circ}$
$\Sigma\left(x^{2} p\right)=25.1326$
$\sqrt{N}=4.79583$ Mean: $\overline{x^{\prime}}=$
 $\left.\begin{array}{rl}\overline{x^{2}} & =\frac{1.0927}{\left(\overline{x^{\prime}}\right)^{2}}=-1.0547 \\ \left(s_{x}\right)^{2} & =-0.0380\end{array}\right\}$ True Mean $\bar{x}=1.0270 \quad\left(s_{x}\right)^{2}=$
$N /(N-1)=$ $\qquad$ Standard Deviation: $s_{x}=$ 0.1950
II. Parameters:

```
\(\sigma_{n}=1.0811\)
\(1 / a=s_{x} / \sigma_{n}=0.1804\)
\(1 /(a \sqrt{N})=(1 / a) / \sqrt{N}=0.0376\)
```

                                \(\bar{y}_{H}=\quad 0.5283\)
    $\bar{y}_{n}(1 / a)=0.0953$
$u=\bar{x}^{\mp} \bar{y}_{\text {m }}(1 / 0)=0.9317 \quad$ mode
$\xrightarrow{\text { NOTE: Upper sign used for maxima, }}$
III Line of Expected Extremes:
$\begin{array}{ccccc}x=u \pm(1 / a) y= & 0.9317 & + & 0.1804 \\ y: & -2.00 & 0.00 & 3.00 & 5.00 \\ 4.60\end{array}$

v. Half-width of 0.68269 Confidence Bond, $\sigma_{x, m}=\sigma_{y, m} \sqrt{N} /(a \sqrt{N})=\left(\sigma_{y, m} \sqrt{N}\right)[(1 / a) / \sqrt{N}]$ :

| $\Phi(x):$ | .150 | .200 | .300 | .400 | .500 | .600 | .700 | .800 | .850 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\sigma_{y, m} \sqrt{N}:$ | 1.255 | 1.243 | 1.268 | 1.337 | 1.443 | 1.598 | 1.835 | 2.241 | 2.585 |

    \(\sigma_{x, m}: .0472 .0467 .0477 .0503 .0543 .0601 .0690 .0843 . .0972\)
    Far largest value, \(\Delta_{x, n}=1.141\) (1/a) \(=\)
    For next-ta-largest value, $\Delta_{x, N-1}=.759[\mathrm{~N} /(\mathrm{N}-1)](1 / a)=$ 0.1369

ㅍ. Expected Extreme, in $T$ periods (years, etc.): $x_{T}=x_{10}+Z_{T}\left(x_{100}-x_{10}\right)$ : $x_{100}-x_{10}=$ $\qquad$


Ploce:
Data: - NACA - Langley Sample No. III

Computer: WRTE Date: 5/12/52

$$
3
$$


a Values of $B_{p}=\left[\left(6 / \pi^{2}\right) \sigma^{2}\left({ }_{G}\right)\right]^{\frac{1}{2}}$ are based on empirical sampling methods. See Appendix $E$.
$\stackrel{b}{ }$ For explanation of the quantity of the form $(1 / k) Q_{m}$ appearing in $\Delta_{0}$, see equation ( $E, 8$ ) in Appendix $E$ and accompanying discussion.
$\varsigma$ Based on assumption of normality for order-statistics estimator. For discussion, see Appendix , Section 1.
d. Applies only for $P \geq n /(n+1)=.909,952$, 968 for $n=10,20,30$, respectively. See footnote to page 97 (Appendix D).

## TABLE IX

Empirical saupling values of finst and second monents of sample mean y and standard deviation sor samples of $x=10,20,30$, compared with the corresponding theoretical value where obtainable
(Discussed in Appendix E)

| Estimate of | $\begin{gathered} n=10 \\ (1200 \text { smples }) \end{gathered}$ |  | $\begin{gathered} B=20 \\ (600 \text { samples }) \end{gathered}$ |  | $\begin{gathered} n=30 \\ (400 \text { samples) } \end{gathered}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Empirical values | Theoretical values | Empirical palues | Theoretical values | Empirical values | Theoretical values |
| $E(y)$ | 0.5698 | 0.5772 | 0.5698 | 0.5772 | 0.5698 | 0.5772 |
| $\sigma^{2}(\bar{y})$ | 0.1663 | 0.1645 | 0,0884 | 0.0822 | 0.0535 | 0.0548 |
| E(\$) | 1.1656 |  | 1.2211 |  | 1.2459 |  |
| $\sigma^{2}(\mathrm{~s})$ | 0.1321 |  | 0.0775 |  | 0.0513 |  |
| $\sigma\left(\frac{1}{y}, \mathfrak{3}\right)$ | 0.0800 |  | 0.0438 |  | 0.0297 |  |



Figure 2 - Density function $f(x)$ for extreme-value distribution
with parameters $\beta=1, u=0: f(x)=\exp \left(-x-e^{-x}\right)$.
(4dapted from reference 18, Figure 2 )
(x) よ ‘uotqounJ א7tsuəa





Figure 4.-Graphical analysis of a sample of 23 maximum acceleration increments by method of order statistics.
(Data from Worksheet 2)

Figure 5.-Comparison of order-statistics and Gumbel
methods of analyzing a sample of 23 maximum acceleration increments, showing 68-percent control curves. (8 obsorvations at lower end omitted to avoid crowding.)
(Data from Worksheet 2 and Table VII, Part B)
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## THE NATIONAL BUREAU OF STANDARDS

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[^0]:    * The technical terms used here are defined and discussed in the main text, and they can be located with the aid of the list of symbols in Section 3.

[^1]:    ＊That is，$W e$ are concerned with the transformed parameters（ $\bar{\xi}_{2} \beta^{\prime}$ ），obtained from the original parameters $(u, \beta)$ by the linear transformation $\xi=u+\beta, y, \beta^{\prime}=\beta$ 。We shall henceforth give attention only to the first parameter \％，disregarding the second parameter $\beta^{0}$ of the transformed pair $\left(\frac{\xi}{3}, \beta^{1}\right)$ ） Whenever it should become necessary to refer to $\beta^{\prime}$ 。 however，the prime will be dropped for simplicity． （See second footnote to page 26）．

[^2]:    \% The variate $x$ is sometimes referred to as the original or "unreduced" variate。

[^3]:    * These variances are equal because they depend only upon $m, P$, and $\beta$, which are constant for all the subgroups of the same sample.

[^4]:    * If the observations are not available in their original order, it will first be necessary to randomize them by use of a table of random numbers.

[^5]:    \# I.e., coordinate paper with one scale (x) uniformly spaced and the other (y) distorted in such manner that the extreme-value distribution $\exp \left(-\theta^{-} \mathrm{Y}\right)$ will plot as a straight line.

[^6]:    * Since these calculations are mainly of theoretical interest, they have been omitted in order to keep this report from becoming unduly long.

[^7]:    * The notation in the table cited differs from that used here: $E\left(y_{f}\right)$ in this report corresponds to $E\left(y_{n-i}\right)$ in the table.

[^8]:    *When (as will generally be the case) the ranks $\lambda_{n}$, $\mu \mathrm{n}$, 2 n are not integers, they will be defined to be the nearest integers to these quantities.

[^9]:    * These considerations assune that the sample of data is already at hand, perhaps by a survey already made, such as the Thunderstorm Project mentioned in reference 18. Of course, if it is a question of planning for the securing of data, it is desirable to use as efficient an estinator as possible, but in that case, the investigation will rarely be sufficiently extensive to provide sampleslarge enough for the method described in this Appendix to be applicable.

[^10]:    Cramér-Rao lower bound is $Q_{L B}=Q_{0} / n$, where $Q_{0}=\left(0.60793 y_{P}^{2}+0.51404 Y_{P}+I_{0} 10866\right) \beta^{2}$, $n$ is sample size。
    H** These lines give the variances and efficiencies for the ordermstatistics estimator of the parameter u.
    *** The variances for $P=1$ are all infinite. Expressing them by means of the dominant term in $Q_{n}$ permits

[^11]:    *For explanation of $Q$, see equation ( $E_{\circ} 8$ ) in Appendix $E$ and accompanying discussion.

